1. The risk neutral measure for this problem is:

\[ Q[S(k+1) = S(k)u|S(k)] = q = \frac{(1 + r) - d}{u - d} = 0.4 \]

(a) (Compound option). Consider the second option (that expires at \( T = 3 \)) first. The payoff for this option at \( T = 3 \) is:

\[ [S(3) - K_3]^+] = \begin{cases} (337.5 - 100)^+ = 237.5, & uu \ \\
(150 - 100)^+ = 50, & uud, duu or udu \ \\
(67 - 100)^+ = 0, & udd, udu, or ddu \ \\
(29.6 - 100)^+ = 0, & ddd. \end{cases} \]

The possible prices for this option at \( t = 1 \) are:

\[ \pi_1([S(3) - K_3]^+) = (1 + r)^{-2}E^Q[S(3) - K_3]^+|S(1)] \]

\[ = \begin{cases} 62, & S(1) = S(0)u \ \\
8, & S(1) = S(0)d \end{cases} \]

The \( t = 0 \) price of the compound option is given by pricing the \( t = 1 \) payoff

\[ [\pi_1 - K_1]^+] = \begin{cases} (62 - 40)^+ = 22, & S(1) = S(0)u \ \\
(8 - 40)^+ = 0, & S(1) = S(0)d \end{cases} \]

(The option gives the holder the right to buy a contract worth \( \pi_1 \) for the price 40). Therefore:

\[ \pi_0 = (1 + r)^{-1}E^Q[\pi_1 - K_1]^+] = (0.4)(62 - 40)^+ + (0.6)(8 - 40)^+ = 8.8. \]

2. \( t_k = kT/N \)

\[ \int_0^T g(t)dt \triangleq \mathbb{E} \sum_{k=0}^{N-1} g(t_k)[W(t_{k+1}) - W(t_k)] \]

For \( j < k \):

\[ \mathbb{E}[g(t_k)[W(t_{k+1}) - W(t_k)]g(t_j)[W(t_{j+1}) - W(t_j)]] = \mathbb{E}[g(t_k)g(t_j)[W(t_{k+1}) - W(t_k)]\mathbb{E}\{W(t_{k+1}) - W(t_k)|\mathcal{F}_j\}] = 0 \]
since $E[W(t_{k+1}) - W(t_k)] = 0$. Also:

$$E \left[ g(t_k)^2 [W(t_{k+1}) - W(t_k)]^2 \right] = E \left[ g(t_k)^2 E \left( [W(t_{k+1}) - W(t_k)]^2 \mid F_{t_k} \right) \right] = E \left[ g(t_k)^2 (t_{k+1} - t_k) \right]$$

since $W(t)$ is a standard B.M. Therefore:

$$E \left\{ \left[ \int_0^T g(t) dt \right]^2 \right\} = \sum_{k=0}^{N-1} E \left[ g(t_k)^2 (W(t_{k+1}) - W(t_k))^2 \right]$$

$$= \sum_{k=0}^{N-1} E g(t_k)^2 (t_{k+1} - t_k)$$

$$= E \int_0^T g(t)^2 dt$$

3. Suppose:

$$dx(t) = \mu dt + \sigma dW(t), \quad x(0) = x_0.$$

Let: $Z(t) = f(t, x(t))$ where $f(t, x) = e^{\alpha x}$.

$$f_t(t, x) = 0, \quad f_x(t, x) = \alpha e^{\alpha x}, \quad f_{xx}(t, x) = \alpha^2 e^{\alpha x}$$

Ito’s formula gives:

$$dZ(t) = \left\{ \alpha e^{\alpha x(t)} \mu + \frac{1}{2} \alpha^2 e^{\alpha x(t)} \sigma \right\} dt + \alpha e^{\alpha x(t)} \sigma dW(t)$$

$$= Z(t) \left( \mu + \frac{1}{2} \alpha^2 \sigma^2 \right) dt + Z(t) \alpha \sigma dW(t)$$

4. Suppose that $x(t)$ is the solution of:

$$dx(t) = \alpha x(t) dt + \sigma x(t) dW(t), \quad x(0) = x_0.$$  

i.e. $x(t)$ has a log-normal distribution: $x(t) = \exp\{(\alpha-(1/2)\sigma^2)t+\sigma W(t)\}$. Suppose $f(x) = x^2$ and $Z(t) = f(x(t))$. Then:

$$f_x(x) = 2x, \quad f_{xx}(x) = 2$$

so by Ito’s formula:

$$dZ(t) = \left\{ 2\alpha x(t)^2 + \sigma^2 x(t)^2 \right\} dt + 2x(t)^2 \sigma dW(t)$$

$$= Z(t) \left( 2\alpha + \sigma^2 \right) dt + 2Z(t) \sigma dW(t)$$

and (unsurprisingly) $Z(t)$ also has a log-normal distribution.
5. Define:

\[ dX(t) = \alpha X(t)dt + \sigma X(t)dW(t). \]  

(1)

Define \( Z(t) = 1/X(t) \). By Ito’s formula:

\[
dZ(t) = \left\{ -\frac{1}{X(t)^2} \alpha X(t) + \frac{1}{2} \frac{2}{X(t)^3} \sigma^2 X(t)^2 \right\} dt - \frac{1}{X(t)^2} \sigma X(t)dW(t) \\
= Z(t)\{-\alpha + \sigma^2\}dt - Z(t)\sigma dW(t)
\]

Comparing with the structure of (1) we see that \( Z(t) \) is log-normal with:

\[
Z(t) = \exp \left\{ \left( -\alpha + \frac{1}{2} \sigma^2 \right) t - \sigma W(t) \right\}
\]

6. Let:

\[ dX(t) = \sigma(t)dW(t), \quad X(0) = 0 \]

where \( \sigma(t) \) is a deterministic function of time. Define \( Z(t) = \exp\{iuX(t)\} \) where \( u \in \mathbb{R} \). Ito’s formula gives:

\[
dZ(t) = iue^{iuX(t)}\sigma(t)dW(t) + \frac{1}{2} (iu)^2 e^{iuX(t)}\sigma(t)^2 dt \\
= iu\sigma(t)Z(t)dW(t) - \frac{1}{2} u^2 \sigma(t)^2 Z(t)dt
\]

Since \( Z(t) \) is bounded and (assume) \( \sigma(t) \) is bounded (e.g. continuous will do) it follows that \( iu\sigma(t)Z(t) \) is in \( L^2 \). Therefore \( \int_0^t iu\sigma(s)Z(s)dW(s) \) is a martingale and has zero expectation. After taking expectations, the SDE becomes:

\[
\mathbb{E}Z(t) = 1 - \frac{u^2}{2} \int_0^t \sigma(s)^2 \mathbb{E}Z(s) ds
\]

Setting \( \phi(t) \equiv \mathbb{E}Z(t) \) we obtain the following ODE:

\[
\dot{\phi}(t) = -\frac{u^2}{2} \sigma(t)^2 \phi(t), \quad \phi(0) = 1
\]

Solving for \( \phi(t) \) we obtain:

\[
\mathbb{E} \exp \left\{ iuX(t) \right\} = \mathbb{E}Z(t) = \phi(t) = \exp \left\{ -\frac{u^2}{2} \int_0^t \sigma(s)^2 ds \right\}
\]

which is the characteristic function of a normal r.v. with mean zero and:

\[ \text{Var}(X(t)) = \int_0^t \sigma(s)^2 ds \]