1. Assume that the stock and bond price processes are solutions of:

\[
    dS(t) = S(t) \left\{ \mu dt + \sigma dW \right\} \\
    dB(t) = rB(t)dt
\]

where the parameters \( \mu \), \( r \) and \( \sigma \) are constants and \( W(\cdot) \) is a one-dimensional (standard) Brownian motion. Derive the price of a European put option (i.e payoff \( \Phi(S(T)) = (K - S(T))^+ \)) by evaluating the expectation in the risk-neutral representation of the price of this option:

\[
p(t, S) \triangleq e^{-r(T-t)} N[-d_2(t, S)] - SN[-d_1(t, S)]
\]

where

\[
N[x] \triangleq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy
\]

is the standard normal distribution function and:

\[
d_1(t, s) = \frac{(r + \frac{1}{2} \sigma^2)(T-t) + \ln s}{\sigma \sqrt{T-t}} \\
n_2(t, s) = d_1(t, s) - \sigma \sqrt{T-t}
\]

2. For given functions \( \mu(t, x), \sigma(t, x), k(t, x) \) and \( \Phi(x) \), consider the following boundary value problem in the domain \([0, T] \times \mathbb{R}\):

\[
\frac{\partial F}{\partial t} + \mu(t, x) \frac{\partial F}{\partial x} + \frac{1}{2} \sigma(t, x)^2 \frac{\partial^2 F}{\partial x^2} + k(t, x) = 0 \\
F(T, x) = \Phi(x).
\]

Show that:

\[
F(t, x) = E_{t,x}[\Phi(X_T)] + E_{t,x} \int_{t}^{T} k(s, X(s)) ds,
\]
where $X(s)$ is the solution of the following SDE:

$$
\begin{align*}
\frac{dX(s)}{ds} &= \mu(s, X(s))ds + \sigma(s, X(s))dW(s) \\
X(t) &= x
\end{align*}
$$

Use this result to solve the following PDE:

$$
\frac{\partial F}{\partial t} + \frac{1}{2} x^2 \frac{\partial^2 F}{\partial x^2} + x = 0
$$

$$
F(T, x) = \ln(x^2)
$$

3. (a) Consider the standard Black-Scholes model:

$$
\begin{align*}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma dW(t) \\
\frac{dB(t)}{r} &= dt
\end{align*}
$$

(assume for simplicity that $r$, $\mu$, and $\sigma$ are constants) and a European contingent claim $\chi \in \mathcal{F}_T$ that expires at $T$ and has the form $\chi = \Phi(S(T))$. Denote that arbitrage free price process of $\chi$ by $\Pi(t)$ (i.e. $\Pi(t) = F(t, S(t))$ where $F$ is the solution of the Black-Scholes PDE). Show that the dynamics of $\Pi(t)$ under the risk-neutral measure $Q$ is the solution of an SDE of the form:

$$
\frac{d\Pi(t)}{\Pi(t)} = r\Pi(t)dt + g(t)d\bar{W}(t)
$$

for some process $g(t)$.

Hint: Use the risk-neutral dynamics of $S(t)$ together with the PDE satisfied by $F(t, S)$ and Ito’s formula.

(b) Show that the process $Z(t) = \frac{\Pi(t)}{B(t)}$ under the risk-neutral measure $Q$ is a martingale. i.e. Show that:

$$
\frac{dZ(t)}{Z(t)} = \sigma_Z(t)d\bar{W}(t).
$$

Derive the expression for $\sigma_Z(t)$ in terms of $F$ and its derivatives.