Stochastic Inventory Control
University of California, Berkeley
Spring 2004
Homework 1 - Solutions

• 1.5 (a)

\[ v(y) = p \mathbb{E} \min(D, y) - cy \]
\[ = p \mathbb{E} \left\{ D - \max(D - y, 0) \right\} - cy \]
\[ = p \mu - cy - p \mathbb{E}(D - y)^+ \]

(b) Immediate

• 1.9 (a)

\[ v(y) = p \mathbb{E} \min(D, y) - cy \]
\[ = p \left\{ \int_0^y \xi \phi(\xi) d\xi + y \Phi(y) \right\} - cy \]
\[ = p \Phi(y) \int_0^y \xi \frac{\phi(\xi)}{\Phi(y)} d\xi + py \Phi(y) - cy \]

(b) \( \Phi(S) \) satisfies

\[ \Phi(S) = \frac{p - c}{p} \]

From (a)

\[ v(S) = p \mathbb{E}(D \mid D \leq S) \left( \frac{p - c}{p} \right) + pS \left\{ 1 - \frac{p - c}{p} \right\} - cS \]
\[ = (p - c) \mathbb{E}(D \mid D \leq S) \]

(c) Optimal profit is the marginal profit per item multiplied with the expected number of items sold, given that the demand does not exceed supply.

• 1.11 Discrete demand \( D \in \{x_1, x_2, \cdots\} \) with \( 0 \leq x_1 \leq x_2 \leq \cdots \). Denote \( q_i \triangleq \mathbb{P}[D = x_i] \).

Assume without loss of generality that \( q_i > 0 \) for all \( i \). Assume that the standard assumption that \( p > c > 0 \) holds.

(a) Observe that

\[ v(y) = -cy + p \mathbb{E} \min(y, D) \]
\[ = pg \left\{ \frac{p - c}{p} - \Phi(y) \right\} + p \sum_{i:x_i \leq y} x_i q_i \]
Since
\[ \Phi(y) = \sum_{i: x_i \leq y} q_i \]
it follows that \( v(y) \) is piecewise linear with points of non-differentiability at \( \{x_1, x_2, \cdots \} \). The only possible points of discontinuity are \( \{x_1, x_2, \cdots \} \). However
\[
v(x_i^+) - v(x_i^-) = \lim_{y \downarrow x_i} v(y) - \lim_{y \uparrow x_i} v(y)
\]
\[
= \left[ px_i \left\{ \frac{p - c}{p} - \sum_{k=1}^{i} q_k \right\} + p \sum_{k=1}^{i} x_k q_k \right] - \left[ px_i \left\{ \frac{p - c}{p} - \sum_{k=1}^{i-1} q_k \right\} + p \sum_{k=1}^{i-1} x_k q_k \right]
\]
\[
= 0
\]
i.e. \( v(y) \) is piecewise linear and continuous in \( y \). Finally, since \( \Phi(y) \to 1 \) and
\[
p \sum_{x_i \leq y} x_i q_i \to \mu
\]
as \( y \to \infty \) it follows from \( p > c \) that \( v(y) \to -\infty \) as \( y \to \infty \). Therefore, \( v(y) \) must have a maximum at one of the points \( \{x_1, x_2, \cdots \} \).

(b) Since
\[
v(x_i) = py \left\{ \frac{p - c}{p} - \Phi(x_i) \right\} + p \sum_{x_k \leq x_i} x_k q_k = py \left\{ \frac{p - c}{p} - \sum_{k=1}^{i} q_k \right\} + p \sum_{k=1}^{i} x_k q_k
\]
it follows that
\[
v(x_{i+1}) - v(x_i) = p(x_{i+1} - x_i) \left\{ \frac{p - c}{p} - \Phi(x_i) \right\}.
\]
Noting that \( \Phi(x_1) \leq \Phi(x_2) \leq \cdots \) it follows that the optimal stock level is the smallest \( i \) such that
\[
\Phi(x_i) \geq \frac{p - c}{p}.
\]

(c) Since (by part (a)) the optimal order level \( y \) occurs at one of the potential demand levels \( \{x_1, x_2, \cdots \} \) and \( \Phi(y) \) can only change value at one of these potential demand levels, it follows from (b) that the optimal order level is the smallest \( y \) such that:
\[
\Phi(y) \geq \frac{p - c}{p}.
\]