Stochastic Comparison of Discounted Rewards

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Abstract

It is well known that the expected exponentially discounted total reward for a stochastic process can also be defined as the expected total undiscounted reward earned before an independent exponential stopping time (let us call this the stopped reward). Feinberg and Fei recently showed that the variance of the discounted reward is smaller than the variance of the stopped reward. We strengthen this result to show that the discounted reward is smaller than the stopped reward in the convex ordering sense.

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Consider the following definitions of Feinberg and Fei (2009).

\[ J_1 = \int_0^\infty e^{-\alpha t} r_t dt + \sum_{n=1}^\infty e^{-\alpha T_n} R_n \]

where \( r_t \) is an \( \mathcal{F}_t \)-adapted stochastic process representing the reward rate at time \( t \), \( \mathcal{F}_t \) is an increasing filtration on a probability space \((\Omega, \mathcal{F}, P)\), \( T_n \) is an \( \mathcal{F}_t \)-adapted stopping time, and \( R_n \) is an \( \mathcal{F}_{T_n} \)-adapted stochastic sequence representing lump sum rewards, and

\[ J_2 = \int_0^T r_t dt + \sum_{n=1}^{N(T)} R_n \]

where \( T \sim \exp(\alpha) \), independent of \( \mathcal{F}_\infty \), and \( N(t) = \sup\{ n : T_n \leq t \} \) is the number of stopping times before \( T \). Then \( J_1 \) represents the total discounted reward, and \( J_2 \) represents the total undiscounted reward earned before an exponential time \( T \). It is well known that \( EJ_1 = EJ_2 \), and Feinberg and Fei (2009) showed that \( Var(J_1) \leq Var(J_2) \). We show the following more general result, where for two random variables \( X \) and \( Y \), \( X \leq_{cx} Y \) if and only if \( Ef(X) \leq Ef(Y) \) for all convex functions \( f \).

**Theorem** \( J_2 \geq_{cx} J_1 \).

**Proof.** Let us make the dependency on \( T \) explicit and write \( J_2(T) \). We have that \( E[J_2(T)|\mathcal{F}_\infty] = J_1|\mathcal{F}_\infty \) so \( J_2(T)|\mathcal{F}_\infty \geq_{cx} J_1|\mathcal{F}_\infty \), because, from Jensen’s inequality, for any random variable \( X \), \( X \geq_{cx} EX \). Therefore, for any convex function \( f \),

\[ E[f(J_2(T))|\mathcal{F}_\infty] \geq_{cx} E[f(J_1)|\mathcal{F}_\infty] \]

so

\[ E[f(J_2(Y))] = E[E[f(J_2(T))|\mathcal{F}_\infty]] \geq_{cx} E[f(J_1)|\mathcal{F}_\infty] = E[f(J_1)], \]

and therefore \( J_2(T) \geq_{cx} J_1 \). \( \square \)

**References**