Linear Tariffs with Quality Discrimination

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The Bell Journal of Economics
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Linear tariffs with quality discrimination

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When a good or service is offered in a variety of qualities with corresponding marginal prices, a quality-based allocation of consumption units by buyers is induced. This enables a monopolist supplier to achieve partial discrimination among buyers with different consumption preferences. We consider the situation in which a monopoly supplier, with complete information about the distribution of customer preferences, selects a generalized two-part tariff consisting of a single fixed subscription charge and quality dependent marginal charges. We analyze consumer behavior and optimal pricing strategies with and without positive demand externalities.

1. Introduction

Background. The economics of goods or services that are offered at multiple quality levels provide some interesting extensions of standard microeconomics. If the tariff consists of a fixed subscription fee and a variety of quality-dependent marginal prices, a customer's purchase decision has two stages. He must decide whether to subscribe, and then, if he subscribes, he must select a distribution of consumption quantities over the various quality levels. These decisions depend on the subscription fee, the quality versus price differentials, and the customer's utility function, which may vary from one customer to another. In markets such as telecommunications, where positive externalities are present, subscribers' benefits and consequently their purchase decisions are also influenced by the total number of subscribers.

Linear tariffs in markets with demand externalities have been analyzed in several economic contexts. Monopoly pricing conditions were investigated by Oi (1971). Models for telephone communication services involving mutual subscriber benefit were developed by Artie and Averous (1975), Squire (1973), and Littlechild (1975). The effect of tariff structure on the formation of a critical mass of subscribers has been analyzed by Rohlfis

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Support for this work was provided by the Xerox Corporation and by the National Science Foundation (grants IST-8108350 and SES-8108226). Discussions with our colleagues, Robert Gibbons and Charles Feinstein, have been helpful in the course of this research. We would like to thank the referees and the Editorial Board for two very detailed reviews, which have significantly improved the paper. We are particularly grateful to Alvin Kleinerman, who read all versions of the article in detail, contributed many valuable suggestions, and expedited the communications of the review process.

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Quality dependent pricing designed to discriminate through customer self-selection was formulated and analyzed by Mussa and Rosen (1978). Their model assumes a separable utility function, and each customer has one unit of consumption, which he assigns to some quality level. Slokey (1979) also considered a one-unit consumption model, in which the timing of consumption is variable. A fundamental idea in these models is the use of information on buyers' preferences that is provided through their consumption choices. This enables a supplier to achieve partial price discrimination while offering the same tariff to all buyers. The notion of partial discrimination through self-selection has also appeared in the context of nonlinear quantity dependent pricing in Spence (1977) and in Goldman, Leland, and Sibley (1977). Tariffs which are based on simultaneous self-selection of quality and quantity of consumption have been considered recently by Mirman and Sibley (1980), who analyze nonlinear tariffs that are offered at multiple discrete quality levels.

**Scope of our analysis.** We consider the situation in which a monopoly supplier offers a good or service in different quality levels, with the marginal price for each quality level being a constant. A single fixed charge may also be imposed for subscription to the service, i.e., for the right to purchase at the posted unit prices. Income effects are ignored. Different consumption unit types will be defined in such a way that preferences are uniform across all consumers. This is achieved by assuming that demand variations due to quality and unit price changes can be captured by a functional which is independent of consumer type. Total potential consumption of the various unit types, however, may vary by consumer type and may depend upon the number of subscribers. Thus, demand externalities may influence preferred consumption levels. Each consumer compares the total derived benefit with the total charges to decide whether subscription is warranted. It is assumed that the supplier knows all the consumption preferences in the population and thus the demand function for each consumer type.

Our analysis generalizes those previously mentioned in that a fixed subscription charge and positive demand externalities are included. Also, by viewing quality levels as a continuum, we are able to identify those regions of quality levels which should be offered and those which should be excluded by a monopoly supplier. In practice, quality levels would, of course, be discrete, but the continuum analysis indicates the pricing, subscription levels, and externality effects that would result for a "large" number of discrete qualities.

In Section 2, we present the specific model used in our analysis and motivate the assumptions that we make. Then, in Sections 3 and 4, we derive optimality conditions for the consumers and for the supplier and compare these with analogous conditions that have been obtained in other economic contexts. Next, we extend the optimality conditions to include positive demand externalities. In Section 6, we consider a particular example, and we use the example to compare the usage volume, tariff, net revenue, and total consumer surplus that result with and without quality discrimination. Section 7 contains concluding remarks.

The underlying structure of our model was suggested by the markets for communications services that are supplied in a variety of qualities. In conventional telephone service, for example, quality differentiation might correspond to the various times for placing long distance calls, such as daytime, afternoon, late night, weekend, etc., with the assumption that preferences for calling times are uniformly held. The recently offered facsimile transmission services also provide an excellent example. Tariffs vary linearly in quantity, but customers are permitted to choose different priority levels, for correspondingly adjusted marginal charges, to insure that urgent messages are delivered more quickly. Consumption unit types could be defined by urgency levels, which would be measured
in terms of the customers' willingness to pay for priority delivery for each particular type. Classes of mail delivery also provide examples in the postal service market and in private parcel delivery services. Tariffs which are linear in quantity prevail for the most part in all of these examples. The reasons for linear tariffs include tradition and customer convenience as well as economic causes such as the presence of resale markets or the supplier's inability to monitor and control purchase sizes. Fixed subscription charges are explicit in some cases, but also may arise through the monthly charges for a terminal, facsimile device or other equipment required to obtain the service. Multiple or quality dependent fixed charges may also arise in this way, but we have not included these in our analysis because of the analytical difficulties they create.

2. Model formulation

In general, the utility that a subscriber derives is determined by the quantity of units he consumes at each quality level. We assume that quality is indexed by a variable $s$, $s_0 \leq s < \infty$, where smaller values of $s$ are always preferred to larger ones. In communications, one might think of $s$ as speed of delivery, with $s_0$ being some bound on the best quality level offered by the supplier. Similarly, consumer types are indexed by $t$, which will be treated as a continuous variable. Without loss of generality we may assume that the index $t$ is uniformly distributed on the interval $[0, 1]$.\footnote{For example, suppose consumers were indexed by their income level $I$, with cumulative distribution $G(I)$. Then one could define a new index $t = 1 - G(I)$, so that the index corresponding to a particular consumer denotes the fraction of consumers whose income is higher than his. The new index has the desired property of being uniform on the unit interval.}

Consider the demand for each particular quality $s$ as a function of its unit price $p$. Each consumer type $t$ will have a demand function

$$d(s, p, t) = \text{quantity of units demanded by consumer type } t, \text{ when a single quality } s \text{ is offered at marginal price } p.$$ 

We shall assume that a change in the quality level $s$ can always be compensated for by a corresponding change in the marginal price $p$, such that demand remains the same. Furthermore, we shall assume that the compensation adjustment is the same for all consumer types. Mathematically, this can be stated as

**Assumption 1.** There exist functions $g(s, p)$ and $\tilde{Q}(z, t)$ such that the demand function can be written as

$$d(s, p, t) = \tilde{Q}(z, t), \quad \text{where} \quad z = g(s, p),$$

and the function $g(s, p)$ is independent of $t$. Furthermore, the level curves of $g$ and $\tilde{Q}$, in the respective $s$, $p$ and $z$, $t$ planes, do not cross.

That is, demand depends only upon the index value $z$ and the consumer index $t$, and not upon the particular $s$, $p$ values. The index $z$ serves as a numeraire on the $s$, $p$ indifference curves. Without loss of generality, we may assume that $z \in [0, 1]$ and $\partial \tilde{Q} / \partial z > 0$. This implies, from the definition of $Q$, that $\partial z / \partial p < 0$ and $\partial z / \partial s < 0$.

Let $\hat{u}(s, t, d)$ represent the inverse demand function $p = \hat{u}(s, t, d)$, which gives the marginal willingness to pay of customer $t$ for the $d$th unit of demand at quality level $s$. Assumption 1 implies that this inverse demand function can be decomposed as shown by the following lemma.

**Lemma 1.** For any given $s$, the inverse demand function consistent with Assumption 1 is given by

$$\hat{u}(s, t, d) = u(s, z) = g^{-1}(s, z).$$
where \( z = \tilde{Q}^{-1}(t, d) \), and the inverse functions of \( g \) and \( \tilde{Q} \) are taken with respect to \( p \) and \( z \), respectively.

**Proof:** By definition, \( \bar{u}(s, t, d) \) must satisfy

\[
d(s, \bar{u}(s, t, d), t) = d.
\]

Also,

\[
g(s, \bar{u}(s, z)) = z = \tilde{Q}^{-1}(t, d).
\]

But by Assumption 1 we have

\[
d(s, u(s, z), t) = \tilde{Q}(g(s, u(s, z))), t) = \tilde{Q}(\tilde{Q}^{-1}(t, d), t) = d.
\]

Hence, \( u(s, z) \) satisfies the required condition on the unique inverse demand function. \( Q.E.D. \)

It follows from the above that the demand function characterized by Assumption 1 can be equivalently represented by the functions \( u(s, z) \) and \( \tilde{Q}(z, t) \), where \( z \) in both functions is a common index. It also follows that \( du/d\tilde{z} < 0 \) and \( du/ds < 0 \). Since \( du/d\tilde{z} < 0 \), \( z \) can be thought of as a kind of priority index on units of consumption, with \( z = 0 \) corresponding to the highest priority and \( z = 1 \) corresponding to the lowest. Given this interpretation of \( z \), we then have

\[
u(s, z) = \text{willingness to pay for consuming each unit of type } z \text{ at quality level } s\]
and

\[
Q(z, t) = \partial \tilde{Q}(z, t)/\partial z = \text{density function for units of type } z \text{ for customer type } t.
\]

The functions \( Q(z, t) \) and \( u(s, z) \) can be estimated directly based on consumers' preferences in the marketplace. First, a priority index \( z, 0 \leq z \leq 1, \) is assigned to unit types, so that smaller \( z \) values indicate more preferred units. In communications services, for example, the index \( z \) could indicate the urgency of delivering particular units. The key assumption is that the priority ranking of units is invariant to quality level changes. If the overall market sensitivity of willingness-to-pay to quality variations is known, the function \( u(s, z) \) might be selected to match that sensitivity for the market as a whole, assuming the same function applies to all consumer types. Variations by individual type could be accounted for through the choice of \( Q(z, t) \). If the functions \( u(s, z) \) and \( Q(z, t) \) are determined in this way, the form of the demand function in Assumption 1 is implicitly assumed. Hence, \( Q \) and \( u \), which will be used in our subsequent analysis, fully specify the demand function \( d \).

**Customer's optimal quality selection.** The consumption choice pattern of each subscriber can be characterized in terms of a density function \( \theta(s, z) \), specifying his quantity distribution of units of type \( z \) over the various quality levels, i.e.,

\[
\theta(s, z)ds = \text{quantity of units of type } z \text{ consumed at quality levels in the interval } [s, s + ds].
\]

The total benefit or willingness to pay of customer \( t \) for a consumption distribution \( \theta(s, z) \) is:

\[
U(t; \theta(s, z)) = \int_0^1 \int_{s_0}^\infty \theta(s, z)u(s, z)dscdz,
\]

since \( u(s, z) \) is the marginal utility or willingness to pay for consuming one unit of type \( z \) at quality level \( s \). The consumption distribution \( \theta(s, z) \) by customer \( t \) clearly must satisfy

\[
\int_{s_0}^\infty \theta(s, z)ds \leq Q(t, z) \quad \text{for all } \quad z \in [0, 1].
\]
For the supplier, we assume that regulatory constraints or lack of information prevent price discrimination on the basis of consumer or consumption unit type. Thus, the tariff structure is nondiscriminatory with respect to these indices and is based only on quality level. Specifically, we consider tariffs composed of quality dependent, but quantity invariant, marginal prices \(m(s)\) and a single fixed subscription charge \(F\). (The function \(m(s)\) is assumed to be twice continuously differentiable.) Thus, the total charge for a consumption distribution \(\theta(s, z)\), would be of the form

\[
F + \int_0^1 \int_{z_0}^{z} \theta(s, z)m(s)dsdz.
\] (1)

Consider now the consumer’s problem of selecting his optimal \(\theta(s, z)\). This selection can be determined separately for each \(z\), since \(F\) affects only the subscription decision. Thus, a consumer who maximizes benefit minus cost will select an optimal consumption distribution \(\theta^*(s, z)\) which solves the problem

\[
\max_{\theta(s, z)} \int_{z_0}^{z} \theta(s, z)[u(s, z) - m(s)]ds,
\]
subject to \(0 \leq \int_{z_0}^{z} \theta(s, z)ds \leq Q(t, z)\).

The “optimal” quality level \(s(z)\) for each \(z\) is defined by

\[
s(z) = \arg \max_{s \geq 0} [u(s, z) - m(s)].\]
(2)

Since both the objective function and the constraint are linear, it is clear that the customer should concentrate the maximum quantity \(Q(t, z)\) at \(s(z)\), unless \(u(s(z), z) - m(s(z)) < 0\), in which case the optimal consumption level for unit type \(z\) is 0. Thus, we can state that

\[
\theta^*(s, z) = \begin{cases} 
Q(t, z) & \text{if } s = s(z) \text{ and } u(s(z), z) \geq m(s(z)); \\ 
0 & \text{otherwise.}
\end{cases}
\]

Consequently, the total benefit that a subscriber of type \(t\) derives from optimal consumption of his type \(z\) units is

\[
U(t; \theta^*(s, z), z) = \begin{cases} 
Q(t, z)u(s(z), z) & \text{if } u(s(z), z) \geq m(s(z)); \\ 
0 & \text{otherwise.}
\end{cases}
\] (3)

\[
\square \quad \text{Monotonicity property for optimal qualities.} \quad \text{We stated previously that } z \text{ is an indicator of the customer’s relative willingness to pay for that particular unit type. We shall assume that this relative preference intensifies as quality levels improve and diminishes for lower quality levels. Stated mathematically, we have}
\]

**Assumption 2.** For all \(s\) and \(z\), the per unit utility function \(u(s, z)\) satisfies \(\partial^2 u/\partial s \partial z > 0\).

The motivation for this sort of monotonicity assumption is illustrated as follows. Suppose there is a finite set of service quality levels \(s_0, s_1, \ldots, s_n\). Let the \(\{s_i\}\) be in order of decreasing service quality, where choosing \(s_n\) corresponds to not using the service. Given the marginal tariffs \(m(s_i) = m_i\), the consumer’s optimal behavior is simply to choose

\[
\max_i \{u(s_i, z) - m_i\}, \text{ for each } z.
\] (4)

Such a situation is illustrated in Figure 1, for the special case in which \(u(s, z)\) is linear in \(z\). This example has the desirable property that more valuable units (lower \(z\) values) are assigned to better service qualities. This gives rise to a set of break points \(z_0, z_1, \ldots, z_{n+1}, \ldots\).
with $z_0 = 0$ and $z_{n+1} = 1$, such that units with index $z \in [z_i, z_{i+1})$ are assigned to service quality $s_i$. Much of our analysis becomes intractable unless this "well-ordered" selection pattern holds. As shown by the following lemma, this form for the optimal policy results if and only if Assumption 2 holds.

**Lemma 2.** Let $s_1$ and $s_2$ be the respective optimal service levels for $z_1$ and $z_2$, i.e., $s_1 = s(z_1)$, $s_2 = s(z_2)$, then $(s_1 - s_2)(z_1 - z_2) > 0$ if and only if $u_{z2} = \partial^2 u/\partial s \partial z > 0$.

**Proof:** Since $s_1, s_2$ are optimal for $z_1, z_2$, respectively, we have

$$u(s_1, z_1) - m_1 > u(s_2, z_1) - m_2$$

and

$$u(s_2, z_2) - m_2 > u(s_1, z_2) - m_1 .$$

Adding the inequalities and rearranging, we have

$$[u(s_1, z_1) - u(s_1, z_2)] - [u(s_2, z_1) - u(s_2, z_2)] > 0 .$$

If $(s_1 - s_2)(z_1 - z_2) > 0$, we may divide by this quantity. Letting $z_1 \to z_2$ and then $s_1 \to s_2$ establishes that $u_{z2} > 0$.

On the other hand, applying the Mean Value Theorem twice, first to the function $\Phi(s) = u(s, z_1) - u(s, z_2)$ to obtain

$$0 < \Phi(s_1) - \Phi(s_2) = (s_1 - s_2)[u_\xi(z_1) - u_\xi(z_2)]$$

and then again, we obtain $(s_1 - s_2)(z_1 - z_2)u_{z2}(\xi, \eta) > 0$, for some $\xi, \eta$ such that $s_1 \leq \xi \leq s_2$, $z_1 \leq \eta \leq z_2$. If $u_{z2} > 0$ for all values, the converse is proved. Q.E.D.

Thus, more valuable consumption units are always assigned to higher service quality levels,
no matter what the \( m_i \) are. There may, however, be certain service qualities \( s_i \) that are never selected.

\( \square \) Necessary conditions for optimal customer behavior. In addition to producing the above ordering, Assumption 2 allows us to use first-order necessary conditions to obtain \( s(z) \). The customer's selection of the optimal service quality \( s \) for any given \( z \) must satisfy the first-order necessary condition for (2),

\[
u_s = \partial u(s, z)/\partial s = m'(s),
\]

as long as \( s \) is an interior point with regard to any constraints. If we view \( m'(s) \) as the charge for receiving the additional service improvement from \( s + ds \) to \( s \), (5) is a standard optimal consumption condition. As we proceed with our analysis, treating successive service qualities as incremental improvements will continue to be a useful interpretation.

Consider the trajectory \( s(z) \), \( 0 \leq z \leq 1 \), determined by condition (5), and suppose that \( u \) and \( m \) are such that \( s'(z) \) exists. From the Implicit Function Theorem we have that

\[
s' = -(\partial^2 u/\partial s^2)\partial s/\partial z - m'' \leq 0 \text{ since this is a second-order necessary condition for a local maximum. The numerator is negative when Assumption 2 holds. Thus } s' > 0, \text{ i.e., monotonicity in the continuous case holds if and only if the first-order condition (5) yields a maximum. An equivalent condition for } s'(z) > 0 \text{ is that there is an inverse function } z(s) \text{ such that } z(s), s \text{ are optimal pairs and } 0 \leq z'(s) < \infty. \text{ This inverse function } z(s) \text{ is the continuous analog of the break points } z_0, z_1, \ldots, z_m \text{ and will be a convenient way to express the optimal policy.}
\]

3. Determining the subscriber set and the qualities selected

- Quality levels offered. Since the function \( u(s, z) \) is independent of the customer type, the endpoints of the optimal trajectory \( z(s) \) as well as the trajectory itself will be independent of \( t \). At each endpoint, the optimal trajectory terminates in one of two ways. As \( z(s) \) moves in the direction of decreasing \( z \) and \( s \), either \( z = 0 \) or \( s = s_0 \) will eventually occur, yielding a lower terminal condition. As \( s \) and \( z \) increase, either \( z = 1 \) will be reached or an \( s^* \), \( z(s^*) \) pair will arise such that \( u(s^*, z^*) = m(s^*) \). Since \( \partial u(s, z)/\partial z < 0 \), we know that

\[
u(s, z) - m(s(s^*) - m(s) \leq u(s^*, z^*) - m(s^*) = 0 \text{ for all } z > z^*.
\]

Thus the constraint \( u(s, z(s)) \geq m(s) \) becomes binding at most once. The two possible forms for the lower endpoint of \( z(s) \) are illustrated in Figure 2. The corresponding equa-

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**Figure 2**

**Two cases for the lower endpoint of the optimal trajectory**

![Diagram](image-url)
tions are as follows for respective lower and upper endpoint pairs \((z_*, s_*)\) and \((z^*, s^*)\):
\[
\frac{\partial u(z_*, s_*)}{\partial s} - m'(s_*) = 0 \quad \text{and} \quad z_* \geq 0, \quad s_* = s_0
\tag{6}
\]
or
\[
\frac{\partial u(z_*, s_*)}{\partial s} - m'(s_*) > 0 \quad \text{and} \quad z_* = 0, \quad s_* \geq s_0
\tag{7}
\]
and
\[
u(z^*, s^*) - m(s^*) = 0 \quad \text{and} \quad z^* \leq 1
\tag{8}
\]
or
\[
\frac{\partial u(z^*, s^*)}{\partial s} - m(s^*) > 0 \quad \text{and} \quad z^* = 1.
\tag{9}
\]

Once the endpoints have been determined from (6)-(9), the optimal trajectory must satisfy
\[
\frac{\partial u(s, z(s))}{\partial s} = m'(s) \quad \text{and} \quad z'(s) \geq 0 \quad \text{for} \quad s_* \leq s \leq s^*.
\tag{10}
\]

As we mentioned previously, certain intervals of qualities within \([s_*, s^*]\) may not be selected, and these would correspond to regions in which \(z'(s) = 0\). These "gaps" may appear even when \(m(s)\) is selected optimally by the seller. In optimizing the seller's tariff, the location of the gaps will be quite apparent as illustrated in the next section. Outside the range \([s_*, s^*]\), of course, no consumption occurs.

\[\square \text{ Determining the marginal subscriber.} \quad \text{Once the optimal allocation} \quad s(z) \text{ or } z(s) \text{ has been determined, we can calculate the maximal consumer surplus } CS(t) \text{ for each customer. Customer } t \text{ will subscribe if and only if:}
\]
\[
CS(t) = \int_{0}^{z^*} Q(t, z)[u(s(z), z) - m(s(z))]dz - F \geq 0.
\tag{11}
\]

The set of consumer types \(t\) for which (11) holds is known as the subscriber set. The analysis gets quite complex unless we can guarantee that the subscribers with smaller \(t\) indices are always more likely to subscribe. The simplest way of achieving this is to require that potential demand is uniformly larger for all \(z\) values as \(t\) decreases. Thus, we introduce the following additional assumption.

Assumption 3. For all \(t, z\), we have \(\frac{\partial Q}{\partial t} < 0\).

If \(F = 0\), this assumption is unnecessary since all customers would subscribe in any case. For \(F > 0\), however, it guarantees that the set of subscribers is of the form \([0, y]\), \(0 \leq y < 1\).

This also indicates the analytical difficulties that would arise if separate fixed charges were attached to each quality level. Sufficient monotonicity assumptions would be needed so that subscriber sets are intervals with regard to each quality level; otherwise the problem would be intractable. Thus we chose not to pursue this generalization.

4. Seller's optimal tariff

\[\square \quad \text{Given the buyer's selection rules, which we just determined, the seller wishes to select a tariff plan } \{F, m(s)\} \text{ that maximizes his profit, aggregated over all buyers. We assume that the subscribing customers correspond to a set of indices } [0, y] \text{ as described above. We also assume that the supplier knows the distribution of potential consumption } Q(t, z) \text{ and the marginal utility } u(s, z), \text{ but does not discriminate on the basis of the indices } t \text{ or } z.
\]

We formulate and solve the general continuous case which has fixed charge \(F\) and variable charge \(m(s)\). The seller's net revenue for a subscriber set \([0, y]\) is given by
\[
\Pi(y) = \int_{0}^{y} \int_{0}^{z^*} Q(t, z)[m(s(z)) - c(s(z))]dzdt + (F - k)y,
\tag{12}
\]
where $k$ is the fixed cost of supplying a subscriber and $c(s)$ is a marginal quality-dependent supply cost. The inner integral has been expressed using the customer's quality selection function $s(z)$. Assuming optimal buyer behavior, this is obtained from (6)–(9). When $F$ is optimally selected for any $y < 1$, condition (9) reduces to
\begin{equation}
F = \int_0^{s^*} Q(y, z)[u(s(z), z) - m(s(z))]dz,
\end{equation}
for the marginal subscriber $y$.

In formulating the supplier's optimization problem, the marginal prices $m(s)$ do not depend upon $t$, and thus all buyers will select the same optimal fractional allocation function $s(z)$. The endpoints $(s_*, s_*)$ and $(z^*, s^*)$ are determined according to (6)–(9), and they may vary with $y$, but not with $t$.

We may now use (13) to substitute for $Fy$ in (12). Defining the total potential demand,
\begin{equation}
T = T(y, z) = \int_0^y Q(t, z)dt \quad \text{and} \quad q = yQ(y, z),
\end{equation}
we may express the supplier's optimization problem as
\begin{equation}
\max_{m, z, y} \Pi(y) = \int_0^{z^*} [T(m - c) + q(u - m)]dz - ky,
\end{equation}
subject to $u_t = m'$ for $s_* \leq s \leq s^*$ and (6)–(9).

Now introduce the variable change $z = s(z)$, where $s(z)$ is the inverse of the function $s(z)$. This inverse is defined only for $s_* \leq z \leq s^*$, so that the integral over $[0, s_*]$ must be accounted for separately. Since $u_t = m'$ is to yield a maximum, we require in addition that $z(s) > 0$ for $s_* \leq s \leq s^*$, as discussed previously in Lemma 2. Assuming for the moment that this constraint is not active, we can restate the problem as
\begin{equation}
\max_{m, z, y} \int_{s_*}^{s^*} [T(m - c) + q(u - m)]z'ds + K(z_*),
\end{equation}
where
\begin{equation}
K(z_*) = \int_0^{s_*} \{q(u(s, s_0) - m(s_0)) + T(m(s) - c(s_0))\}dz - ky.
\end{equation}
This is subject to $u_t = m'$ along the trajectory $s(z)$, $s$, for $s_* \leq s \leq s^*$, and the boundary constraints (6)–(9).

Integrating all terms in (15) except $qu$ by parts and substituting $u_t$ for $m'$, we obtain the equivalent problem
\begin{equation}
\max_{z, y} \int_{s_*}^{s^*} [T_j(c' - u_t) + qu_t + quz']ds + \int_{s_*}^{s^*} K(z_*)d*,
\end{equation}
where $T_j$ and $q_j$ denote the indefinite integrals
\begin{equation}
T_j = \int T(y, z)z'\,dz = \int T(y, z)dz \quad \text{and} \quad q_j = \int q(y, z)z'\,dz = \int_0^y q(y, z)dz.
\end{equation}
For $s_* < s < s^*$, the necessary conditions for the optimal trajectory $s(z)$ may be obtained by applying the Euler condition $\partial L/\partial z = d(\partial L/\partial z')/ds$ to the integrand $L$ in (16). The terms involving endpoints subject to (6)–(9) will be optimized in the Appendix by applying a transversality condition. The left-hand side of the Euler condition is
\begin{equation}
\partial L/\partial z = T(c' - u_t) + qu_t + (q_j - T_j)u_t + z'\partial(qu)/\partial z,
\end{equation}
while the right-hand side gives
\begin{equation}
d(\partial L/\partial z')/ds = d(qu)/ds = qu + [\partial(qu)/\partial z]z'.
\end{equation}
Upon equating these terms and simplifying, we obtain the condition

\[ 0 = (T_i - q_i)u_{se} + T(u_s - c'), \quad \text{for} \quad s_e < s < s^*, \quad (17) \]

or

\[ (1 - R)T_i(d/dT_i(u_{se})) = 1 - c'/u_s, \quad (18) \]

where

\[ R = q_i/T_i. \]

Equation (18) looks similar to the standard elasticity condition for a monopolist seller, except for the scale factor \( 1 - R \) on the left-hand side. The factor \( R = q_i/T_i \) is the ratio of the smallest subscriber’s volume to the average subscriber’s volume, and it is analogous to the scale factor obtained for the single quality case by Oi (1971). It can be explained as follows. For (13) to be satisfied for the same fixed \( y \), as \( m' \) is changed, it can be seen that only the fraction \( 1 - R \) of the net revenue change is actually retained by the supplier. That is, a price increase \( \Delta m' \) resulting in increased revenue \( T_i \Delta m' \) requires that \( q_i \Delta m' \) be refunded through reduced fixed charges if (13) is to remain valid. Thus, the scaling factor \( 1 - R \) accounts for the fixed charge rebate required to maintain the same level of market penetration \( y \), while varying the marginal service price schedule.

Equation (18) is the functional relationship for a rational buyer’s allocation \( z(s) \) that maximizes the supplier’s profit. Since it involves only known functions, it can, in principle, be solved pointwise for the optimal \( z(s), s \) pairs. If the resulting solution \( z(s) \) satisfies \( z'(s) \geq 0 \), the optimal marginal price plan \( m(s) \) can then be determined from \( u_s = m' \) in the range \( s_e \leq s \leq s^* \). Solutions violating \( z'(s) \geq 0 \) can be modified into feasible optimal solutions, as will be illustrated in the next section.

By considering the service quality differences as a market in service improvements, equation (18) can be interpreted as a family of elasticity conditions for a monopoly supplier. For improving service quality from \( s + ds \) to \( s \), the respective marginal charges and costs are \( m'(s) \) and \( c'(s) \). For any fixed \( s \), let \( D = D(m') \) be the demand for quality improvement as a function of its price \( m' \). Then the standard optimal monopoly pricing condition would be

\[ -D/(m'dD/dm') = 1 - c'/m'. \quad (19) \]

In determining \( z(s) \) from (18) with \( u_s = m' \), we are in effect determining the optimal price \( m' \). In fact, there is a one-to-one correspondence since \( dz/dm' = 1/u_s > 0 \). The demand for quality improvement at level \( s \) is simply the total consumption for all \( z \leq z(s) \), since these units are assigned to quality \( s \) or better.

Thus,

\[ D(m'(s)) = \int_0^\mu \int_0^{z(s)} Q(t, z)dzdt = T_i(y, z(s)) \]

and

\[ dD/dm' = (dT_i/T)(dz/dm') = T_i/u_{se}. \]

Since \( u_s = m' \), it can be seen that (18) has exactly the form (19), except for the scale factor \( 1 - R \).

Specific assumptions concerning the form of the potential demand function \( Q(t, z) \) or the subscription charge \( F \), lead to special cases of equation (18) which vary with respect to \( R \). Table 1 summarizes some interesting variations. We note that if all subscribers are the same (Case 2), then \( R = 1 \). Consequently, by (18), service improvements are priced at marginal cost, i.e., \( m' = c' \). The implication is that when all customers are identical, the supplier should collect all his profits through the fixed charge \( F \). The opposite situation, \( F = 0 \), results when either there is a priori no fixed charge, or the smallest subscriber has negligible volume.
5. Externalities and other extensions

Now consider the case in which the decision of some subscribers not to join, i.e., to
select zero consumption at zero cost, affects other subscribers. To analyze this, we shall
assume that each subscriber’s per unit consumption utility function \( u(s, z) \) remains the
same and that externalities appear only in terms of consumption volume. In a commu-
nication network, for example, each subscriber can communicate with only the fraction
of his total volume that involves other subscribers who have joined the network. When
only a subset of the potential subscribers joins, the potential communication volume for
subscribers and thus the perceived benefits of subscription are correspondingly reduced.\(^2\)

To obtain workable formulations of this problem, the externality effect must be
defined in such a way that subscribers will join in some prescribed order as the service
grows and leave in the reverse order if it contracts. For this purpose, we shall use a
construction similar to that introduced by Oren and Smith (1981), which leads to a
subscription preference ranking consistent with the customer index \( t \). This preference
ranking implies that subscriber sets are always of the form \([0, y]\) for some \( 0 \leq y \leq 1 \)
and can therefore be fully characterized by the index \( y \) denoting the marginal subscriber.
Specifically, we define for every index \( t \) and subscription level \( y \), a density function
\( Q(t, y, z) \) such that:

\[
Q(t, y, z)dz = \text{total potential consumption volume of type } [z, z + dz] \text{ for a subscriber}
\]

with index \( t \), given that all the customers whose indices are in \([0, y]\) subscribe.

Then we introduce the following stronger version of Assumption 3.

**Assumption 4.** The function \( Q(t, y, z) \) is monotonically decreasing in \( t \) for any \( y \) and \( z \).

Equivalently, this assumption implies that a ranking based on the volume that can be
realized for any subscriber set \([0, y]\) would assign the same \( t \) index to each subscriber as
does the ranking for the full subscription case. Although this is often a reasonable as-
sumption, it is not difficult to construct examples in which it is violated (Rohls, 1974).

Since we have assumed that externality effects appear only in terms of the volume
that can be sent and not in the function \( u \), the necessary conditions for profit maximiza-
can be modified in a fairly simple manner to include these effects. Letting

\[
T(y, z) = \int_0^y Q(t, y, z)dt \quad \text{and} \quad q(y, z) = yQ(y, y, z),
\]

replace \( T \) and \( q \), respectively, we can repeat the derivations of equations (13)–(19) step
by step for the externality case. The maximization with respect to \( y \) is to be performed

\(^2\text{There is, of course, an opposite type of externality due to congestion, which reduces the subscriber's benefit. This aspect, however, is beyond the scope of this article. We assume that the supply capacity is sufficiently large so that this effect is insignificant.}\)
TABLE 2  
Special Cases with Externalities

\[ Q = Q(t, y, z) \quad q_t = \int_0^t y Q(t, y, z) dz \]
\[ T_t = \int_0^t \int_0^t Q(t, y, z) dt dz \]
\[ Q = Q(t, y) \quad T_t = z T(y) = z \int_0^t Q(t, y) dt \]
\[ Q = Q(y, z) \quad R = 1, \quad c' = u_z = m' \]

last, and is, of course, more complicated in the externality case. The general necessary conditions for an optimal \( y \) are so complex algebraically that little can be said about the general solution. A specific example is solved for \( y \) in Section 6. Special cases are summarized in Table 2 above.

\[ \square \quad \text{Quality gaps. If the constraint } z'(s) \geq 0 \text{ is violated by the solution of (18), an optimal solution can be obtained by a construction like the one employed by Mussa and Rosen (1978, p. 313). Such a violation of the monotonicity of } z(s) \text{ can occur if the marginal cost increase } c'(s) \text{ in some quality range exceeds the increase in marginal utility } u_z(s, z) \text{ for some unit type } z \text{ assigned to a quality in that range. In such a case, the term on the right-hand side of (18) becomes negative and for (18) to hold, } T_t \text{ must be negative. This can happen, however, only if } z' \text{ is negative. To avoid such violations of the monotonicity constraint on } z(s), \text{ the constraint must be included explicitly in the formulation of the optimization problem. It can be verified that the resulting Euler condition will then yield a modified form of (17), involving a Lagrange multiplier } \lambda(s) \text{ corresponding to the constraint, namely} \]

\[ \lambda' = (T_t - q_t) u_{zz} + T(u_z - c'), \quad \lambda z' = 0, \quad z' \geq 0, \text{ for } s_\ast < s < s^\ast. \quad (20) \]

These conditions result from adding the term \( \lambda z' \) to the integrand in (16).

Let \( Z(s) \) be the solution for \( \lambda' = 0 \). There may be an interval in which \( Z'(s) < 0 \) as illustrated in Figure 3. Around any such interval, we shall select points \( s_\ast, S^\ast, \) and \( z_\ast \) with \( z_\ast = Z(s_\ast) = Z(S^\ast) \) such that the interval where \( Z'(s) < 0 \) is contained in \([S_\ast, S^\ast]\) as shown in Figure 3. The \( S_\ast, S^\ast, z_\ast \) will be selected so that the modified function,

\[ \text{FIGURE 3} \]
\[ \text{MODIFYING } Z(s) \text{ TO OBTAIN A FEASIBLE OPTIMAL TRAJECTORY} \]
satisfies the necessary conditions (20).

Since \( z'(S_\ast) \) and \( z'(S^\ast) \) are nonzero, we know that \( \lambda(S_\ast) = \lambda(S^\ast) = 0 \). Thus we obtain the condition that

\[
\int_{S_\ast}^{S^\ast} \lambda(s) ds = \int_{S_\ast}^{S^\ast} [(T_c - q_c) u_c(s, z_c) + T u(s, z_c) - c(s)] ds
\]

\[
\bigg| \left. [(T_c - q_c) u_c(s, z_c) + T u(s, z_c) - c(s)] \right|_{S_\ast}^{S^\ast} = 0,
\]

where

\[ T_c = T(y, z) \quad q_c = q(y, z) \]

Since the selection of \( z_c \) uniquely determines the \( S_\ast \) and \( S^\ast \), as illustrated in Figure 3, the condition above will determine the solution. It can be verified that the resulting modified trajectory \( z(s) \) satisfies (20). The effect of such a modification is to exclude an interval \([S_\ast, S^\ast]\) of "inefficient" quality levels. This requires that the subscribers will be indifferent between assigning unit type \( z_c \) to qualify level \( S_\ast \) or \( S^\ast \), i.e.,

\[ u(S_\ast, z_c) - m(S_\ast) = u(S^\ast, z_c) - m(S^\ast) \]

This condition determines the price differential \( m(S_\ast) - m(S^\ast) \) across the excluded quality interval. To prevent the purchase of quality levels in this interval, the supplier can either not offer these qualities or discourage subscribers from using them by setting the price \( m(s) \) so that

\[ u(s, z_c) - m(s) < u(S_\ast, z_c) - m(S_\ast) \quad \text{for} \quad S_\ast < s < S^\ast. \]

6. An example

In this section we illustrate the results obtained in this article and compare the implications of two-part tariffs with and without quality discrimination. This is done for a particular specification of the utility, cost, and potential volume functions which combines the model employed by Mussa and Rosen (1978) and the example given by Oren and Smith (1981). We assume a marginal utility function of the form

\[ u(z, s) = w(1 - z)(1 - s) \]

and a cost function,

\[ c(s) = c(1 - s)^2 \]

where \( 0 \leq s \leq 1 \) and \( c \geq w/2 \).

We further assume that the potential volume is given by

\[ Q(t, y, z) = Q(t, y) = 2T(1 - t)y(2 - y). \]

This specification, introduced in Oren and Smith (1981), represents a situation in which the potential communication volume between any two customers \( t \) and \( \tau \) is uniformly distributed across message types and is proportional to \((1 - t)\) and \((1 - \tau)\). The parameter \( T \) is the total potential volume (per subscriber) in a fully penetrated market, i.e., \( y = 1 \). When \( y \leq 1 \), the total potential volume is given by:

\[ T(y) = \int_0^y Q(t, y) dt = Ty^2(2 - y)^2 \]

and

\[ R(y) = \frac{yQ(y, y)}{T(y)} = 2(1 - y)/(2 - y). \]
The solution of this example for a fixed market penetration \( \gamma \) is summarized in the left side of Table 3. Considerable algebraic manipulation, using the results we derived above, is required to obtain the expressions for the solutions. The details of the calculations are available from the authors upon request. The right side of Table 3 summarizes the comparable results for the case where only a single quality level \( s_1 \in [s^*, s^*] \) is offered. For comparison, we assume a marginal cost \( c(s_1) = c(1 - s_1)^2 \) and utility \( u(z) = w(1 - z)(1 - z) \). This is a special case of the example solved by Oren and Smith (1981). The optimal marginal tariff \( m_1 \) was obtained from the results given in Oren and Smith (1981) and turns out to be equal to \( m^*(s_1) \). (This correspondence holds for the entire class of cases in which \( u(z, s) = (1 - z)u(s) \), for arbitrary \( u(s) \) and \( c(s) \).) The marginal message type \( z \tilde{=} \), is obtained from the condition \( u(z) = m_1 \). The expressions for \( F_1(y) \), \( \Pi_1(y) \), and \( CS_1(y) \) result from the multiple quality formulas evaluated at the single quality \( s_1 \), with the values of \( m_1 \) and \( z \tilde{=} \).

Our results show that the range of message types served, \( 0 \leq z \leq z \tilde{=} \), in the single quality case is always less than or equal to the range in the multiple quality case. The marginal tariff \( m_1 = m(s_1) \) is equivalent for that particular quality level. The fixed charge, the net revenue, and the consumer surplus all involve the scale factor \( \varphi(s_1) \), which for \( s_1 \in [s^*, s^*] \) satisfies \( 0 \leq \varphi(s_1) \leq 8/9 \). (The maximum value of \( \varphi(s_1) \) occurs at \( s_1 = 1 - w/3c \).) Thus, for the same market penetration level \( \gamma \), the usage, the consumer surplus, and the net revenue are increased by quality discrimination. When the fixed cost \( k = 0 \), the net revenue, fixed charge, and the consumer surplus increase in the same proportion. For \( k > 0 \), \( \Pi_1(y)/\Pi(y) < \varphi(s_1) \); that is, in the presence of a fixed cost, quality discrimination improves net revenue by a higher percentage than it increases consumer surplus. For the average quality level \( s_1 = 1 - w/4c \), \( z \tilde{=} = 0.75z^* \) and \( \varphi(s_1) = 0.84 \), i.e., quality discrimination will increase usage by 33%, consumer surplus by 19%, and net revenue by at least 19%.

The above comparison was based on a fixed market penetration \( \gamma \). Actually, the supplier determines \( \gamma \) by optimizing the net revenue, which may lead to different \( \gamma \) values with and without quality discrimination. Maximizing \( \Pi(y) \) yields the first-order necessary condition

\[
y(2 - y)^2(4 - 5y) = 24k_c/T_w^2.
\]  
(26)

### Table 3  
Comparing Multiple and Single Quality Results for Fixed \( \gamma \)

<table>
<thead>
<tr>
<th>Multiple Qualities: ( 0 \leq s &lt; 1 )</th>
<th>Single Quality: ( s_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Optimal Consumption</strong></td>
<td></td>
</tr>
<tr>
<td>( z(s) = \frac{[1 - (2c/w)(1 - s)]/[2 - R(y)]}{1 - (2c/w)(1 - 1 - s^*)} )</td>
<td>use quality ( s_2 \leq s_1 \leq s^* )</td>
</tr>
<tr>
<td>( s_2 = 1 - (w/2c)z \leq s \leq 1 - s^* )</td>
<td>for all ( z ) such that</td>
</tr>
<tr>
<td>( 0 \leq z \leq z^* = 1/[2 - R(y)] )</td>
<td>( 0 \leq z \leq z^* = 1 - (c/w)(1 - s_1)z^* )</td>
</tr>
<tr>
<td><strong>Optimal Tariff</strong></td>
<td></td>
</tr>
<tr>
<td>( m(s) = {c(1 - z)^2 + w[1 - R(y)]/[2 - R(y)]/[2 - R(y)]} )</td>
<td>( m_1 = m(s_1) )</td>
</tr>
<tr>
<td>( F_1(y) = w^2Q(y)/12c[2 - R(y)] )</td>
<td>( F_1(y) = \varphi(s_1)F(y) ) where</td>
</tr>
<tr>
<td>( \varphi(s_1) = \delta[1 - s_1c/w][1 - (1 - s_1)c/w]^2 )</td>
<td>( \Pi_1(y) = \varphi(s_1)\Pi(y) - (1 - \varphi(s_1))ky )</td>
</tr>
<tr>
<td><strong>Supplier’s Net Revenue</strong></td>
<td></td>
</tr>
<tr>
<td>( \Pi(y) = -ky + w^2T(y)/12c[2 - R(y)] )</td>
<td>( \Pi(y) = \varphi(s_1)\Pi(y) - (1 - \varphi(s_1))ky )</td>
</tr>
<tr>
<td><strong>Total Consumer Surplus</strong></td>
<td></td>
</tr>
<tr>
<td>( CS(y) = w^2T(y)[1 - R(y)]/12c[2 - R(y)]^2 )</td>
<td>( CS(y) = \varphi(s_1)CS(y) )</td>
</tr>
</tbody>
</table>
Similarly, if we maximize \( \Pi(y) \), we obtain the same condition, but with \( k \) replaced by \( k/\Phi(s_i) \). In general, equation (26) has two solutions in the interval \([0, y]\), but the second-order conditions show that only the larger root corresponds to a maximum. Furthermore, the value of this root, giving the optimal penetration level, decreases as the right-hand side increases. This is intuitive, since as the cost of supply \( k \) increases, there is less incentive to the supplier to extend service to small users. But, since \( \Phi(s_i) < 1 \), it follows that \( k/\Phi(s_i) \geq k \), which implies that the optimal market without quality discrimination is smaller. The market penetration level in the two cases is the same (\( y^* = .8 \)) only when \( k = 0 \).

7. Conclusion

Several general observations can be made regarding the results obtained in this article. First, viewing successive quality levels as incremental improvements over poorer quality levels is a fruitful approach. This allows both the buyer’s and seller’s conditions to be interpreted as standard economic criteria, which must hold independently at each separate quality level. Indeed, the seller’s optimality condition (18) allows us to optimize within each of the incremental markets separately as long as \( z(s) \geq 0 \) is not violated. Since a pointwise optimal solution always can be modified to a feasible optimal one, this independence is useful in general. Also, there is a one-to-one correspondence between the optimal seller induced allocation \( z(s) \) and the optimal price level \( m(s) \) in each incremental market.

The seller uses the pricing tools of the fixed charge and linear quality based charges to induce buyer self-selection, a phenomenon observed previously. The relative use of these tools is governed by the degree to which customers differ, with customer similarity favoring the use of the fixed charge. As illustrated by the example in Section 6, quality discriminating pricing leads to greater market efficiency and wider customer participation in a monopoly supplied market. While the numbers we have obtained depend on the particular specification used, one expects that these observations hold under more general conditions. Often multiple qualities are offered but not distinguished, as in the case of random delays in delivery of communications. Quality discrimination then merely involves the introduction of priority levels and monitoring techniques so that quality based pricing can be implemented. Recent developments in microprocessors have greatly reduced the cost of implementing priorities and monitoring consumption in many industries.

An important assumption of this article is the single fixed subscription charge. Conceivably, in a discrete quality case, one may have a separate fixed charge for each quality level. Unfortunately, in the continuous case, such a tariff structure will induce “bunching” of unit types at several discrete quality levels, and this would destroy most of the mathematical structure we exploited in our analysis. We believe, therefore, that such a generalization can be more easily investigated in a discrete quality framework. Another possible generalization of the tariff structure explored in this article is to have a fixed charge that depends upon the highest quality level used by the subscriber. Such a tariff is appropriate when the subscription charge includes the lease of equipment (e.g., a terminal) and the highest service quality determines the type of equipment needed by the subscriber. This type of tariff will induce subscribers to truncate the quality spectra according to their individual utility functions. One can then consider the problem of determining the optimal nonlinear fixed charge outlay simultaneously with the optimal quality differential pricing. The detailed implications of such generalizations require further investigation. As a general rule, however, one expects that the added flexibility in such tariffs would result in higher efficiency and profits.
Appendix

We shall now derive the optimal endpoint conditions for the objective function in (15). To simplify the algebra, we present the case in which $T_l = zT$, $q_T = zq$. The general case is obtained in a similar manner. Let

$$L = L(s, z) = [T(c' - u_s) + qu_T]z + quz^*$$

and

$$\Phi^* = [Tm^* - c(s^*) + qu(s^*, z^*) + \lambda_1(1 - z^*) + \lambda_2[u(s^*, z^*) - m^*] + \lambda_3(1 - z^*)[u(s^*, z^*) - m^*].$$

(A2)

The expression $\Phi^*$ contains all terms involving the upper boundary variables $z^*$, $s^*$. The first term in $\Phi^*$ corresponds to the constant terms (16) that resulted from integration by parts, while the remaining terms with Lagrange multipliers $\lambda_1$, $\lambda_2$, and $\lambda_3$ result from the endpoint constraints (9).

There is an additional set of terms $\Phi_*$ corresponding to conditions at the endpoint involving $s_*$ and $z_*$, which also includes the term $K(z_*)$ from (15). Since the necessary condition (18) uniquely determines the optimal trajectory $z(s)$ between $s_*$ and $s^*$, the lower endpoint conditions can be derived independently from the upper ones. In fact, since (17) must hold at $s_*$, $z_*$, these endpoint conditions can be obtained directly from (6)–(7). The trajectory $z(s)$ simply continues in the direction of decreasing $s$ until either $s = s_0$ or $z(s) = 0$. If $s = s_0$ occurs first, we have $s_0 = s_*$ and $z_0 = z(s_0) = 0$. If $z(s) = 0$ occurs first, we have $z(s_*) = 0 = z_*$ and $s_* = s_0$. Thus, no optimization of $\Phi_*$ is required.

Returning to (A1) and (A2), we must optimize with respect to $s^*$, $z^*$, $m^*$, subject to the various constraints. The transversality conditions for $L$ and $\Phi^*$ (Elsgoltz, 1962, p. 114) are

$$L - z'L_z + \partial \Phi^*/\partial s^* = 0$$

(A3)

$$L_z + \partial \Phi^*/\partial z^* = 0.$$  

(A4)

From (A3) we obtain

$$[-Tu_T + qu_T + Tc'(s^*)]z^* - Tc'(s^*)z^* + \lambda_1 u_T + \lambda_3 (1 - z^*)u_T = 0.$$  

Simplifying, we find, since $u_T < 0$, that

$$\lambda_2 + \lambda_3 (1 - z^*) = (T - q)z^*.$$  

Since $T > q$, we have that $z^* < 1$ implies $\lambda_2 > 0$. By complementarity, this means that $u^* = m^*$ for $z^* < 1$. For $z^* = 1$, $u^* = m^*$ by (9). Thus $u^* = m^*$.

From (A4) we obtain

$$qu^* + T(m^* - c^*) - qm^* - \lambda_1 u_T + \lambda_2 u_T + \lambda_3 (1 - z^*)u_T = 0.$$  

Using $u^* = m^*$, this reduces to

$$T(u^* - c^*) + z^*(T - q)u_T^* = \lambda_1.$$  

(A5)

Again by complementarity either $\lambda_1 = 0$ or $z^* = 1$. Thus for $z^* < 1$

$$-(u^* - c^*)/u_T^* = (1 - R)z^*.$$  

(A6)

By continuity, we also have that (17) holds at the endpoint giving

$$- [u_T^* - c'(s^*)]/u_T^* = (1 - R)z^*.$$  

(A7)

Thus, the optimal trajectory $z(s)$ is continued in the direction of increasing $s$ until either
\( z(s) = 1 \) (in which case \( z(s^*) = 1 = z^* \), or the condition

\[
[u^+ - c'(s^*)]/u^+_x = (u^* - c^*)/u^+_x
\]

occurs. This determines both \( s^* \) and \( z^* \) in this case. The solution to the optimization problem is then completed by determining \( m^* = u(s^*, z^*) \).

References


