Product Line Pricing

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Pricing a Product Line*

We study the problem of setting prices for a line of products that are partial substitutes for each other. The main innovation is an application of nonlinear pricing to capture the role of price breaks designed to achieve market segmentation. We focus on products interpreted as "machines" that differ along a single dimension of "quality" interpreted as marginal operating cost. The pricing problem is, How should the terms of leases be designed to maximize the lessor's total profit? The net price to the customer may be a nonlinear function of the usage rate. Our results show how to compute an optimal price schedule.

I. Introduction

Many firms confront the recurring problem of pricing strategy for a line of products that are partial substitutes for each other. The difficulty of this problem stems from the complex interactions among the pricing decisions on the several products. A low price for one product may siphon demand away from another as customers realize the advantages of substituting one for the other. Or quantity discounts may induce customers to switch products. Confronting the problem in its full complexity, one realizes that all of the prices and terms must be set jointly to take full account simultaneously of the many interactions.

In this paper we address a special case of the product-line pricing problem in which the prices of successively higher quality products can be determined sequentially. Our aims are quite practical. For example, for one class of models an explicit computational procedure has been programmed for a computer so that a pricing manager sitting at an interactive terminal with a visual display can readily explore the implications of a variety of assumptions. Alternatively, market data generated by a current pricing strategy can be used as input to predict ways in which profits might be improved by altering the array of

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prices. Presently, however, we can not offer empirical evidence on the practical success of the pricing strategies generated by this procedure.

The special case that we consider allows ample complexity in some respects but severely restricts matters in other respects. On the positive side:

1. the model is specially designed to allow quantity discounts of any sort;
2. the model takes account specifically of the diversity of characteristics among potential customers; and
3. the model considers the effects of competition among the client firm's products as well as from products offered by other firms.

On the negative side we impose the following restrictions to simplify the analysis:

1. products are differentiated along a single dimension of quality; the significant attribute of quality (interpreting the product as a machine) is the marginal operating cost per unit of a standardized measure of output;
2. customer characteristics are arrayed along a single one-dimensional continuum, and no two customers respond alike to the same set of prices;
3. competitors' product lines and prices are taken to be fixed and their responses are not taken into account.

These assumptions are common in the literature on nonlinear pricing (cf. Goldman, Leland, and Sibley [1977], Spence [1977], or Mussa and Rosen [1978]; our formulation differs from Mirman and Sibley [1980] who assume that customers buy some of each product offered, among other differences). In companion papers we relax these assumptions in the case of a single product, but our work has not yet used weaker versions of these assumptions in a product-line context.[1]

The assumptions have been made in order to pose a reasonably tractable formulation of the pricing problem that still addresses an important and frequently encountered practical situation. We anticipate that the methods developed here have applications in situations marked by greater complexity than we admit into the present formulation.

In order to appreciate the range of applications the reader might consider some illustrative product lines that fit our assumptions fairly well: office machines (e.g., copiers), computational equipment (e.g., personal computers, terminals), transportation vehicles (e.g., cars, trucks, RVs). We mention these because they involve "machines" with fairly standard measures of output (e.g., copies per hour) and

1. In Oren, Smith, and Wilson (1980) and (1982*a*) we study models in which products have two attributes. In Oren et al. (1983) we develop the theory of Cournot oligopoly among firms using nonlinear price schedules. In (1982*b*) and (1982*c*) we develop other applications.

easily monitored unit costs (cents per copy); also, they are frequently leased. For the model to be formulated in Section II the pricing policy has a natural interpretation when the machine is to be leased. For example, the lease for a copier may specify a rental rate that depends on the actual usage so that the net price reflects a quantity discount based on the total number of copies made.

In Section II we formulate one version of the model and carry out the analysis in Section III. The results are illustrated with a worked-out example in Section IV. In subsequent sections we extend the model to encompass several refinements.

The reader who is not interested in the mathematical analysis may prefer to skip through the text and then examine the compact summary at the end of each section.

II. Formulation of the Basic Model

In this section we lay out the components of the basic model. This requires mathematical specifications of the products' attributes, the customers' characteristics, and the competitive environment in the market. We pay particular attention to the interpretations of the various assumptions that are made in the formulation.

A few caveats are needed to set the stage. First, the set of products is fixed; we do not address the problems of product design and differentiation or the composition of the product line. Similarly, competitors' products and prices are fixed; we do not anticipate their reactions.[2] The products are taken to be partial substitutes for each other in a particular way: Each performs the same basic task for the customer but the products differ according to their costs of producing a standardized unit of output. (These costs might include direct material inputs, labor requirements, and maintenance, as well as imputed costs depending on the speed of operation, error rate, etc.) Stretching terminology somewhat, we say that a product with a lower operating cost has a higher "quality." Conventional notions of quality can be adapted to this interpretation if the units of output are defined appropriately, as we shall elaborate later.

Second, the distribution of characteristics among the population of potential customers is taken to be fixed and known. We do not include random factors or uncertainty, nor do we consider intertemporal variations in customers' behavior—the model is cast in a static environment. Income effects are assumed to be negligible factors in customers' decisions; we think of the customers as firms. We also assume that there are many potential customers, indeed so many that the popula-

2. One can, however, use the model in a repetitive fashion, taking the viewpoint of different firms so as to predict the response of each one to the pricing policies of the others.

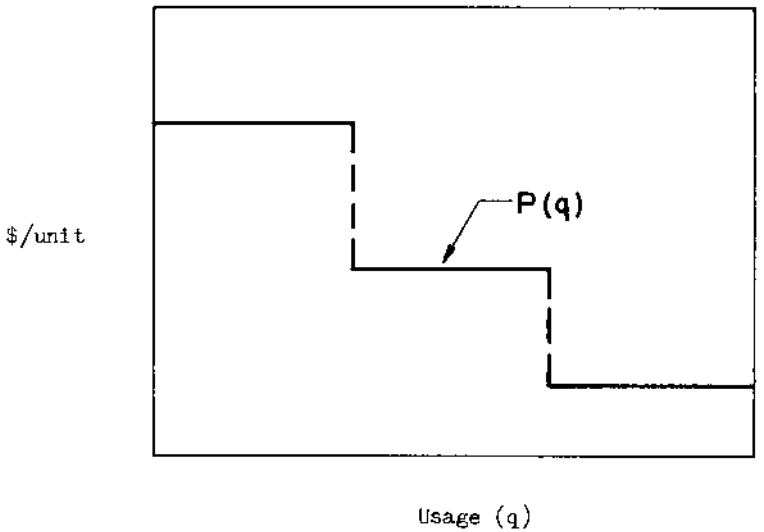

FIG. 1.—A block declining tariff

tion can be described conveniently as a continuum. Using a continuum is mainly a simplification allowing the use of calculus; one can, however, recast the model with a finite population and use instead the methods of mathematical programming without changing any results appreciably.

Finally, a key feature of our formulation is that we admit pricing rules that may be nonlinear in arbitrary ways. In practice this means that quantity rebates of any sort are possible. (See fig. 2 below.) We view this freedom of choice in the design of the pricing policy as a desirable addition to the theory of pricing strategy. It eliminates the constraints on the form of the pricing policy that have usually been employed for no compelling reason.[3] Ordinarily, of course, price breaks are offered at discrete intervals of usage; one price prevails for the first 1,000 units, another for the next 5,000, and so on. A "block declining tariff," as it is usually called, of this sort is illustrated in figure 1. Such a pricing policy makes the customer's payment a piecewise-linear function of usage. This format may be easier to understand or administer. Here, though, we interpret such a payment schedule as an approximation to whatever nonlinear payment schedule is optimal in the absence of administrative considerations.

It is important to realize that any payment schedule dependent on usage can be feasible only if the customer's usage can be controlled, monitored, or predicted. The products with which we have been con-

3. One wonders how many marketing managers responsible for pricing are aware of the numerous occasions on which they pay nonlinear prices for airline tickets, long-distance telephone calls, electricity, etc.

cerned in our work all have metering devices (the counter on a copier, the time clock on a terminal connection, the odometer on a car). Moreover, it must be that the user does not have free access to a resale market; otherwise, nonlinear pricing policies attract arbitrageurs. If any wholesalers are involved, we consider them to constitute the "customers"; simultaneous wholesale and retail sales present special difficulties that we avoid in this exposition.

We now describe the details of the formulation.

Products

Each firm has a line of products. The firms are indexed by $j = 1, 2, \ldots, n$, and the products are indexed by $i = 1, 2, \ldots, m$. Firm j's product line is a subset L_j of the set of products. One further possibility is that a customer may choose not to lease any machine at all, so we denote this possibility by a dummy firm $j = 0$ with a dummy product $i = 0$ that is its entire line [$L_0 = \{0\}$]. Each product is described for the firm by its total cost of producing units of the standardized output. Let $C_i(q)$ be the total cost of producing q units with the ith machine. We divide this cost into its constituents as follows. First, $C_i(0) = 0$, since no machine is needed if no output is required. Second, if $q > 0$ then it will suffice here for expository purposes to assume that $C_i(q) = F_i + V_i \cdot q$. That is, the total cost is a fixed component F_i and variable component with marginal cost V_i. If costs are accounted for periodically, say monthly, then of course F_i must be an amortized value and V_i must include the cost of usage-dependent maintenance, customer services, and depreciation. (For the dummy product, set $F_0 = 0$ and $V_0 = \infty$.) In practice the customer may incur portions of these costs; however, it simplifies matters to suppose that the lessor reimburses the customer's incurred expenses. Thus, we can assume that the customer considers only that the product is described by the payment schedule in the lease, whereas the lessor's calculations rely on the total costs.

Leases and Payment Schedules

Since we allow nonlinear pricing, the terms of the lease for product i are summarized by a function $R_i(q)$ that specifies the lessee's payment, and the lessor's revenue, for each possible usage rate q. Figure 2 depicts a concave payment schedule that offers quantity discounts for usage rates exceeding a base load q^*. Every customer has the opportunity to lease machine i according to this payment schedule: we exclude the possibility that the lessor can discriminate among customers according to their identities. The envelope of the products' payment schedules, $R(q) = \min_i R_i(q)$, is the salient concern of each customer. We assume that each R_i, and therefore R, is concave so that there is no incentive for a customer to lease more than one machine.[4]

4. The verification of concavity is a technical matter addressed later.

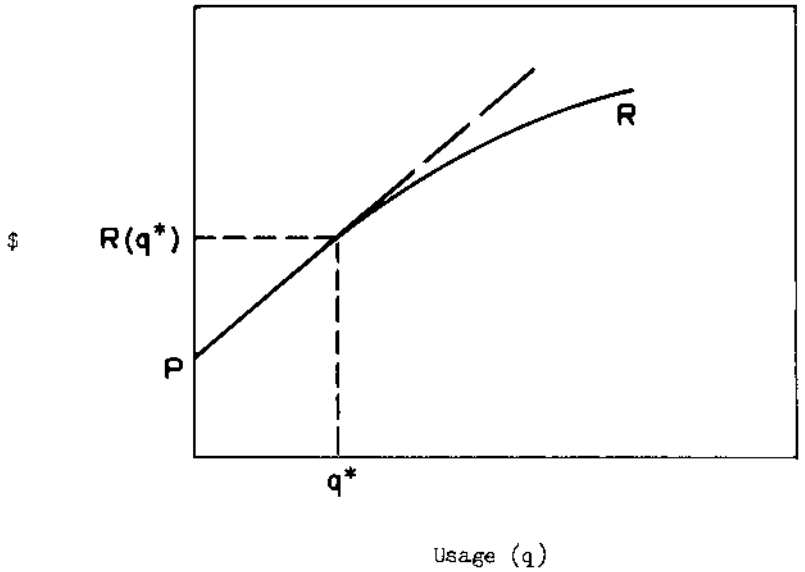

FIG. 2.—A concave payment schedule

Customers

There are many customers, each with different characteristics summarized in an index t indicating the customer's type. We suppose that there is a continuum of types with indices arrayed in an interval $t_0 \leq t \leq t_1$. The fraction of the population whose types are less than an index t is given by a distribution function $H(t)$ that is assumed to be continuous and strictly increasing. It simplifies notation to let $s = H(t)$ be this fraction, so that $t = H^{-1}(s)$, and then s is uniformly distributed on the interval $0 \leq s \leq 1$. If $s = H(t)$ then s is t's fractile ranking in the population's set of types. Thus we can use s to denote a customer's rank or type equally well. The transformation $t \rightarrow s$ is represented in figure 3.

The customer of rank s is described by his total benefit $U(q, s)$ obtained from q units of the standardized output in each accounting period. Operationally, this means that if customer s leases machine i then he will choose the usage rate $q_i(s)$ that is the value of q that maximizes its net benefit $U(q, s) - R_i(q)$. Similarly, in deciding among machines he will choose that machine $i(s)$ indexed by the value of i that maximizes $U[q_i(s), s] - R_i[q_i(s)]$ among the optimized net benefits of the available machines. The locus of net benefits from different selections by a typical customer is traced in figure 4. In practice, of course, the customers' demand behavior must be estimated from market data and in Section III we show how this can be done.

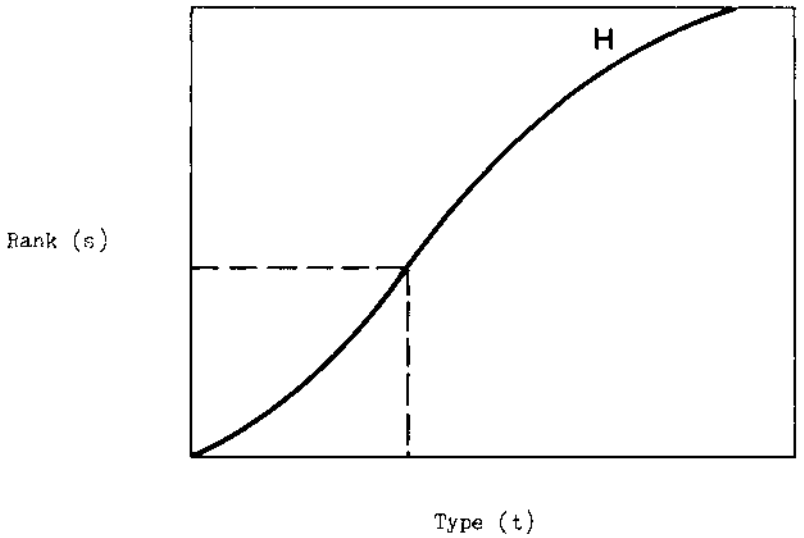

FIG. 3.—Transforming customers' types to ranks

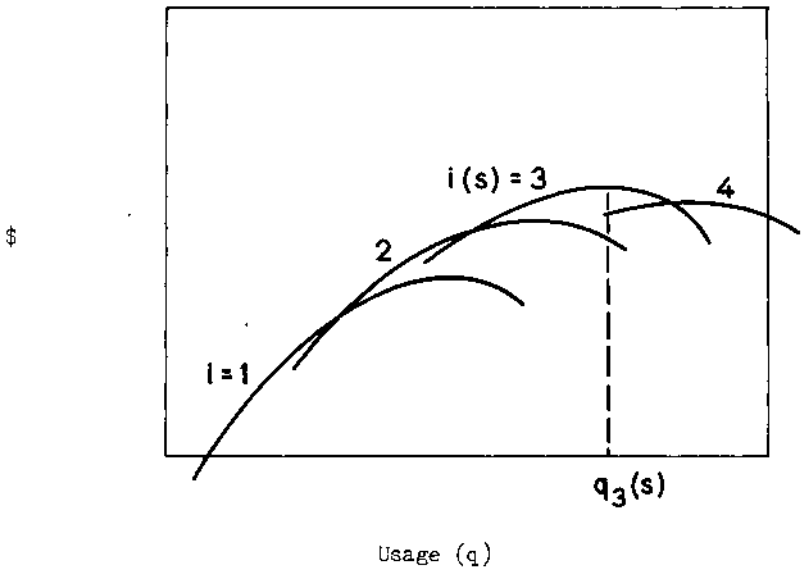

FIG. 4.—Net benefits of a typical customer

It will be convenient to have notation for the customer's marginal benefit, or willingness to pay, for an incremental unit of output: $u(q, s) = \partial U(q, s)/\partial q$. We assume that the total benefit function is an increasing and concave function of the usage rate; consequently, $u(q, s) > 0$ and $\partial u(u, s)/\partial q < 0$. Since there are no income effects the total benefit is simply the sum or integral of the cumulative marginal benefits:

$$U(q, s) = \int_0^q u(x, s)dx.$$

A key assumption of the present analysis is that no two customers respond in the same way. This assumption can be expressed in technical terms as the requirement that $\partial u(q, s)/\partial s \neq 0$ for every q and s. To be specific, we adopt the convention that $\partial u(q, s)/\partial s > 0$. This convention means that it is customers of higher rank who are willing to pay more for additional units of output. In particular, if the payment schedule $R_i(q)$ for machine i is increasing and concave then $q_i(s) < q_i(\hat{s})$ whenever $s < \hat{s}$. This feature is shown in figure 5. In such a circumstance a customer's rank can be inferred from the rank order of his usage. This assumption fits the circumstances in a wide variety of practical applications but it is not universal. It will be evident later how this assumption can be relaxed when one uses aggregated demand data.

The customers' first-order conditions (depicted in fig. 5) for their optimal usage selections must be supplemented by appropriate second-

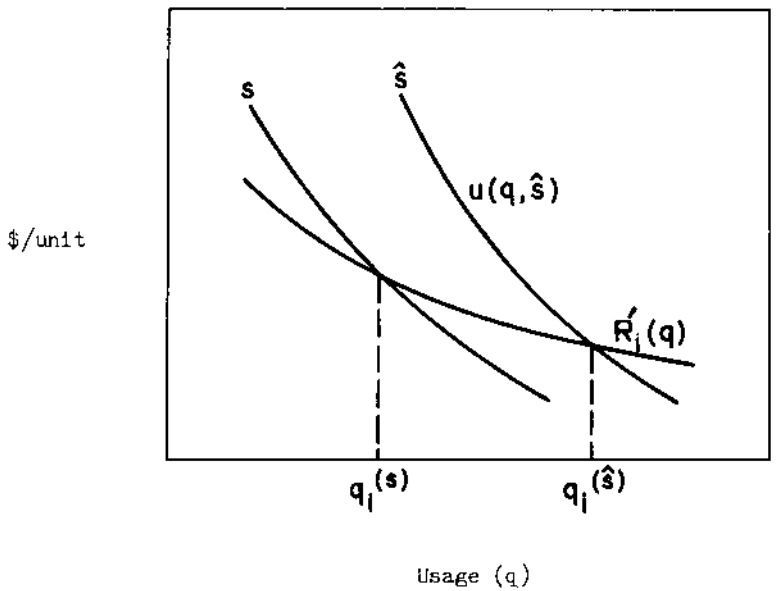

FIG. 5.—Usage selections by customers of different ranks

order conditions; for example, the marginal payment schedule $R_i'(q)$ must intersect the willingness-to-pay functions from below. These conditions are discussed in the Appendix and shown to be satisfied for the firms' optimal concave payment schedules.

Firms

Each firm picks a payment schedule for each machine in its product line. A firm is not, of course, assured that it will lease every machine in its product line. If another firm has a machine with uniformly lower costs then optimal pricing policies will lead inexorably to the inferior machine being undercut and no customers will lease it. The several machines in firm j's product line give it an efficient cost potential that is

$$C^j(q) = \min_{i \in L_j} C_i(q)$$

for each usage rate q. That is, the efficient cost potential is obtained by assigning to each usage rate the machine that is least costly in producing that many units of output. Similarly, among all firms the efficient cost potential is

$$C(q) = \min_{1 \leq j \leq n} C^j(q)$$

for each usage rate q. Thus C is the lower envelope of the cost functions associated with the individual machines, as shown in figure 6.

As a first step we simply delete those products that are not cost

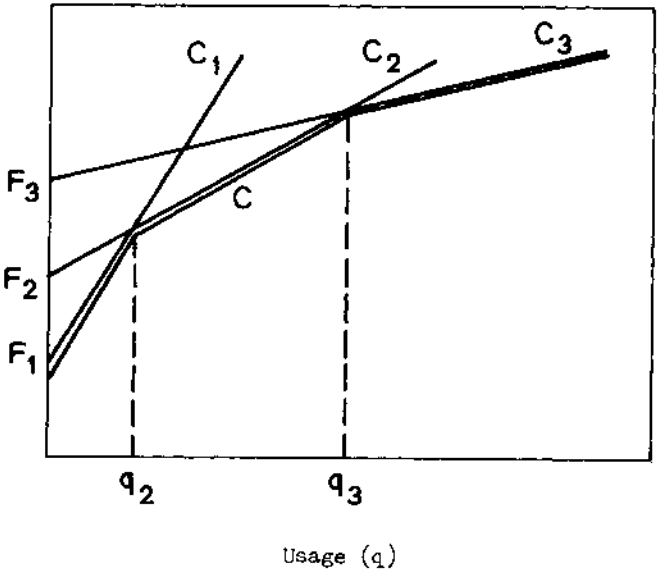

FIG. 6.—The efficient cost envelope

efficient for some usage rate. Having done this we can conclude that each machine i is efficient for some usage rates in an interval $q_i \leq q \leq q^i$. A customer s whose usage is in this interval will presumably have chosen to lease machine i if the firms' pricing policies induce efficient choices of machines. Actually, however, one can show that a somewhat weaker hypothesis is the correct one: It is true that no inefficient machine will find customers, but it may be that not every efficient machine finds customers. This is because the payment schedules offered by firms will in general exceed actual costs in order to exploit the opportunities for monopoly profits. This is a general result in the theory of monopolistic competition among firms with differentiated products, and later on we will show how the set of products that survives in the market is determined.

According to our earlier assumption, the customers' ranks are reflected in their usages. Thus, those customers who lease machine i will have ranks in an interval $s_i \leq s \leq s^i$, and every customer whose rank is in this interval will choose to lease machine i. Thus, machine i obtains the market share $s^i - s_i$ if this difference is positive.

Putting this together, a firm's profit on leases of one of its machines i will be the sum or integral of its net revenues from all customers leasing that machine:

$$\int_{s_i}^{s^i} \{R_i[q_i(s)] - C_i[q_i(s)]\} ds.$$

And its profit from its entire product line will be

$$\sum_{i \in L_j} \int_{s_i}^{s^i} \{R_i[q_i(s)] - C_i[q_i(s)]\} ds.$$

We suppose that each firm j chooses the terms of the leases for its products so as to maximize the value of this formula for the total profit from its product line. In doing this it takes as fixed the lease terms offered by its competitors. Its objects of choice are the payment schedules; we shall show later how the market shares are determined.

One may be impressed with the seeming complexity of this formulation of the pricing problem. The payment schedule $R_i(q)$ for machine i might be a quite complicated function of the usage rate. Each customer's usage rate $q_i(s)$ depends on the payment schedule in intricate ways. Moreover, the market share $s^i - s_i$ that machine i captures depends on the lease terms for other machines in the firm's product line as well as on the terms offered by competitors for their machines.

There are, however, some simplifying features in our formulation that redeem the situation and make the pricing problem fairly tractable. In Section III we show how to construct the solution. The features that ease the task most sharply are those reflecting the assumptions about

customers' behavior. Customers are described by a one-dimensional index of rank-ordered usage, and each customer values only a machine's comparative efficiency (albeit according to the payment schedules offered by firms rather than true costs) in producing units of output. The main consequence of these structural assumptions is that a machine's market segment is simply an interval from the spectrum of customer ranks corresponding to a "volume band" of usage rates. And a corollary is that for each machine its salient competition is from those two machines with adjacent market segments—one that is contiguous below and one that is contiguous above.

To express these ideas succinctly we will index the products so that machines with adjacent market segments are indexed successively. Thus we choose the indexing of products so that $s^i = s_{i+1}$. With this convention the dummy product captures the market segment $0 = s_0 \leqslant s \leqslant s_1$; machine 1, the segment $s_1 \leqslant s \leqslant s_2$; . . . ; and machine m, the segment $s_m \leqslant s \leqslant s_{m+1} = 1$. Thus, each machine i captures a market segment that corresponds to a volume band comprising an interval $q_i(s_i) \leqslant q \leqslant q_i(s_{i+1})$ of usage rates. By offering more favorable terms on a lease of machine i, the lessor may attract customers away from machines $i - 1$ and $i + 1$, but not from machines "further away" than that, unless the price cut is sufficiently large to exclude machine $i - 1$ or $i + 1$ entirely from any positive market share. The focus on market segments comprising volume bands and the predominance of competition between products with adjacent bands are in fact common features of the markets to which this model applies.

Before we close this section it is worth emphasizing that the interpretation of the variable q as a usage rate or quantity of output is not at all necessary. One can as well apply the model to situations in which q is interpreted as an index of the quality of the product as a consumption good for the customer. In this case, suppose that each customer has an inelastic demand for one unit of the consumption good at a reservation price $U(q, s)$ that depends on the quality index and on the customer's rank; thus, $u(q, s)$ is the customer's willingness to pay for a marginal increment of quality. Similarly, $C^j(q)$ is firm j's cost of supplying an item of quality q (by technique i if $C^j[q] = C_i[q]$). Thus, the entire formulation carries over intact to the case that quality has the conventional interpretation and customers' demands are inelastic.[5] The formulation is summarized in table 1.

5. This is the interpretation in the original work by Mussa and Rosen (1978) on linear pricing in the context of a monopolist with a continuum of products differentiated along a continuum of qualities. In Oren et al. (1980, 1982*b*) we study monopoly linear and nonlinear pricing in models in which customers purchase various quantities over a spectrum of qualities. In Oren et al. (1982*a*) one of the two dimensions of product attributes can be interpreted as a measure of quality. The multiproduct formulation in Mirman and Sibley (1980) is also subject to a quality interpretation.

TABLE 1 Summary of the Formulation

Index	Formulations	
Products (i):		
Cost	$C_i(q) = F_i + V_i \cdot q$	$q > 0$
Lease	$R_i(q)$	$q > 0$
Market segment	$s_i \leq s \leq s^i = s_{i+1}$	
Volume band	$q_i \leq q \leq q^i$	$q_i \leq q_i(s_i)$ $q_i(s^i) \leq q^i$
Leases:		
Payment	$R_i(q)$	
Marginal		
Price	$p_i(q)$	
Customers [$s = H(t)$]:		
Benefit	$U(q, s)$	
Marginal benefit	$u(q, s)$	
Usage	$q_i(s)$	$i = i(s)$
Payment	$R_i[q_i(s)]$	
Firms (j):		
Product line	L_j	
Profit	$\sum_{i \in L_j} \int_{s_i}^{s^i} \{R_i[q_i(s)] - C_i[q_i(s)]\} ds$	

NOTE.—For the dummy product ($i = 0$) and firm ($j = 0$): $R_0(q) = C_0(q)$, $F_0 = 0$, $V_0 = \infty$.

III. Analysis of the Basic Model

In this section we analyze the basic model formulated in Section II and provide a complete characterization of the solution. We urge the reader who is not interested in the mathematical details to skip to the summary of the results at the end of the section. Also, it may be useful to look ahead to see how we apply the results to the solution of a particular example in Section IV.

Our task is to determine the optimal lease terms that a firm should offer. One takes as fixed the terms offered by competitors, but of course all of the firm's own products can be included in the optimization. We first pursue the analysis in a simplified setting and then in Section V show how the methods can be extended to construct the entire solution.

The simplified setting employs two restrictive assumptions that are then relaxed in the subsequent analysis of the general case. The first assumption is that the client firm faces only two competing products on either end of a sequence of adjacent volume bands. Thus, we can take the products to be indexed by $i = 0, 1, \ldots, m, m + 1$, of which the firm's product line is (taking the firm's index to be j) $L_j = \{1, \ldots, m\}$, and we take as fixed the payment schedules $R_0(q)$ and $R_{m+1}(q)$ offered by the lessors of machines 0 and $m + 1$. Of course either machine 0 or $m + 1$ could be a dummy machine. The second assumption is that the optimal payment schedule $R_i(q)$ for each machine i is concave and differentiable over the interval $q_i \leq q \leq q_{i+1}$ of usages by its lessees. This assumption is a working hypothesis whose truth we will be able later to verify.

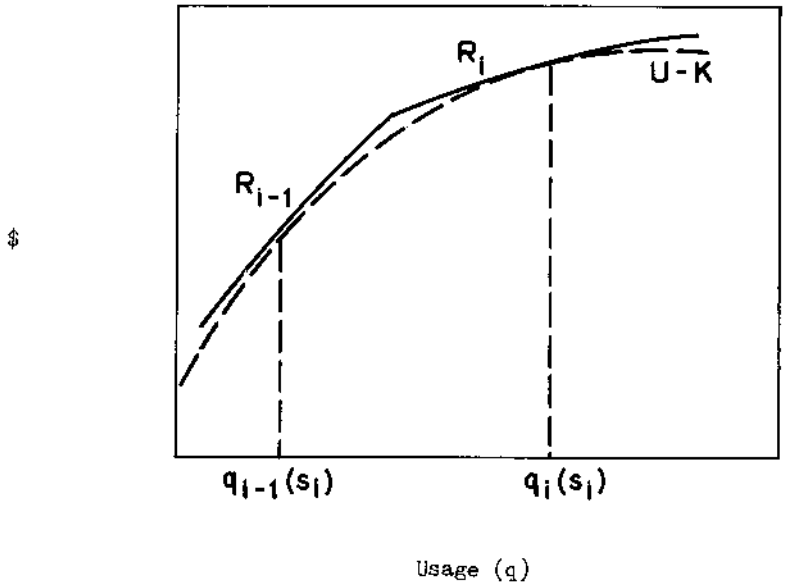

FIG. 7.—Determination of the market segment boundary

Our analysis of the firm's pricing problem is based on two key properties of the model that we now explain. The first property is that if machine i is to capture the market segment $s_i \leq s \leq s_{i+1}$, then it must be that customer s_i is indifferent between leasing machines i and $i - 1$; similarly, s_{i+1} is indifferent between leasing machines i and $i + 1$ (unless, of course, $s_i = 0$ or $s_{i+1} = 1$). In figure 7 we use a downward translation of the customer's benefit function to illustrate this property. This indifference relation is expressed by the equation:

$$U[q_{i-1}(s_i), s_i] - R_{i-1}[q_{i-1}(s_i)] = U[q_i(s_i), s_i] - R_i[q_i(s_i)]; \quad (1)$$

or if $i - 1 = 0$ is the dummy machine, then

$$0 = U[q_1(s_1), s_1] - R_1[q_1(s_1)]. \quad (1')$$

The second property is that for each customer s its usage $q_i(s)$ if it leases machine i is determined by the condition that the customer anticipates usage up to the point that its willingness to pay no longer exceeds the marginal charge for an increment of usage. This condition is expressed by the equation

$$u(q, s) = p_i(q), \quad (2)$$

where $q = q_i(s)$ and $p_i(q) = dR_i(q)/dq$ is defined to be the price for an increment in the usage rate. The condition (2) is depicted in figure 8.

This second property can be recast in an advantageous form. Recall our assumption that higher-ranked customers always choose higher usage rates. This means that if a constant price p were charged for units

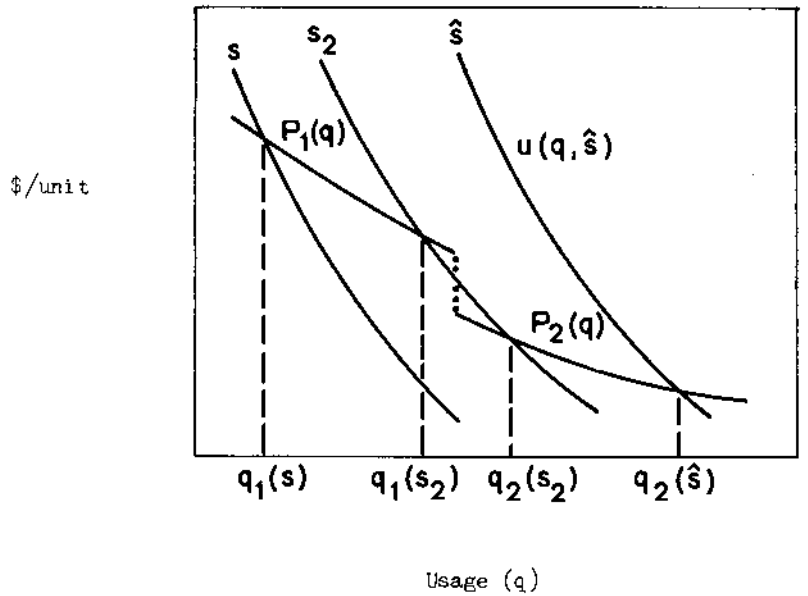

FIG. 8.—Determination of customer's usage rate

of usage then the fraction of customers whose usage would exceed a level q would be given by a function $S(q, p) = 1 - s$, where s satisfies $u(q, s) = p$. Similarly, if $p(q) = dR(q)/dq$ is declining then $S[q, p(q)]$ is the fraction with usages at least q. This feature is particularly important for the practical task of gathering data in order to implement the model. Often the raw data that are available consist of a tabular array indicating the number (or fraction) of the customers choosing each usage at each of several prices. Such an array is, in effect, a representation of selected values of the function S. Consequently, we will base our subsequent analysis on the key role played by this summary representation of the demand data. In the case of nonlinear pricing that we address, $S[q, p_i(q)]$ is the fraction of potential customers whose usage rate would exceed q if they leased machine i with a payment schedule $R_i(q)$ having a corresponding marginal price schedule $p_i(q)$ for incremental units. (This uses the assumption that the payment schedule is concave so that the marginal price schedule is not decreasing.) In figure 9 we represent $S(q, p)$ for high and low values of p.

We now address the problem of determining the optimal pricing policy. The firm's profit will be

$$\Pi_j = \sum_{i=1}^{m} \int_{s_i}^{s_{i+1}} \{R_i[q_i(s)] - C_i[q_i(s)]\} ds.$$

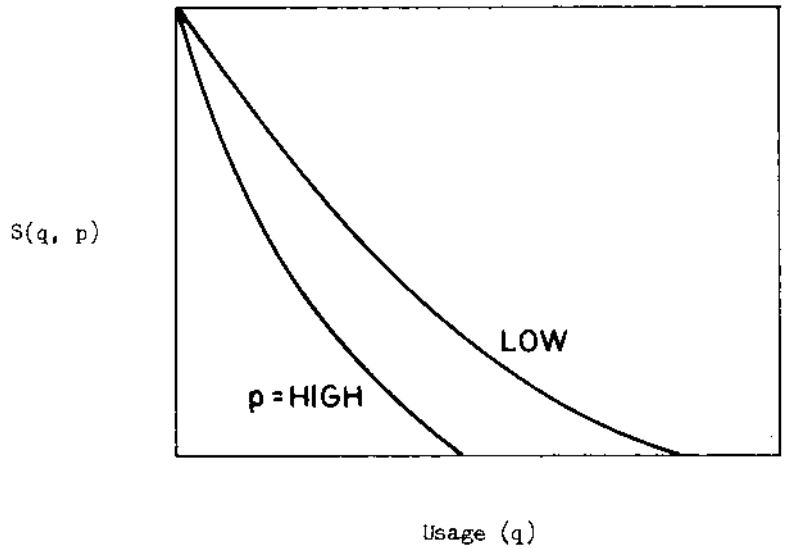

FIG. 9.—Demand date for high and low prices

Choosing the payment schedules to maximize this expression turns out to be quite easy after we transform it suitably. The first step is to integrate by parts in each term and then to substitute the relations (1) or (1′) and then (2). Also, we can change the variable of integration from s to q so as to utilize the function $S(q, p)$. When this is done we get the following alternative formula for the firm's profit, which we shall use hereafter:

$$\begin{aligned}\Pi_j = {} & (1 - s_1) \cdot \{U[q_1(s_1), s_1] - U[q_0(s_1), s_1] + R_0[q_0(s_1)] - C_1[q_1(s_1)]\} \\ & + (1 - s_{m+1}) \cdot \{U[q_{m+1}(s_{m+1}), s_{m+1}] - U[q_m(s_{m+1}), s_{m+1}] \\ & - R_{m+1}[q_{m+1}(s_{m+1})] + C_m[q_m(s_{m+1})]\} \\ & - \sum_{i=2}^{m}(1 - s_i) \cdot (\{U[q_i(s_i), s_i] - C_i[q_i(s_i)]\} \\ & - \{U[q_{i-1}(s_i), s_i] - C_{i-1}[q_{i-1}(s_i)]\}) \\ & + \sum_{i=1}^{m} \int_{q_i(s_i)}^{q_i(s_{i+1})} S[q, p_i(q)] \cdot [p_i(q) - V_i]dq.\end{aligned} \tag{3}$$

In this formula, note that $U[q_0(s_1), s_1] = 0$ and $R_0[q_0(s_1)] = 0$ if machine 0 is actually the dummy machine; also, if there is no machine $m + 1$ of higher quality than machine m then $s_{m+1} = 1$, so the second and third lines drop out. To see how the integration by parts is done take the case $m = 1$:

$$\begin{aligned}\Pi_j &= \int_{s_1}^{s_2} \{R_1[q_1(s)] - C_1[q_1(s)]\}ds \\ &= -[1 - s_2] \cdot \{R_1[q_1(s_2)] - C_1[q_1(s_2)]\} \\ &\quad + [1 - s_1] \cdot \{R_1[q_1(s_1)] - C_1[q_1(s_1)]\} \\ &\quad + \int_{s_1}^{s_2} [1 - s]d\{R_1[q_1(s)] - C_1[q_1(s)]\},\end{aligned}$$

where, changing the variable of integration to q, the latter differential is

$$d\{R_1[q_1(s)] - C_1[q_1(s)]\} = [p_1(q) - V_1]dq,$$

and using (2) the integrand $1 - s$ changes to $S[q, p_1(q)]$. Last, using (1) in the first line above yields the desired expression corresponding to (3) for the case $m = 1$.

The last line in (3) is the key one and it has an interpretation that is worth emphasizing. A fraction $S[q, p_i(q)]$ of the potential customers will have usages of q or more and from each of these customers the firm will obtain a profit of $p_i(q) - V_i$ on the qth unit of their usage, even if they lease machines of one of the higher qualities $i + 1, \ldots, m$. (This is not exactly true if machine $m + 1$ is not a dummy machine but this is compensated for by the second and third lines of [3].) The reason for this is that lessees choosing higher quality machines will pay for increments of usage at lower levels via the payment schedules $R_{i+1}(q)$, etc. This situation is depicted in figure 10 for the special case that the payment schedules happen to be linear. In each instance the higher quality machine commands a premium in the fixed part of the lease payment that captures the imputed charges for all increments of usage at lower levels than the range for which the given machine is the optimal choice for the customer.

The great advantage of the complicated-looking profit formula (3) is that it breaks the problem of determining the optimal pricing policy into several independent and manageable parts. One subproblem is to determine the appropriate marginal price schedules $p_i(q)$ for increments of usage for each machine i. From the last line of (3) one sees that $p_i(q)$ should be that value of p that maximizes the integrand $S(q, p) \cdot [p - V_i]$; namely, $p_i(q)$ should satisfy the first-order condition[6]

$$\partial S[q, p_i(q)]/\partial p \cdot [p_i(q) - V_i] + S[q, p_i(q)] = 0 \tag{4}$$

and also the corresponding second-order condition, for each usage q in the interval $q_i(s_i) \leq q \leq q_i(s_{i+1})$. (This condition can be interpreted as

6. This is not exactly true. As Mussa and Rosen (1978) and other authors have noted and we have treated in detail in Oren et al. (1983), if (4) does not yield a nonincreasing marginal price schedule over some interval $a \leq q \leq b$ then the optimal marginal price schedule is obtained as the one satisfying $p_i(q) = p^*$ over a larger interval $a^* \leq q \leq b^*$ for which (4) is satisfied on average.

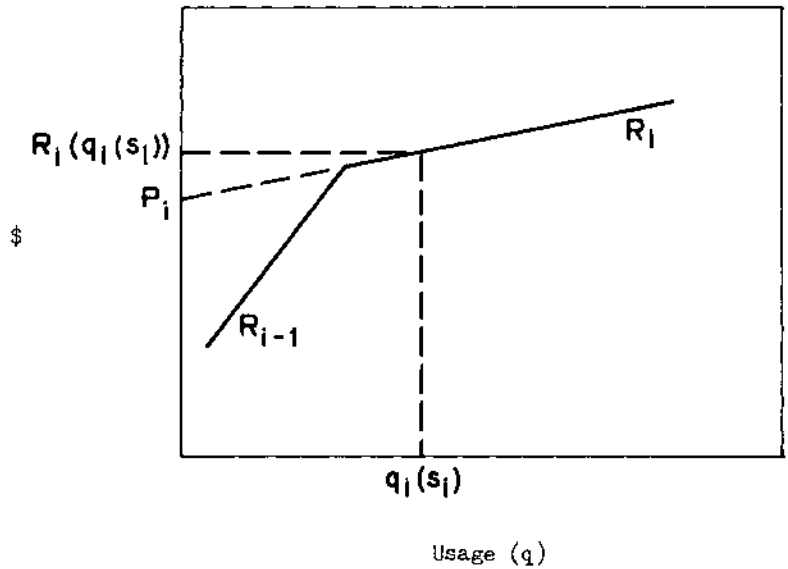

FIG. 10.—Payment schedules

the first-order condition for monopoly pricing of the qth unit under the demand function $S[q, \cdot]$ and marginal cost V_i. What distinguishes the nonlinear pricing problem from the usual linear pricing problem is the interpretation of the demand function; here it is the fraction of customers purchasing at least q units.) Applying this result yields a marginal price schedule that has the form depicted in figure 8.

The remaining task, therefore, is to determine the market segments $[s_i, s_{i+1}]$ for each product i. That is, once we know the s_i we can determine the volume bands $[q_i(s_i), q_i(s_{i+1})]$ from the condition (2) using the marginal price schedule and the total payment schedule at the endpoints from (1) or (1′). If $2 \leq i \leq m$ then the optimal choice of s_i is the one that maximizes the sum of the middle lines in (3). For this maximization the first-order condition (after simplifying by substituting [2] and [4]) is[7]

$$\begin{aligned}(1 - s_i) \cdot \partial\{U[q_{i-1}(s_i), s_i] - U[q_i(s_i), s_i]\}/\partial s \\ = \{U[q_{i-1}(s_i), s_i] - C_{i-1}[q_{i-1}(s_i)]\} - \{U[q_i(s_i), s_i] - C_i[q_i(s_i)]\}.\end{aligned} \tag{5}$$

Of course the second-order condition must also be satisfied. We do not write out the similar conditions for s_1 and s_{m+1} (if $s_{m+1} < 1$) that differ

7. This condition is obtained as a transversality condition. It is equivalent to the integral of (4) between $q_{i-1}(s_i)$ and $q_i(s_i)$. As a technical matter it should be mentioned that one can always ensure that the envelope of the tariffs, $R(q) = \min_i R_i(q)$, is differentiable by the rule that $p(q) = u(q, s_i)$ for $q_{i-1}(s_i) \leq q \leq q_i(s_i)$, as is indicated in fig. 8.

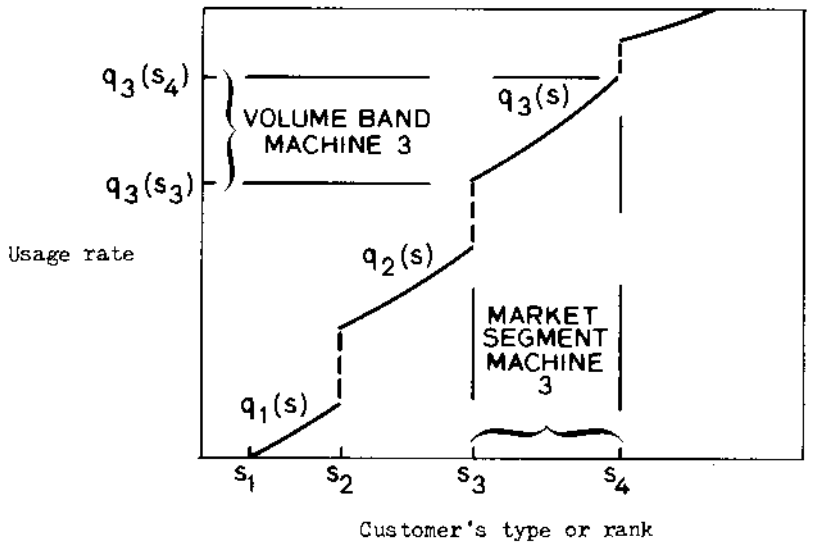

FIG. 11.—Volume bands and market segments

only in that the competitor's payment schedule R_0 or R_{m+1} is taken as fixed.

The determination of the market segments, as above, can also be viewed as determining the levels of the payment schedules (the slopes having been previously determined): the choice of market segments accomplishes exactly that using the recursive formula

$$R_i[q_i(s_i)] = R_{i-1}[q_{i-1}(s_{i-1})] + \int_{q_{i-1}(s_{i-1})}^{q_{i-1}(s_i)} p_{i-1}(q)dq$$
$$+ U[q_i(s_i), s_i] - U[q_{i-1}(s_i), s_i],$$

for $i = 1, \ldots, m$ where R_0 is known.

The relationship between the machines' volume bands and market segments is depicted in figure 11.

In Section IV we show via an example how to apply all of the conditions (1), (2), (4), and (5) to solve for an optimal pricing policy. Then in Section V we return to the present subject to address the complications that arise when a firm's market segments are not adjacent or when it is not optimal to lease all of the machines in the firm's product line.

Table 2 summarizes the conditions for an optimal pricing policy.

IV. Solution of an Example

In this section we illustrate the results from Section III by finding the optimal pricing policy for a particular example in which the customers' benefits are quadratic functions of the usage rate.

TABLE 2 Summary of the Conditions for an Optimal Price Policy

Equation	Description
(2)	Customer's response to the marginal price schedule: $u[q_i(s), s] = p_i[q_i(s)]$; i.e., marginal benefit of usage equals marginal price.
(1)	Lower endpoint of the market segment for machine i: $U[q_{i-1}(s_i), s_i] - R_{i-1}[q_{i-1}(s_i)] = U[q_i(s_i), s_i] - R_i[q_i(s_i)]$; i.e., customer s_i is indifferent between leasing $i - 1$ and i.
(4)	Optimal marginal price schedule: each $p_i(q)$ solves $\partial S[q, p_i(q)]/\partial p \cdot [p_i(q) - V_i] + S[q, p_i(q)] = 0$; i.e., maximize induced profit for qth unit of usage.
(5)	Optimal market segment for machine i: $(1 - s_i) \cdot \partial\{U[q_{i-1}(s_i), s_i] - U[q_i(s_i), s_i]\}/\partial s = \{U[q_{i-1}(s_i), s_i] - C_{i-1}[q_{i-1}(s_i)]\} - \{U[q_i(s_i), s_i] - C_i[q_i(s_i)]\}$; i.e., this is the discrete version of equation 4.

NOTE.—(1) The marginal price schedule is $p_i(q) = dR_i(q)/dq$, required to be nonincreasing. (2) The function S satisfies $S(q, p) = 1 - s$ if $u(q, s) = p$; i.e., it is the fraction of customers with usage at least q if the marginal price of usage is p. Hence $S[q, p_i(q)]$ is the fraction using at least q.

Starting with the corresponding market data that would be observed, we suppose that the fraction of potential customers using at least q at the price p is given by the function

$$S(q, p) = 1 - \frac{q}{1 - p},$$

provided $0 \leq q \leq 1 - p \leq 1$. One can think of this relation as arising from a loglinear regression of $1 - S$ on q and $1 - p$, with the coefficients normalized to unity by appropriate choices of the origins and scales of measurement for usage and revenue.[8] Solving this equation for p and using $s = 1 - S$ yields the imputed willingness to pay for marginal units of the customer of rank s:

$$u(q, s) = 1 - \frac{q}{s},$$

provided $0 \leq q \leq s$. Integrating u with respect to q yields a quadratic function for the customer's benefit.

8. We have not investigated thoroughly the empirical aspects of estimating demand behavior. The key step in any practical application is to estimate the function $S(q, p)$. With data encompassing a wide range of values of q and p, the task is to identify a functional form having the requisite monotonicity properties and yielding a good fit. Previous use of a pricing policy including nonlinear payment schedules will tend to produce data with this richness, but in a first application it is likely that the range of values for which data is available is so small that the identification problem is quite severe. In some cases it may be advisable to adopt an interim pricing policy designed mainly to collect data on customers' demand behavior.

We first derive the optimal marginal price schedule for each machine using the condition (4), and obtain:

$$p_i(q) = 1 - [q(1 - V_i)]^{\frac{1}{2}},$$

provided $V_i < 1$ and $q \leq 1/[1 - V_i]$. One sees here that as required the marginal price is a declining function of the usage rate. An alternative interpretation is that the nominal price is $p_i(0) = 1$ and a rebate of $r_i(q) = (2/3)[q \cdot (1 - V_i)]^{\frac{3}{2}}$ is allowed for each unit of usage if the total usage is q. That is, the payment schedule has the form $R_i(q) = q \cdot [p_i(0) - r_i(q)]$ and its derivative is $p_i(q)$ as above.

The customer of rank s responds to this marginal price schedule with the usage implied by (2): $q_i(s) = s^2 \cdot [1 - V_i]$. Note that customers of higher rank have more usage; in this case usage increases as the square of the rank due to the extra inducement provided by the rebate.

The next step is to determine the optimal market segment for each machine. If the firm offers two machines, $i - 1$ and i, with contiguous market segments, then according to (5) the optimal boundary point is

$$s_i = \frac{2(F_i - F_{i-1})}{(1 - V_{i-1})^2 - (1 - V_i)^2},$$

provided that $F_{i-1} < F_i$ and $V_{i-1} > V_i$, as required for efficiency. Indeed one can verify that if machine i is less costly for usages $q > q_i$, then $q_{i-1}(s_i) < q_i < q_i(s_i)$, as one expects. If machine i is to command a positive market share, moreover, one needs $s_i < s_{i+1}$. In Section V we indicate how to handle situations failing this condition; usually it means that machine i should be deleted from the product line.

Assuming that machine i obtains a positive market share, the next step is to determine the premium in the lease's payment schedule that customers of ranks $s \geq s_i$ must pay in order to get the higher quality machine i. According to (1) this premium is the difference

$$R_i[q_i(s_i)] - R_{i-1}[q_{i-1}(s_i)] = (F_i - F_{i-1}) \cdot (\alpha_i^{-1} - s_i),$$

where $\alpha_i = 1 - (V_{i-1} + V_i)/2$. This premium for moving up to machine i is calculated as the amount that makes customer s_i indifferent about paying the premium and increasing his usage by

$$q_i(s_i) - q_{i-1}(s_i) = s_i^2 \cdot (V_{i-1} - V_i).$$

Actually, this premium turns out to be equivalent to paying $p_{i-1}[q_{i-1}(s_i)]$ for each unit of usage from $q_{i-1}(s_i)$ to q_i, and then $p_i[q_i(s_i)]$ for each unit from q_i up to $q_i(s_i)$, as was depicted in figure 8.

In order to find the payment schedule for usages $q > q_i(s_i)$ of machine i, one integrates (2) to get

$$\begin{aligned} R_i(q) &= R_i[q_i(s_i)] + \int_{q_i(s_i)}^{q} p_i(x)dx \\ &= R_i[q_i(s_i)] - \hat{R}_i[q_i(s_i)] + \hat{R}_i(q), \end{aligned}$$

where

$$\hat{R}_i(q) = q \cdot \{1 - (2/3)[q \cdot (1 - V_i)]^{1/2}\}.$$

In particular, this formula is used to compute the value of $R_i[q_i(s_{i+1})]$ employed in the calculation of the premium for the upgrade from machine i to $i + 1$.

It is important to observe that as a practical matter the seller may want to quote lease terms that specify a two-part tariff plus offers of quantity discounts. In this case the two-part tariff is composed of a nominal (before the quantity discounts that apply once the base usage $q_i[s_i]$ is passed) marginal price $p_i = p_i[q_i(s_i)]$ together with a fixed payment

$$P_i = R_i[q_i(s_i)] - p_i \cdot q_i(s_i),$$

as shown in figure 10.

The two machines 0 and $m + 1$ affect the pricing of machines 1 and m. Either may be real machines offered by competitors or they may be dummy machines introduced to fit our formulation.

Suppose first that there is no machine $m + 1$ of higher quality than machine m. In this case all of the highest ranked customers will lease machine m, since there is no higher quality substitute. In this case the role of equation (5) is played by the specification $s_{m+1} = 1$ for the dummy machine $i = m + 1$. Consequently, the upper limit of the volume band for machine m is the usage $q_m(1) = 1 - V_m$. As one would expect, this is just the usage resulting from pricing the last unit at marginal cost: $p_m[q_m(1)] = V_m$.

If there is no machine of lower quality than machine 1, then the dummy machine 0 represents the customer's option not to lease any machine. In this case s_1 is the lowest rank of customer who chooses to lease at all, and $1 - s_1$ is the market size or penetration. To compute s_1 we solve (5) for the case that $q_0(s_1) = 0$; that is, the next lower quality option for customer s_1 is to forgo any output. In our example this yields $s_1 = (2F_1)^{1/2}/(1 - V_1)$, and then (1) provides the initial lease payment for the first machine:

$$R_1[q_1(s_1)] = \frac{F_1 \cdot [2 - (2F_1)^{1/2}]}{1 - V_1},$$

at the minimal usage $q_1(s_1) = 2F_1/(1 - V_1)$. It is worth noting that the crucial aspect of the pricing policy reflected in this calculation is the size of the market for the product line, since only potential customers with ranks exceeding s_1 will find it worthwhile to lease any machine.

We now study the case in which either machine 0 or machine $m + 1$ is an actual machine offered by a competitor. In this case the pricing policy depends on the payment schedules R_0 and R_{m+1} specified in the competitors' leases. Rather than develop an entire analysis, we illustrate the calculations with an example in which machine 0 leases for a

fixed periodic payment L_0 plus a constant marginal price p_0 per unit of usage; that is, $R_0(q) = L_0 + p_0 q$. (The calculations are quite similar in the general case.)

In our example a customer of rank s who leases machine 0 will choose the usage $q_0(s) = s \cdot [1 - p_0]$, according to (1). In order to find the optimal market segment for machine 1, one can see from (3) that the appropriate condition is the same as (5) except that R_0 substitutes for C_0 exactly as if R_0 were the cost that the firm would incur were it to supply customers by leasing machine 0 from the competitor and then re-leasing it to its own clientele. From this construction one finds that s_1 is the positive root of the quadratic equation

$$\tfrac{1}{2}s_1^2(1 - V_1)^2 - s_1(1 - p_0)^2 = F_1 - L_0 - \tfrac{1}{2}(1 - p_0)^2.$$

The last step is then to determine the basic lease payment from (1), which yields

$$R_1[q_1(s_1)] = F_1 + \tfrac{1}{2}[s_1^2(1 - V_1) - (1 - p_0)^2] - \tfrac{1}{2}s_1^3(1 - V_1)^2,$$

at the minimal usage $q_1(s_1) = s_1^2[1 - V_1]$.

The calculations in the case that machine $m + 1$ is also a product offered by a competitor are quite similar, since again condition (5) with R_{m+1} replacing C_{m+1} is the appropriate condition to determine the market segment captured by machine m.

V. Refinements

We now address several of the matters left unresolved in Section III. Foremost among these is completing the analysis of the product-line pricing problem to handle arbitrary configurations.

In the general case a firm's product line may compete with products that are dispersed throughout the quality spectrum. In our formulation the quality of a product is synonymous with the volume band for which it is efficient in terms of the producer's costs. In turn, the firm's pricing policy translates the volume band into a market segment that is an interval in the spectrum of customers' ranks. Competition among products with adjacent qualities, namely contiguous volume bands and market segments, occurs on the margin between them. If a firm controls both products then it constructs its pricing policy, and thereby the boundary between the market segments,[9] so as to maximize the sum of its profits over the product line. A firm that controls only one of the two products considers only its own profits and naturally tries to encroach on the neighboring market segment via somewhat lower prices. A firm with a single product isolated between two competitors finds it neces-

9. Recall that ordinarily the bands of usages chosen by customers are not actually contiguous—there is a gap between them. The optimal payment schedule induces a shrinkage in each volume band.

sary to compete this way at both ends of its market segment. Most of these observations are evident from the example studied in Section IV.

If the set of products in a firm's product line is assuredly the right one, then the analysis in Section III carries over to the general case. Each firm can partition its product line into clusters of products with adjacent market segments covering an interval of customers' ranks. Each cluster (say it is indexed by $i = 1, \ldots, m$) can be analyzed as in Section III by treating its competitors' substitute products at either end of the interval as the products 0 and $m + 1$.

The more difficult problem is to determine the right set of products to be included in the product lines of the firms. This problem takes two forms. On the one hand, one firm's pricing strategy can eliminate the market segment for another firm's product. On the other hand, within a cluster for one firm, it may be optimal to delete one or more products. In our work we have adopted the ad hoc expedient of deleting some one product with a "negative" market share (as calculated according to the algorithm in Sec. III), and then repeating the procedure until the market shares of the remaining products are all positive. In general, however, this need not be an optimal procedure. The construction of an optimal procedure poses difficult combinatorial problems that we have not overcome in our work to date. (In part we have not pursued the matter vigorously because in practice firms want to retain their full product lines for other reasons, and to do this they may be willing to cut prices below actual costs to retain market share. In this case the pricing of such a product is essentially imposed exogenously, and the present model does not apply to these products, which must be treated exogenously as though they were competitors' products.)

Various refinements of the present model are easily developed. One can, for example, find the price cuts necessary to expand market share by a given amount or, just the reverse, the prices of adjacent competitors that would choke off demand for one's product. We have found that a favorite use of the model is to predict the market share that would be captured by a new product, either with or without the optimal pricing policy supposed by the model. The key role of the function $S(q, p)$ in interpreting demand data and in designing pricing policies seems also to be an insight valued by pricing managers. Indeed, equation (3) as a formula for the firm's profit reveals the essential structure of the product-line pricing problem: If one thinks of the qth unit of usage as a separate market, with the understanding that each customer must buy a qth unit in order to be able to buy any larger number of units, then one sees that the formula (4) for the optimal marginal price schedule is just the familiar one for optimal pricing by a firm with a differentiated product.

A simplifying assumption in the present model is that customers' characteristics are arrayed along a one-dimensional continuum. It is

more usual in practice to describe customers with a vector of socio-demographic characteristics. If products are differentiated along only one dimension, however, this need not yield any substantial gain in generality. For example, if there exists a function mapping the vector of characteristics into a summary attribute that indexes the customers according to their purchase sizes, then the same results apply. The key test is whether it is sufficient to describe demand responses by the function $S(q, p)$ indicating the fraction of potential customers using at least q units at the marginal price p.[10]

We have assumed that products are differentiated along a single dimension of quality synonymous with marginal operating cost. In our companion papers (Oren et al. 1980, 1982*a*) we have studied models in which products are differentiated along two dimensions of quality.[11] The same methods of analysis apply and the results are analogous, but of course matters are somewhat more complicated. The key idea remains the same: The marginal price schedule assigns a price to each incremental "square" comprising an increment of each quality, just as in the one-dimensional case the marginal price schedule assigns a price to each incremental interval of the single quality (which in the present model is simply an incremental interval of quantity).

We have assumed that the payment schedules are required to be concave. In practice this is usually necessary to prevent the development of a resale market. However, if firms can control usage to prevent resale among customers, then in some cases it may be allowable and optimal to have payment schedules with strictly convex portions. In these cases the concavity constraint that we have imposed can be deleted. For further details consult Goldman, Leland, and Sibley (1977).

Some of the mathematical aspects skipped over in Section III are reviewed in the Appendix.

VI. Conclusion

In this paper we have addressed the problem of product-line pricing as an application of the general methodology of nonlinear pricing. Not all product-line pricing problems take the form we have hypothesized, of course, but we anticipate that our approach can be adapted to a wider class of situations by the systematic application of the same sort of ideas. Consequently, it seems worthwhile to spell out here the general principles that guide the analysis.

The initial premise is that the population of potential customers is

10. If this test fails then the mapping from vectors of characteristics into a summary attribute may depend endogenously on the pricing policy. We have not completed the analysis of this general case.

11. A different formulation is studied by Mirman and Sibley (1980).

heterogeneous. The differences in benefits among customers offer the seller an opportunity to improve profits (and in most cases consumer welfare) by offering products with different attributes that appeal to different market segments. With an appropriate pricing policy, a variegated menu of products induces customers to self-select the product best adapted to each one's needs. The seller's profit opportunities arise in at least three forms. First, he can offer finer gradations among the products in order to appeal to specialized tastes among the customers. This is the problem of product design and differentiation, which we have treated only obliquely here. Second, with a given limited spectrum of products in his line, he can offer terms of payment that induce self-selection on the basis of customers' differing responses to the same quality attribute of a product. In this paper we have used a machine's operating cost, manifested in the variable portion of the payment schedule, as a surrogate for the quality attribute, and we have constructed optimal quantity discounts as the mechanism of self-selection. Third, he can adjust up or down the entire payment schedule for a product to obtain the optimal allocation of market segments to his products. Here, we have determined the optimal market segments in conjunction with the pricing policy, and specified the fixed portions of the leases' terms to achieve this segmentation. In the latter case we obtained a result that is a general principle: the initial payment by the lowest ranked customer in the market segment, or equivalently the two-part tariff from which the quantity discounts are subtracted, is determined by accumulating the marginal charges assessed for all lesser usages—in effect, what one can charge as a fixed fee for upgrading to a higher quality product is imputed from the opportunity cost imposed by the terms offered for the lower quality product.

In sum, given a product line with products ordered in a sequence of improving quality, one obtains an "algorithm" for constructing the optimal payment schedules by applying systematically the principles of optimal self-selection, starting with the lowest quality product and then moving up the product line. The initial fixed part of the payment schedule for one product is determined from the terms offered at the upper end of the payment schedule for its predecessor, and from there the optimal discounts are constructed to achieve the most profitable pattern of self-selection among the customers.[12]

From the viewpoint of overall marketing strategy, a salient conclusion from our analysis is that pricing policy can be an instrument of product differentiation. In the context studied here a customer views a

12. These principles are, of course, evident in all treatments of nonlinear pricing, e.g., Mussa and Rosen (1978). The difference here lies in the recognition that limited product differentiation still leaves opportunities for further profit improvements by designing the quantity discounts optimally and by adjusting the fixed part of the payment schedule to achieve optimal market segmentation among products.

product as described by two attributes—its fixed and variable operating costs. An ordinary linear payment schedule offers all customers a single pair of attributes. A nonlinear payment schedule, on the other hand, offers a menu of such pairs. (Each item in the menu is a pair corresponding to the fixed and variable portions of the linear payment schedule that is tangent to the nonlinear payment schedule at a corresponding usage rate.) The menu is so designed that each type of customer selects that pair of attributes from the menu that best serves his needs. Whenever there are several different types of customers a firm's optimal pricing policy will offer a corresponding variety of attribute pairs that appeal differently to the various customer types. A visual analogy is provided by thinking of a machine's cost attributes (F_i, V_i), which is a single point in the two-dimensional space of product attributes, as being "smeared" into a locus of such points that are offered to customers as a menu of alternative price plans among which to choose. Each point on the locus specifies a corresponding linear payment schedule, and the lower envelope of these linear payment schedules determines the nonlinear payment schedule that is optimal for the firm. High-usage customers are in effect choosing a plan with high fixed charges and low variable charges, and low-usage customers a plan with lower fixed charges and higher variable charges. The effect is the same as if the firm had a corresponding variety of machines to offer, obtained through the technological design of differentiated products to obtain a complete product line.

Appendix

In this Appendix we review some of the mathematical aspects that were skipped over in Section III. Recall (see nn. 6 and 7) that by construction the envelope $R(q)$ is constructed to be concave and differentiable. This assures that the function $S(q, p)$ is an accurate summary of the demand data: $S[q, p(q)]$ is the fraction of the customers using at least q, since $\hat{q} \geqslant q$ implies $p(\hat{q}) \leqslant p(q)$ due to the concavity of $R(q)$ and our assumptions on U assure that S is decreasing in both q and p. The concavity and positivity of R also assure that no customer can gain by leasing more than one machine:

$$R(q) + R(\hat{q}) \geqslant 2R\left(\frac{q + \hat{q}}{2}\right) \geqslant R(q + \hat{q}).$$

From the relation $u[q(s), s] = p[q(s)]$ it follows that

$$\partial u/\partial q = \partial p/\partial q - \frac{(\partial u/\partial s)}{(\partial q/\partial s)} < \partial p/\partial q,$$

since $\partial u/\partial s > 0$ by assumption and $\partial q/\partial s > 0$ as shown in connection with figure 5. Hence, U is more concave than R, and therefore for the customer's maximization problem the necessary condition (2) is sufficient as well as necessary. Similarly, the necessary condition (4) is sufficient since the maximand is unimodal on the domain $p > V_i$ and $S > 0$; the same argument applies to the

sufficiency of (5) since it is obtained by integrating (4) over the appropriate interval (see n. 7). The reader interested in verifying these results will find it easiest to express (4) and (5) in terms of the inverse $s_i(q)$ of the map $s \to q_i(s)$:

$$[1 - s_i(q)] \cdot \frac{\partial u[q, s_i(q)]}{\partial s} + u[q, s_i(q)] = V_i, \tag{4'}$$

and similarly for (5), derived as the transversality condition around the point q_i of discontinuity of the marginal cost function. It should be mentioned too that the derivation takes advantage of the fact that the envelope $C(q)$ of the cost function is concave in that the marginal costs V_i decline with i. It remains to show that the index $i(s)$ of the machine selected is a nondecreasing function of the customer rank s. Suppose to the contrary that (i, s') and (j, s) are optimal pairs with $s' > s$ and $j > i$. Then:

$$U[q_j(s), s] - R_j[q_j(s)] \geqslant U[q_i(s), s] - R_i[q_i(s)],$$

$$U[q_i(s'), s'] - R_i[q_i(s')] \geqslant U[q_j(s'), s'] - R_j[q_j(s')].$$

Rearranging terms yields $f(s) - f(s') \geqslant 0$, where

$$f(s) = U[q_j(s), s] - U[q_i(s), s] + R_i[q_i(s)] - R_j[q_j(s)].$$

Applying the mean value theorem and using (2) yields

$$\{U_s[q_j(\xi), \xi] - U_s[q_i(\xi), \xi]\} \cdot (s - s') \geqslant 0,$$

for some ξ satisfying $s \leqslant \xi \leqslant s'$. Applying the mean value theorem a second time to q only yields

$$u_s(\eta, \xi) \cdot [q_j(\xi) - q_i(\xi)] \cdot [s - s'] \geqslant 0,$$

where $q_i(\xi) \leqslant \eta \leqslant q_j(\xi)$. Since $s < s'$ and $q_j(\xi) > q_i(\xi)$, this contradicts our assumption that $u_s > 0$. From this we conclude that the set of customer ranks choosing a given machine i is an interval $[s_i, s^i]$ and that these intervals are ordered with increasing values of i.

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