1 Problem 5.18

(a) For $h$ small,

$$P(s < H < s + h | H > s, X(\tau) = i) = q_i h + o(h)$$

The above is true because for $h$ small there is at most one transition in time $\frac{h}{q_i}$, but there has to be one transition for $H < s + h$. Hence

$$\lim_{h \to 0} \frac{P(s < H < s + h | H > s, X(\tau) = i)}{h} = 1$$

(b) from part (a), $H$ is exponential with rate 1.

2 Problem 5.21

With the number of customers in the shop as the state, we get a birth and death process with:

$$\lambda_0 = \lambda_1 = 3, \mu_1 = \mu_2 = 4$$

Therefore:

$$P_1 = \frac{3}{4} P_0, P_2 = \frac{3}{4} P_1 = \frac{9}{16} P_0.$$

Since $P_0 + P_1 + P_2 = 1$, solving for $P_0$ we get, $P_0 = \frac{16}{37}$.

(a) The average number of the customer in the shop is:

$$P_1 + 2P_2 = \frac{30}{37}$$

(b) The proportion of customers that enter the shop is:

$$\frac{\lambda (1 - P_2)}{\lambda} = 1 - P_2 = \frac{28}{37}$$

(c) Use same procedure as above, to find $P_0 = \frac{64}{97}$ and the proportion of customer that enter the system is

$$1 - P_2 = 1 - P_0(3/8)^2 = \frac{88}{97}$$

The rate of added customers is therefore:

$$\lambda \left( \frac{88}{97} - \frac{28}{97} \right) = 3 \left( \frac{88}{97} - \frac{28}{37} \right) \approx 0.45$$

3 Problem 5.25

In steady state it has the same probability structure as the arrival process. Hence if we include in the departure process those arrivals that do not enter then it is a Poisson process.
4 Problem 5.29

(a) By Poisson thinning, \( N_{ij}(t) := \text{number of chains in state } j \text{ at time } t, \text{ starting originally at state } i \), for \( i, j > 0 \) are independent Poisson random variables (refer to Proposition 2.3.2 (ROSS page 69)). Thus
\[
N_j(t) = \sum_{i=1}^{\infty} N_{ij}(t)
\]
are also independent Poisson random variables.

(b) Adapted from Guan-Cheng Li's solution.

Denote by \( P(\mathbf{n}) \) the limiting probabilities. From (a.) we know that \( \{N_j(t), j \geq 1\} \) are independent Poisson random variables when \( N(0) = 0 \), so the limiting probabilities are necessarily of the form
\[
P(\mathbf{n}) = \prod_{j=1}^{\infty} e^{-\alpha_j} \frac{\alpha_j^{n_j}}{n_j!}
\]
for some \( \alpha_1, \alpha_2, \ldots \). Assume that \( \alpha_j = \frac{\lambda P_j}{P_0 v_0} \) is true. We can verify the vector process's reversibility by checking three cases, the birth of a new chain, which comes from a Poisson process with rate \( \lambda \), the death of a chain, which goes away by being absorbed by state 0, and the change of states, where a particular chain goes from one state to another state. The algebraic computations of these three cases all reduce to the basic case \( P_i v_i P_{ij} = P_j v_j P_{ji} \) which is true because the originally considered Markov chain is itself time reversible.