1. Suppose that stock and bond prices are given by:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t) \\
\frac{dB(t)}{B(t)} &= r dt
\end{align*}
\]

where \( r, \mu \) and \( \sigma \) are constants.

(a) Can the process \( X(t) = S(t)^2 \), \( t \in [0, T] \), represent the price of a traded asset if there are to be no arbitrage opportunities? If not, calculate the arbitrage free price of a contingent claim with terminal payoff \( X = S(T)^2 \).

(b) Show that \( X(t) = S(t)^{-\alpha} \) where \( \alpha = 2r/\sigma^2 \) is the price of a traded asset.

(c) Suppose that the terminal payoff of a contingent claim in this market is given by:

\[
X = \begin{cases} 
  c, & Y(T) \leq c \\
  0, & Y(T) > c
\end{cases}
\]

where \( Y(t) \) is the solution of the SDE:

\[
dY(t) = \alpha dt + \beta dW(t)
\]

and \( Y(0) \) is some known constant. Use the Girsanov transformation and risk-neutral pricing to determine the price \( F(t, y) \) (a function of time \( t \) and the value of \( Y(t) \)) of \( X \).

2. Consider the financial market:

\[
\begin{align*}
\frac{dS(t)}{S(t)} &= \mu(t, S(t)) dt + \sigma(t, S(t)) dW(t) \\
\frac{dB(t)}{B(t)} &= r dt
\end{align*}
\]

Suppose that the payoff of this claim is \( X = \Phi(S(T), Z(T)) \) where:

\[
Z(t) = \int_0^t g(u, S(u)) du
\]
for some known function $g(t, S)$. e.g.

$$X = \max \left[ \frac{1}{T} \int_0^T S(t) dt - K, 0 \right]$$

is an Asian option.

(a) Assuming that the price of this claim has the form $F(t, s, z)$ use Ito’s formula to show that $F$ is the solution of the PDE:

$$\begin{cases} 
F_t + srF_s + \frac{1}{2}s^2\sigma^2 F_{ss} + gF_z - rF = 0 \\
F(T, s, z) = \Phi(s, z)
\end{cases}$$

What is the associated replicating portfolio?

(b) Calculate an expression for $F(t, s, z)$ using risk-neutral pricing.

(c) Show that the pricing formula obtained in part (b) is consistent with that given by the Feynman-Kac formula.