Dynamic Mean-Variance Portfolio Selection with No-Shorting Constraints*

Xun Li†  Xun Yu Zhou‡

Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong
Shatin, N.T.
Hong Kong

Andrew E.B. Lim§

Center for Applied Probability
Columbia University
New York, N.Y., 10027
USA
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Abstract

This paper is concerned with mean-variance portfolio selection problem in continuous-time under the constraint that short-selling of stocks is prohibited.

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†E-mail: <xli@se.cuhk.edu.hk>. Phone: 852-2609-8319.
‡Corresponding author. E-mail: <xyzhou@se.cuhk.edu.hk>. Phone: 852-2609-8320.
§E-mail: <lim@ieor.columbia.edu>.
The problem is formulated as a stochastic optimal linear-quadratic (LQ) control problem. However, this LQ problem is not a conventional one in that the control (portfolio) is constrained to take non-negative values due to the no-shorting restriction, and thereby the usual Riccati equation approach (involving a ‘completion of squares’) does not apply directly. In addition, the corresponding Hamilton-Jacobi-Bellman (HJB) equation inherently has no smooth solution. To tackle these difficulties, a continuous function is constructed via two Riccati equations, and then it is shown that this function is a viscosity solution to the HJB equation. Solving these Riccati equations enables one to obtain explicitly the efficient frontier and efficient investment strategies for the original mean-variance problem. An example illustrating these results is also presented.

Keywords- continuous-time, mean-variance portfolio selection, short-selling prohibition, efficient frontier, stochastic LQ control, HJB equation, viscosity solution

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Abbreviated Title. Mean–Variance Portfolio Selection without Shorting
1 Introduction

Research on portfolio selection dates back to the 1950s with Markowitz’s pioneering work [23] on mean-variance efficient portfolios for a single-period investment. The most important contribution of Markowitz’s work is the introduction of quantitative and scientific approaches to risk management and analysis. When short-selling is not allowed efficient portfolios are obtained computationally via solving a quadratic programming Later, Merton [25] derived an analytical solution to the single-period mean-variance problem under the assumption that the covariance matrix is positive definite and short-selling is allowed.

While it is natural to extend Markowitz’s work to multi-period and continuous-time portfolio selections, these extensions have, by and large, taken a somewhat different tack to Markowitz’s original formulation; see, e.g., [26, 14, 10, 27, 1] for the multi-period case, and [24, 17, 4, 7, 13, 8] for the continuous-time case. Specifically, rather than treating the Var $x(T)$ and $Ex(T)$ of a portfolio as separate quantities and finding the relationship between them, a single quantity, the expected utility of terminal wealth $EU(x(T))$, is considered instead. The utility function $U$ is commonly a power, log, exponential, or quadratic form. One disadvantage of this approach is that the relationship between risk and return is contained only implicitly in the utility function. Hence, it is less clear in general what relationship exists between the risk and the return of the derived policy. It should be noted that mean-variance analysis and expected utility formulation are two different tools for dealing with portfolio selections. As a consequence, optimal portfolios determined by utility functions are usually not mean-variance efficient. One exception is the case of the quadratic utility function; see Duffie and Richardson [7] where this relationship is shown in the setting of the related mean-variance hedging problem. For comparison in the performance of the mean-variance and utility approaches, the reader is referred to [29, 14, 11, 12, 34].

One difficulty in extending Markowitz’s idea to the multi-period or continuous-time settings is that the variance Var $x(T)$ involves a term $[EX(T)]^2$ that is hard to analyze due to its non-separability in the sense of dynamic programming; see [35] for a more detailed discussion on this point. Only recently have Li and Ng [19] faithfully extended Markowitz’s mean-variance model to the multi-period setting by using an idea of embedding the problem into tractable auxiliary problem.

In the paper by Zhou and Li [35], the continuous-time mean-variance problem is studied by incorporating the embedding technique used in Li and Ng [19]. However, the main contribution of [35] is not the explicit mean-variance efficient frontier it obtained per se; rather it is the unifying framework, i.e., that of the stochastic linear-quadratic
(LQ) optimal control, it introduced to solve certain finance problems including the mean-variance portfolio selection. The so-called indefinite stochastic LQ control theory has been developed extensively in recent years (see, e.g., [2, 3, 20, 32]), which in turn provides a powerful tool for solving some finance problems that are linear-quadratic in nature ([18, 35, 21]).

The objective of this paper is to investigate continuous-time mean-variance portfolio selection in the case where short-selling the stocks is not allowed. This belongs to the realm of the so-called constrained portfolio selection, which essentially renders the market incomplete. In the past decade, constrained portfolio selection problem has been extensively studied (see, e.g., [6, 30, 31, 28, 16]). However, again the expected utility model has been mainly adopted. In particular, Xu and Shreve in their two-part paper [30, 31] investigated a utility maximization problem with a no-shorting constraint using a duality analysis.

In this paper we continue to use stochastic LQ control as the framework to study the constrained mean-variance portfolio problem. Compared with [35, 21], a major problem in the present case is that the control (portfolio) is constrained, while the LQ theory typically requires the control to be unconstrained (the reason is that the optimal control constructed through the Riccati equation may not satisfy the control constraint). This means that the elegant Riccati approach does not apply directly. We side step this problem by studying the Hamilton-Jacobi-Bellman (HJB) equation (recall that the Riccati equation is essentially the HJB equation after separating the time and spatial variables). However, the HJB equation has no classical (i.e. smooth) solutions in our case due to the presence of the control constraint. To cope with this difficulty, we first conjecture a continuous solution to the HJB equation via two Riccati equations, and then show that it is indeed the viscosity solution to the equation. Further, using the viscosity verification theorem established in [36], we obtain explicitly the optimal strategy along with the efficient frontier.

The outline of this paper is as follows. In Section 2, we formulate the mean-variance portfolio selection under short-selling prohibition. In Section 3, we study a stochastic LQ control problem of which the portfolio selection is a special case, and obtain the viscosity solution to the corresponding HJB equation along with the optimal feedback control. Section 4 is devoted to the derivation of the efficient investment strategies and the efficient frontier for the portfolio selection problem. In Section 5, we present a numerical example to illustrate the results obtained. Finally, Section 6 concludes the paper.
2 Problem Formulation and Preliminaries

2.1 Notation

We make use of the following notation:

- \( M' \) : the transpose of any matrix or vector \( M \);
- \( \|M\| \) : \( \sqrt{\sum_{i,j} m_{ij}^2} \) of any matrix or vector \( M = (m_{ij}) \);
- \( \mathbb{R}^n \) : \( n \) dimensional real Euclidean space;
- \( \mathbb{R}_+^n \) : the subset of \( \mathbb{R}^n \) consisting of elements with nonnegative components.

The underlying uncertainty is generated by a fixed filtered complete probability space \((\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq0})\) on which is defined a standard \( \{\mathcal{F}_t\}_{t\geq0} \)-adapted \( m \)-dimensional Brownian motion \( W(t) \equiv (W^1(t), \ldots, W^m(t))^t \). Given a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \( \{\mathcal{F}_t|a \leq t \leq b\}(\infty \leq a < b \leq +\infty) \), a Hilbert space \( \mathcal{H} \) with the norm \( \| \cdot \|_{\mathcal{H}} \), define the Banach space

\[
L^2_{\mathcal{F}}(0, T; \mathcal{H}) = \left\{ \varphi(\cdot) \mid \begin{array}{l}
\varphi(\cdot) \text{ is } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable process on } [a, b] \\
\text{and } E \int_a^b \|\varphi(t, \omega)\|_{\mathcal{H}}^2 dt < +\infty
\end{array} \right\}
\]

with the norm

\[
\|\varphi(\cdot)\|_{\mathcal{F}, 2} = E \int_a^b \|\varphi(t, \omega)\|_{\mathcal{H}}^2 dt < +\infty.
\]

2.2 Problem Formulation

We consider a financial market where \( m + 1 \) assets are traded continuously on a finite horizon \([0, T]\). One asset is a bond, whose price \( IP_0(t) \), \( t \geq 0 \), evolves according to the differential equation

\[
\begin{aligned}
&dIP_0(t) = r(t)IP_0(t)dt, \quad t \in [0, T], \\
&IP_0(0) = p_0 > 0,
\end{aligned}
\]

where \( r(t) \) (\( > 0 \)) is the interest rate of the bond. The remaining \( m \) assets are stocks, and their prices are modeled by the stochastic differential equations

\[
\begin{aligned}
&dIP_i(t) = IP_i(t)\{b_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)W^j(t)\}, \quad t \in [0, T], \\
&IP_i(0) = p_i > 0,
\end{aligned}
\]

where \( b_i(t)(> r(t)) \) is the appreciation rate and \( \sigma_{ij}(t) \) is the volatility coefficient. Denote \( b(t) := (b_1(t), \ldots, b_m(t))^t \) and \( \sigma(t) := (\sigma_{ij}(t)) \). We assume throughout that \( r(t), b(t) \)
and $\sigma(t)$ are deterministic, Borel-measurable, and bounded on $[0, T]$. In addition, we assume that the non-degeneracy condition

$$\sigma(t)\sigma(t)\geq \delta I, \quad \forall t \in [0, T],$$

(3)

where $\delta > 0$ is a given constant, is satisfied. Also, we define the relative risk coefficient

$$\theta(t) \triangleq \sigma^{-1}(t)(b(t) - r(t)1),$$

(4)

where $1$ is the $m$-dimensional column vector with each component equal to 1.

Suppose an agent has an initial wealth $X_0 > 0$ and the total wealth of his position at time $t \geq 0$ is $X(t)$, Then it is well-known that $X(t)$, $t \geq 0$, follows (see, e.g., [35])

$$\begin{cases}
    dX(t) = \{r(t)X(t) + \sum_{i=1}^{m}(b_i(t) - r(t))u_i(t)\}dt + \sum_{j=1}^{m} \sum_{i=1}^{m} \sigma_{ij}(t)u_i(t)dW^j(t), \\
    X(0) = X_0,
\end{cases}$$

(5)

where $u_i(t)$, $i = 0, 1, \ldots, m$, denotes the total market value of the agent's wealth in the $i$-th bond/stock. We call $u(t) := (u_1(t), \ldots, u_m(t))$ the portfolio (which changes over time $t$). An important restriction considered in this paper is the prohibition of short-selling the stocks, i.e., it must be satisfied that $u_i(t) \geq 0$, $i = 1, \ldots, m$. On the other hand, borrowing from the money market (at the interest rate $r(t)$) is still allowed; that is, $u_0(t)$ is not explicitly constrained.

Mean-variance portfolio selection refers to the problem of finding an allowable investment policy (i.e., a dynamic portfolio satisfying all the constraints) such that the expected terminal wealth satisfies $EX(T) = d$ while the risk measured by the variance of the terminal wealth

$$\text{Var } X(T) = E[(X(T) - EX(T))^2] = E[(X(T) - d)^2]$$

is minimized.

We impose throughout this paper the following assumption.

**Assumption 2.1** The value of the expected terminal wealth $d$ satisfies $d \geq X_0e^{\int_0^T r(s)ds}$.

**Remark 2.1** Assumption 2.1 states that the investor's expected terminal wealth $d$ can not be less than $X_0e^{\int_0^T r(s)ds}$ which coincides with the amount that he/she would earn if all of the initial wealth is invested in the bond for the entire investment period. Clearly, this is a reasonable assumption.

**Definition 2.1** A portfolio $u(\cdot)$ is said to be admissible if $u(\cdot) \in L^2_T(0, T; \mathbb{R}^m)$. 
Definition 2.2 The mean-variance portfolio selection problem is formulated as the following optimization problem parameterized by $d \geq X_0 \int_0^T r(s)ds$:

$$
\begin{align*}
\min & \quad \text{Var} \ X(T) \equiv E[X(T) - d]^2, \\
\text{subject to} & \quad \begin{cases} 
  EX(T) = d, \\
  u(\cdot) \in L^2_T(0, T; \mathbb{R}_+^m), \\
  (X(\cdot), u(\cdot)) \text{ satisfy equation (5)}. 
\end{cases}
\end{align*}
$$

(6)

Moreover, the optimal control of (6) is called an efficient strategy, and $(\text{Var} \ X(T), d)$, where $\text{Var} \ X(T)$ is the optimal value of (6) corresponding to $d$, is called an efficient point. The set of all efficient points, when the parameter $d$ runs over $[X_0 \int_0^T r(s)ds, +\infty)$, is called the efficient frontier.

Since (6) is a convex optimization problem, the equality constraint $EX(T) = d$ can be dealt with by introducing a Lagrange multiplier $\mu \in \mathbb{R}$. In this way the portfolio problem (6) can be solved via the following optimal stochastic control problem (for every fixed $\mu$)

$$
\begin{align*}
\min & \quad E\left\{[X(T) - d]^2 + 2\mu[EX(T) - d]\right\}, \\
\text{subject to} & \quad \begin{cases} 
  u(\cdot) \in L^2_T(0, T; \mathbb{R}_+^m), \\
  (X(\cdot), u(\cdot)) \text{ satisfy equation (5)}, 
\end{cases}
\end{align*}
$$

(7)

where the factor 2 in front of the multiplier $\mu$ is introduced in the objective function just for convenience. Clearly, this problem is equivalent to the following

$$(A(\mu)) : \quad \min \quad E\left\{\frac{1}{2}[X(T) - (d - \mu)]^2\right\},$$

subject to \begin{align*}
\begin{cases} 
  u(\cdot) \in L^2_T(0, T; \mathbb{R}_+^m), \\
  (X(\cdot), u(\cdot)) \text{ satisfy equation (5)}, 
\end{cases}
\end{align*}

in the sense that the two problems have exactly the same optimal control.

3 A General Constrained Stochastic LQ Problem

The problem $A(\mu)$ formulated in the previous section is a stochastic optimal LQ control problem. This problem has two features which distinguish it from conventional LQ problems. One is that the running cost of this problem is identically zero; that is, it is an indefinite stochastic LQ control problem, the theory of which has been developed
extensively in recent years (see, for example, [2, 3, 20, 32, 33]). The other feature, which also gives rise to the main difficulty of the problem, is that the control is constrained. Therefore, the conventional ‘completion of squares’ approach to the unconstrained LQ problem, which involves the Riccati equation, will no longer apply. In this section, we solve a class of constrained, indefinite stochastic LQ problems of which $A(\mu)$ is a special case.

Consider the controlled linear stochastic differential equation
\begin{equation}
\begin{aligned}
&dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]ds + \sum_{j=1}^{m} D_j(t)u(t)dW^j(t), \quad t \in [s, T], \\
x(s) = y \in \mathbb{R},
\end{aligned}
\end{equation}

where $A(t)$ and $f(t) \in \mathbb{R}$ are scalars, $B(t)' \in \mathbb{R}^m_+$ and $D_j(t)' \in \mathbb{R}^m_+$ ($j = 1, \ldots, m$) are column vectors. In addition, we assume that the matrix $\sum_{j=1}^{m} D_j(t)'D_j(t)$ is nonsingular.

The class of admissible controls associated with (8) is the set $\mathcal{U}[s, T] = L^2_{\mathcal{F}}(s, T; \mathbb{R}^m_+)$. Given $u(\cdot) \in \mathcal{U}[s, T]$, the pair $(x(\cdot), u(\cdot))$ is referred to as an admissible pair if $x(\cdot) \in L^2_{\mathcal{F}}(s, T; \mathbb{R})$ is a solution of the stochastic differential equation (8) associated with $u(\cdot) \in \mathcal{U}[s, T]$. Our objective is to find an optimal $u(\cdot)$ that minimizes the quadratic (terminal) cost function
\begin{equation}
J(s, y; u(\cdot)) = E \left\{ \frac{1}{2} x(T)^2 \right\}.
\end{equation}

The value function associated with the LQ problem (8)–(9) is defined by
\begin{equation}
V(s, y) = \inf_{u(\cdot) \in \mathcal{U}[s, T]} J(s, y; u(\cdot)).
\end{equation}

3.1 Hamilton-Jacobi-Bellman Equation

Since the Riccati equation approach is not applicable in this case, we study the corresponding Hamilton-Jacobi-Bellman (HJB) equation instead, which is the following partial differential equation:
\begin{equation}
\begin{aligned}
&v_t(t, x) + \inf_{u \geq 0} \left\{ v_x(t, x)(A(t)x + B(t)u + f(t)) + \frac{1}{2} v_{xx}(t, x)u'D(t)'D(t)u \right\} = 0, \\
v(T, x) = \frac{1}{2} x^2,
\end{aligned}
\end{equation}

where $D(t)' = (D_1(t)', \ldots, D_m(t)')$. Unfortunately, owing essentially to the non-negativity constraint of the control, the HJB equation does not have a smooth solution, as opposed to the unconstrained case where the solution to the HJB equation is a quadratic function which can be constructed via the Riccati equation. The idea here
is to construct a function, show that it is a \textit{viscosity} solution (see Appendix for the definition) to the HJB equation, and then employ the verification theorem to construct the optimal control. To start, let

\[
\tilde{z}(t) := \arg \min_{z(t) \in [0, \infty)^n} \frac{1}{2} \|(D(t)')^{-1}z(t) + (D(t)')^{-1}B(t)\|_2^2, \tag{12}
\]

and

\[
\tilde{\xi}(t) := (D(t)')^{-1}\tilde{z}(t) + (D(t)')^{-1}B(t)' \tag{13}
\]

(see Lemmas 6.1 and 6.2 in Appendix). Note in particular that \(\tilde{\xi}(t)\) is a column vector \textit{independent} of \(x\). Let \(\bar{P}(t), \bar{g}(t)\) and \(\bar{c}(t)\) respectively denote the solutions of the following differential equations (the first being a special Riccati equation)

\[
\begin{align*}
\dot{\bar{P}}(t) &= [-2A(t) + \|\tilde{\xi}(t)\|^2]\bar{P}(t), \\
\bar{P}(T) &= 1, \\
\bar{P}(t) &> 0, \quad \forall t \in [0, T], \\
\dot{\bar{g}}(t) &= [-A(t) + \|\tilde{\xi}(t)\|^2]\bar{g}(t) - f(t)\bar{P}(t), \\
\bar{g}(T) &= 0, \\
\dot{\bar{c}}(t) &= -f(t)\bar{g}(t) + \frac{1}{2}\|\tilde{\xi}(t)\|^2\bar{P}(t)^{-1}\bar{g}(t)^2, \\
\bar{c}(T) &= 0, 
\end{align*} \tag{14}
\]

and \(\tilde{\bar{P}}(t), \tilde{\bar{g}}(t)\) and \(\tilde{\bar{c}}(t)\) respectively denote the solutions of the following differential equations (the first being \textit{another} special Riccati equation)

\[
\begin{align*}
\dot{\tilde{\bar{P}}}(t) &= -2A(t)\tilde{\bar{P}}(t), \\
\tilde{\bar{P}}(T) &= 1, \\
\tilde{\bar{P}}(t) &> 0, \quad \forall t \in [0, T], \\
\dot{\tilde{\bar{g}}}(t) &= -A(t)\tilde{\bar{g}}(t) - f(t)\tilde{\bar{P}}(t), \\
\tilde{\bar{g}}(T) &= 0, \\
\dot{\tilde{\bar{c}}}(t) &= -f(t)\tilde{\bar{g}}(t), \\
\tilde{\bar{c}}(T) &= 0. 
\end{align*} \tag{15}
\]

In the next subsection, we shall show that

\[
V(t, x) = \begin{cases} \\
\frac{1}{2}\bar{P}(t)x^2 + \bar{g}(t)x + \bar{c}(t), & \text{if } x + e^{-\int_t^T A(s)ds} \int_t^T f(z)e^{\int_z^T A(s)ds}dz \leq 0, \\
\frac{1}{2}\tilde{\bar{P}}(t)x^2 + \tilde{\bar{g}}(t)x + \tilde{\bar{c}}(t), & \text{if } x + e^{-\int_t^T A(s)ds} \int_t^T f(z)e^{\int_z^T A(s)ds}dz > 0, 
\end{cases} \tag{20}
\]
is a viscosity solution of the HJB equation (11), and

\[
 u^*(t, x) = \begin{cases} 
 -D(t)^{-1} \tilde{\xi}(t) \left( x + e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz \right), \\
 0, \\
 -D(t)^{-1} \tilde{\xi}(t) \left( x + e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz \right),
\end{cases}
\]

if \( x + e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz \leq 0, \)

is the associated optimal feedback control.

### 3.2 Value Function and Optimal Control

This subsection is devoted to verifying the aforementioned results. First we show that \( V \) constructed in (20) is a viscosity solution to the HJB equation (11).

We start with equation (14). Clearly

\[
 \overline{P}(t) = e^{\int_t^T (2A(s) - \|\tilde{\xi}(s)\|^2) ds}
\]

is the solution of (14). Note in particular that the constraint \( \overline{P}(t) > 0 \) is automatically satisfied. Defining \( \overline{\eta}(t) := \frac{\overline{g}(t)}{\overline{P}(t)} \), it follows from (14) and (15) that

\[
 \dot{\overline{\eta}}(t) = \frac{\overline{P}(t) \dot{\overline{g}}(t) - \dot{\overline{P}}(t) \overline{g}(t)}{\overline{P}(t)^2} = \frac{A(t) \overline{P}(t) \overline{g}(t) - f(t) \overline{P}(t)^2}{\overline{P}(t)^2} = A(t) \overline{\eta}(t) - f(t).
\]

Solving this equation with \( \overline{\eta}(T) = 0 \) yields

\[
 \overline{\eta}(t) = e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz.
\]

Hence,

\[
 \overline{g}(t) = \overline{P}(t) \overline{\eta}(t) = e^{\int_t^T (A(s) - \|\tilde{\xi}(s)\|^2) ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz.
\]

Substituting these expressions into (16), we obtain

\[
 \dot{\tilde{v}}(t) = -f(t) \overline{g}(t) + \frac{1}{2} \|\tilde{\xi}(t)\|^2 \overline{P}(t)^{-1} \overline{g}(t)^2,
\]

\[
 = \left[ -f(t) + \frac{1}{2} \|\tilde{\xi}(t)\|^2 e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz \right] \\
 \cdot e^{\int_t^T (A(s) - \|\tilde{\xi}(s)\|^2) ds} \int_t^T f(z) e^{\int_z^T A(s)ds} dz.
\]
Therefore,
\[
\bar{v}(t) = \int_t^T \left[ f(v) - \frac{1}{2} \|\xi(v)\|^2 e^{-\int_v^T A(s)ds} \int_v^T f(z) e^{\int_s^T A(s)ds} dz \right] e^{\int_t^T (A(s) - \|\xi(s)\|^2)ds} \int_v^T e^{\int_s^T A(s)ds} dz dv.
\]

Now we define the region \(\Gamma_1\) in the \((t, x)\)-plane as
\[
\Gamma_1 := \{ (t, x) \in [0, T] \times \mathbb{R} \mid x + e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_s^T A(s)ds} dz < 0 \}.
\]

In \(\Gamma_1\), \(V\) as given by (20) is sufficiently smooth for the terms in (11) to be well defined, with
\[
V_t(t, x) = \frac{1}{2} \dot{\bar{P}}(t)x^2 + \dot{\bar{g}}(t)x + \dot{\bar{c}}(t), \quad V_x(t, x) = \bar{P}(t)x + \bar{g}(t), \quad V_{xx}(t, x) = \bar{P}(t).
\]
Substituting them into the left-hand side (LHS) of (11), we obtain
\[
\text{LHS} = V_t(t, x) + V_x(t, x)[A(t)x + f(t)] + \inf_{u \geq 0} \left\{ \frac{1}{2} V_{xx}(t, x) u'D(t)'D(t) u + V_x(t, x) B(t) u \right\}
\]
\[
= \left[ \frac{1}{2} \dot{\bar{P}}(t)x^2 + \dot{\bar{g}}(t)x + \dot{\bar{c}}(t) \right] + [\bar{P}(t)x + \bar{g}(t)][A(t)x + f(t)]
\]
\[
+ \inf_{u \geq 0} \left\{ \frac{1}{2} \bar{P}(t) u'D(t)'D(t) u + [\bar{P}(t)x + \bar{g}(t)] B(t) u \right\}
\]
\[
= \left[ \frac{1}{2} \dot{\bar{P}}(t) + A(t) \bar{P}(t) \right] x^2 + \left[ \dot{\bar{g}}(t) + A(t) \bar{g}(t) + f(t) \bar{P}(t) \right] x + \left[ \dot{\bar{c}}(t) + f(t) \bar{g}(t) \right]
\]
\[
+ \bar{P}(t) \inf_{u \geq 0} \left\{ \frac{1}{2} u'D(t)'D(t) u + [x + \bar{g}(t)] B(t) u \right\}.
\]  
(24)

By using Lemma 6.2 of Appendix with \(\alpha = -[x + \bar{g}(t)] > 0\), it follows that the minimizer of (24) is achieved by
\[
u^*(t, x) = -D(t)^{-1} \xi(t)[x + \bar{g}(t)]
\]
\[
= -D(t)^{-1} \xi(t) \left[ x + e^{-\int_t^T A(s)ds} \int_t^T f(z) e^{\int_s^T A(s)ds} dz \right].
\]  
(25)

Substituting \(u^*(t, x)\) back into (24) and noting (14), (15) and (16), it immediately follows that \(\text{LHS} = 0\). This implies that \(V\) satisfies the HJB equation (11) in \(\Gamma_1\).

**Remark 3.1** Although the minimizer (25) of (24) involves the parameter \(\xi(t)\), as defined by (12) and (13), it is important to recognize that \(\xi(t)\) does not depend on \(x\). In particular, this means that \(\bar{P}(t)\), \(\bar{g}(t)\) and \(\bar{c}(t)\), which also depend on \(\xi(t)\), do not depend on \(x\). Hence, the expressions for \(V_t(t, x)\), \(V_x(t, x)\) and \(V_{xx}(t, x)\) do not involve terms of the form \(\bar{P}_x(t), \bar{g}_x(t)\) and \(\bar{c}_x(t)\), etc. It is precisely for this reason that closed form expressions for the value function can still be obtained.
Next we proceed to the region $\Gamma_2$ defined by
\[
\Gamma_2 := \{(t, x) \in [0, T] \times \mathbb{R} \mid x + e^{-\int_t^T A(s) ds} \int_t^T f(z) e^{\int_z^T A(s) ds} dz > 0\}.
\]
Similar to the derivations for the previous case, we obtain
\[
\begin{aligned}
\tilde{P}(t) &= e^{2 \int_t^T A(s) ds}, \\
\tilde{g}(t) &= e^{\int_t^T A(s) ds} \int_t^T f(z) e^{\int_z^T A(s) ds} dz, \\
\tilde{c}(t) &= \int_t^T f(v) e^{\int_v^T A(s) ds} \int_v^T f(z) e^{\int_z^T A(s) ds} dz dv, \\
\tilde{\eta}(t) &= \frac{\tilde{g}(t)}{\tilde{P}(t)} = e^{-\int_t^T A(s) ds} \int_t^T f(z) e^{\int_z^T A(s) ds} dz.
\end{aligned}
\]
In $\Gamma_2$, $V$ is once again sufficiently smooth for the derivatives in (11) to be well defined, and
\[
V_t(t, x) = \frac{1}{2} \tilde{P}(t)x^2 + \tilde{g}(t)x + \tilde{c}(t), \quad V_x(t, x) = \tilde{P}(t)x + \tilde{g}(t), \quad V_{xx}(t, x) = \tilde{P}(t).
\]
Substituting into the left-hand side (LHS) of (11), we obtain
\[
\begin{align*}
\text{LHS} &= V_t(t, x) + V_x(t, x)[A(t)x + f(t)] + \inf_{u \geq 0} \left\{ \frac{1}{2} V_{xx}(t, x) u'D(t)D(t)u + V_x(t, x)B(t)u \right\} \\
&= \left[ \frac{1}{2} \frac{\dot{P}(t)}{P(t)} x^2 + \frac{g(t)}{P(t)} x + \frac{c(t)}{P(t)} \right] + \frac{\tilde{P}(t)x + \tilde{g}(t)}{\tilde{P}(t)} [A(t)x + f(t)] \\
&\quad + \inf_{u \geq 0} \left\{ \frac{1}{2} \tilde{P}(t)u'D(t)D(t)u + [\tilde{P}(t)x + \tilde{g}(t)]B(t)u \right\} \\
&= \left[ \frac{1}{2} \frac{\dot{P}(t)}{P(t)} x^2 + \frac{\dot{g}(t)}{P(t)} x + A(t)\tilde{P}(t) \right] x^2 + \left[ \frac{\dot{g}(t)}{P(t)} + A(t)\tilde{g}(t) + f(t)\tilde{P}(t) \right] x + \left[ \frac{\dot{c}(t)}{P(t)} + f(t)\tilde{g}(t) \right] \\
&\quad + \tilde{P}(t)\inf_{u \geq 0} \left\{ \frac{1}{2} u'D(t)D(t)u + [x + \tilde{\eta}(t)]B(t)u \right\}.
\end{align*}
\]
(26)

Since $x + \tilde{\eta}(t) > 0$, the minimizer of (26) is
\[
u^*(t, x) = 0.
\]
(27)

Substituting $u^*(t, x)$ into (26), it is easy to show that $V$ satisfies the HJB equation (11) in $\Gamma_2$.

Finally, the switching curve $\Gamma_3$ defined by
\[
\Gamma_3 := \{(t, x) \in [0, T] \times \mathbb{R} \mid x + e^{-\int_t^T A(s) ds} \int_t^T f(z) e^{\int_z^T A(s) ds} dz = 0\}
\]
is where the non-smoothness of \( V \) occurs. Firstly, a direct calculation shows that
\[
V(t, x) = \frac{1}{2} \bar{P}(t)x^2 + \bar{g}(t)x + \bar{c}(t) = \frac{1}{2} \bar{P}(t)x^2 + \check{g}(t)x + \check{c}(t) = 0 \quad \text{on } \Gamma_3.
\]
Therefore, \( V(t, x) \) is continuous at \( (t, x) \in \Gamma_3 \). In addition, we also easily obtain
\[
\begin{align*}
V_t(t, x) &= \frac{1}{2} \bar{P}(t)x^2 + \check{g}(t)x + \check{c}(t) = \frac{1}{2} \check{P}(t)x^2 + \check{g}(t)x + \check{c}(t) = 0, \\
V_x(t, x) &= \bar{P}(t)x + \check{g}(t) = \check{P}(t)x + \check{g}(t) = 0.
\end{align*}
\]
That is, \( V(t, x) \) is also continuously differentiable at points on \( \Gamma_3 \). However, \( V_{xx} \) does not exist on \( \Gamma_3 \), since \( \bar{P}(t) \neq \check{P}(t) \). This means that \( V \) does not possess the necessary smoothness properties to qualify as a classical solution of the HJB equation (11). For this reason, we are required to work within the framework of viscosity solutions. From Definition 6.1 in Appendix, it can be shown that for any \( (t, x) \in \Gamma_3 \),
\[
\begin{align*}
D_{t, x}^{1/2, +} V(t, x) &= \{0\} \times \{0\} \times [\check{P}(t), +\infty), \\
D_{t, x}^{1/2, -} V(t, x) &= \{0\} \times \{0\} \times (-\infty, \bar{P}(t)].
\end{align*}
\]
For the HJB equation (11), we define \( G(t, x, u, P) = p[A(t)x + B(t)u + f(t)] + \frac{1}{2} Pu' D(t)' D(t)u. \) For any \( (q, p, P) \in D_{t, x}^{1/2, +} V(t, x) \), where \( (t, x) \in \Gamma_3 \), we have
\[
q + \inf_{u \geq 0} G(t, x, u, P) = \inf_{u \geq 0} \left\{ \frac{1}{2} Pu' D(t)' D(t)u \right\} \geq \inf_{u \geq 0} \left\{ \frac{1}{2} \check{P}(t)u' D(t)' D(t)u \right\} = 0.
\]
Therefore, \( V \) is a viscosity sub-solution of the HJB equation (11). On the other hand, for \( (q, p, P) \in D_{t, x}^{1/2, -} V(t, x) \) where \( (t, x) \in \Gamma_3 \), we have
\[
q + \inf_{u \geq 0} G(t, x, u, P) = \inf_{u \geq 0} \left\{ \frac{1}{2} Pu' D(t)' D(t)u \right\} \leq \inf_{u \geq 0} \left\{ \frac{1}{2} \bar{P}(t)u' D(t)' D(t)u \right\} = 0.
\]
Therefore, \( V \) is also a viscosity super-solution of the HJB equation (11). Finally, it is easy to see that the terminal condition \( V(T, x) = \frac{1}{2} x^2 \) is satisfied. Hence, it follows from Definition 6.1 that \( V(t, x) \) is a viscosity solution of the HJB equation (11). Moreover, for any \( (t, x) \in \Gamma_3 \), take \( (q^*(t, x), p^*(t, x), P^*(t, x), u^*(t, x)) := (0, 0, \check{P}(t), 0) \in D_{t, x}^{1/2, +} V(t, x) \times U[s, T] \), then
\[
q^*(t, x) + G(t, x, u^*(t, x), p^*(t, x), P^*(t, x)) = 0.
\]
It then follows from the verification theorem in [36, Theorem 3.1] that \( u^*(t, x) \) defined by (21) is the optimal feedback control.

4 Efficient Strategies and Efficient Frontier

In this section we apply the general results established in the previous section to the problem \( A(\mu) \) formulated in Section 2. Set
\[
x(t) = X(t) - (d - \mu).
\]
Problem $A(\mu)$ is equivalent to the following problem

$$\min E\left\{ \frac{1}{2} x(T)^2 \right\},$$  \hspace{1cm} (29)

subject to

$$\begin{cases}
    dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]dt + \sum_{j=1}^{d} D_j(t)u(t)dW^j(t), \\
x(0) = X_0 - (d - \mu),
\end{cases}$$  \hspace{1cm} (30)

where $u(\cdot) \in L^2_T(0, T; \mathbb{R}_+^m)$ and

$$\begin{cases}
    A(t) = r(t), \\
    B(t) = (b_1(t) - r(t), \ldots, b_m(t) - r(t)), \\
    f(t) = (d - \mu)r(t), \\
    D_j(t) = (\sigma_{1j}(t), \ldots, \sigma_{mj}(t)).
\end{cases}$$  \hspace{1cm} (31)

Now, corresponding to (12) and (13), set

$$\bar{\pi}(t) := \arg \min_{\pi(t) \in [0, \infty)^m} \frac{1}{2} \| \sigma(t)^{-1}\bar{\pi}(t) + \sigma(t)^{-1}(b(t) - r(t)1) \|^2,$$  \hspace{1cm} (32)

and

$$\bar{\theta}(t) := \sigma(t)^{-1}\bar{\pi}(t) + \sigma(t)^{-1}(b(t) - r(t)1).$$  \hspace{1cm} (33)

### 4.1 An Optimal Strategy

Before analyzing the efficient frontier of the original portfolio selection problem (6), we first present the optimal investment strategy for the problem $A(\mu)$. The optimal control obtained in (21) translates into the following strategy:

$$u^*(t, X) \equiv (u_1^*(t, X), \ldots, u_m^*(t, X))'$$

$$= \begin{cases}
    -(\sigma(t)')^{-1}\bar{\theta}(t)\left[ x + (d - \mu)(1 - e^{-\int_t^T r(s)ds}) \right], & \text{if } x + (d - \mu)(1 - e^{-\int_t^T r(s)ds}) \leq 0, \\
    0, & \text{if } x + (d - \mu)(1 - e^{-\int_t^T r(s)ds}) > 0,
\end{cases}$$

$$= \begin{cases}
    -(\sigma(t)\sigma(t)')^{-1}[\bar{\pi}(t) + (b(t) - r(t)1)]\left[ X - (d - \mu)e^{-\int_t^T r(s)ds} \right], & \text{if } X - (d - \mu)e^{-\int_t^T r(s)ds} \leq 0, \\
    0, & \text{if } X - (d - \mu)e^{-\int_t^T r(s)ds} > 0.
\end{cases}$$  \hspace{1cm} (34)

**Theorem 4.1** An optimal investment strategy to the problem $A(\mu)$ is given by (34).
4.2 Efficient Frontier

In this subsection, we derive the efficient frontier for the portfolio selection problem (6), i.e., we specify the relation between the variance and the expected value of the terminal wealth for every efficient strategy. First of all, note

\[
E\left\{\frac{1}{2}x(T)^2\right\} = E\left\{\frac{1}{2}[X(T) - (d - \mu)]^2\right\}
= E\left\{\frac{1}{2}[X(T) - d]^2\right\} + \mu[EX(T) - d] + \frac{1}{2}\mu^2.
\]

Hence, for every fixed \(\mu\), we have

\[
\min_{u(\cdot) \in \mathcal{U}[0,T]} E\left\{\frac{1}{2}[X(T) - d]^2 + \mu[EX(T) - d]\right\}
= \min_{u(\cdot) \in \mathcal{U}[0,T]} E\left\{\frac{1}{2}x(T)^2\right\} - \frac{1}{2}\mu^2
= V(0, x) - \frac{1}{2}\mu^2
= \frac{1}{2}P(0)x^2 + g(0)x + c(0) - \frac{1}{2}\mu^2
= \frac{1}{2}P(0)[X_0 - (d - \mu)]^2 + g(0)[X_0 - (d - \mu)] + c(0) - \frac{1}{2}\mu^2,
\]

where \(P(\cdot), g(\cdot)\) and \(c(\cdot)\) are specified in (20). Now, if \(X_0 - (d - \mu)e^{-\int_0^T r(s)ds} \leq 0\), we have a concave quadratic function in \(\mu\)

\[
\min_{u(\cdot) \in \mathcal{U}[0,T]} E\left\{\frac{1}{2}[X(T) - d]^2 + \mu[EX(T) - d]\right\}
= \frac{1}{2}P(0)[X_0 - (d - \mu)]^2 + \tilde{g}(0)[X_0 - (d - \mu)] + \tilde{c}(0) - \frac{1}{2}\mu^2
= \frac{1}{2}e^{-\int_0^T |\tilde{\theta}(s)|^2ds} \left[X_0e^{\int_0^T r(s)ds} - (d - \mu)\right]^2 - \frac{1}{2}\mu^2,
\]

If \(X_0 - (d - \mu)e^{-\int_0^T r(s)ds} > 0\), we have a linear function in \(\mu\)

\[
\min_{u(\cdot) \in \mathcal{U}[0,T]} E\left\{\frac{1}{2}[X(T) - d]^2 + \mu[EX(T) - d]\right\}
= \frac{1}{2}\tilde{P}(0)[X_0 - (d - \mu)]^2 + \tilde{g}(0)[X_0 - (d - \mu)] + \tilde{c}(0) - \frac{1}{2}\mu^2
= \frac{1}{2}\left[X_0e^{\int_0^T r(s)ds} - (d - \mu)\right]^2 - \frac{1}{2}\mu^2
= \frac{1}{2}\left(X_0e^{\int_0^T r(s)ds} - d\right)^2 + \left(X_0e^{\int_0^T r(s)ds} - d\right)\mu.
\]

Therefore we conclude that under the optimal investment strategy (34) the optimal cost for the problem (7) is

\[
\min_{u(\cdot) \in \mathcal{U}[0,T]} \left\{[X(T) - d]^2 + 2\mu[EX(T) - d]\right\}
= \begin{cases} 
  e^{-\int_0^T |\tilde{\theta}(s)|^2ds} \left[X_0e^{\int_0^T r(s)ds} - (d - \mu)\right]^2 - \mu^2, & \text{if } X_0 - (d - \mu)e^{-\int_0^T r(s)ds} \leq 0, \quad (35) \\
  \left[X_0e^{\int_0^T r(s)ds} - (d - \mu)\right]^2 - \mu^2, & \text{if } X_0 - (d - \mu)e^{-\int_0^T r(s)ds} > 0.
\end{cases}
\]

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Note that the above value still depends on the Lagrange multiplier \( \mu \). To obtain the optimal value (i.e. the minimum variance \( \text{Var} X(T) \)) and optimal strategy for the original portfolio selection problem (6) one needs to maximize the value in (35) over \( \mu \in \mathbb{R} \) according to the Lagrange duality theorem [22]. A simple calculation shows that (35) attains its maximum value \( \frac{(d - X_0 e^{\int_0^T r(s)ds})^2}{e^{\int_0^T \|\theta(s)\|^2 ds} - 1} \) at \( \mu^* = \frac{d - X_0 e^{\int_0^T r(s)ds}}{1 - e^{\int_0^T \|\theta(s)\|^2 ds}} \). (Note that in the calculation we made use of that fact that
\[
X_0 - (d - \mu^*) e^{-\int_0^T r(s)ds} = \frac{d e^{-\int_0^T r(s)ds} - X_0}{e^{-\int_0^T \|\theta(s)\|^2 ds} - 1} \leq 0
\]
due to Assumption 2.1.)

The above discussion leads to the following theorem.

**Theorem 4.2** The efficient strategy of the portfolio selection problem (6) corresponding to the expected terminal wealth \( E X(T) = d \), as a function of time \( t \) and wealth \( X \), is

\[
u^*(t, X) = (\nu_1^*(t, X), \ldots, \nu_m^*(t, X))',
\]

\[
= \begin{cases} 
- (\sigma(t)\sigma(t)')^{-1}[\pi(t) + (b(t) - r(t)1)] \left[ X - (d - \mu^*) e^{-\int_t^T r(s)ds} \right], & \text{if } X - (d - \mu^*) e^{-\int_t^T r(s)ds} \leq 0, \\
0, & \text{if } X - (d - \mu^*) e^{-\int_t^T r(s)ds} > 0,
\end{cases}
\]

(36)

where \( \mu^* = \frac{d - X_0 e^{\int_0^T r(s)ds}}{1 - e^{\int_0^T \|\theta(s)\|^2 ds}} \). Moreover, the efficient frontier is

\[
\text{Var} X(T) = \frac{(d - X_0 e^{\int_0^T r(s)ds})^2}{e^{\int_0^T \|\theta(s)\|^2 ds} - 1} = \frac{(EX(T) - X_0 e^{\int_0^T r(s)ds})^2}{e^{\int_0^T \|\theta(s)\|^2 ds} - 1}. \quad (37)
\]

**Remark 4.1** The form of the efficient strategy (36) suggests that it should put all the money in the bond if the current wealth is large enough. It also follows from (36) that the mutual fund theorem does not hold under the short-selling prohibition, because the fraction of wealth in stocks in an efficient portfolio depends on the total wealth of the investor. However, the failure of mutual fund theorem is natural as the underlying market is essentially incomplete.
Remark 4.2 The efficient frontier in the mean–standard-deviation diagram is still a straight line. To be specific, let \( \sigma_{X(T)} \) be the standard deviation of the terminal wealth, then (37) gives
\[
EX(T) = X_0 e^{\int_0^T r(s)ds} + \sqrt{e^{\int_0^T \|\tilde{\theta}(s)\|^2 ds} - 1} \sigma_{X(T)},
\]
which is also called the capital market line.

5 An Example

In this section, a numerical example is presented to demonstrate the results in the previous section. Let \( m = 3 \). The interest rate of the bond and the appreciation rate of the \( m \) stocks are \( r = -\frac{2}{100} \) and \( (b_1, b_2, b_3)' = \left(\frac{4}{100}, \frac{5}{100}, \frac{6}{100}\right)' \), respectively, and the volatility matrix is \( \sigma = \begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix} \). Then we have \( \sigma^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} \) and \( (b_1 - r, b_2 - r, b_3 - r)' = \left(\frac{2}{100}, \frac{3}{100}, \frac{4}{100}\right)' \). Hence, \( \theta := \sigma^{-1}(b - r1) = \left(\frac{-2}{100}, \frac{3}{100}, \frac{6}{100}\right)' \). Obviously, \( s(\pi) \triangleq \frac{1}{2}\|\sigma^{-1} \pi + \theta\|^2 \) over \([0, \infty)^m \) has a unique minimizer \( \bar{\pi} = \left(\frac{2}{100}, 0, 0\right)' \) with the minimum value \( s(\bar{\pi}) = \frac{1}{2}\|\sigma^{-1} \bar{\pi} + \theta\|^2 = \frac{9}{4000} \). Then we have
\[
\|\bar{\theta}\|^2 = \|\sigma^{-1} \bar{\pi} + \theta\|^2 = \frac{9}{2000}
\]
and
\[
(\sigma \sigma')^{-1}[\bar{\pi} + (b - r1)] = \left(0, \frac{3}{100}, \frac{9}{100}\right)'.
\]
Therefore, Theorem 4.2 implies that an efficient strategy is
\[
u^*(t, X) \equiv (u^*_1(t, X), u^*_2(t, X), u^*_3(t, X))' = \begin{cases} \\
\begin{bmatrix} 0 \\ \frac{3}{100} \\ \frac{9}{100} \end{bmatrix} \left[(d - \mu^*)e^{\frac{2}{100}(t-T)} - X\right], & \text{if } X - (d - \mu^*)e^{\frac{2}{100}(t-T)} \leq 0, \\
0, & \text{if } X - (d - \mu^*)e^{\frac{2}{100}(t-T)} > 0,
\end{cases}
\]
where \( \mu^* = \frac{d - X_0 e^{\frac{2}{100}T}}{1 - e^{\frac{2}{100}T}} = \frac{d - X_0 e^{\frac{2}{100}T}}{1 - e^{\frac{2}{100}T}} \). The efficient frontier is
\[
\Var X(T) = \frac{(d - X_0 e^{\frac{2}{100}T})^2}{e^{\frac{2}{100}T} - 1} = \frac{(EX(T) - X_0 e^{\frac{2}{100}T})^2}{e^{\frac{2}{100}T} - 1}.
\]
6 Conclusion

This paper investigates a continuous-time mean-variance portfolio selection problem where short-selling is not allowable. The efficient strategies and efficient frontier are derived explicitly based on stochastic LQ control technique and viscosity solution theory. This also demonstrates that stochastic LQ control is a powerful framework to treat some finance problems.

An immediate open problem is to extend the results in this paper to the case where all the market coefficients are random processes. This is a challenging problem because the HJB equation becomes a backward stochastic partial differential equation due to the randomness of coefficients whose viscosity solution theory is still largely unexplored.

Appendix

We list here some basic results from the theory of viscosity solutions and convex analysis which are referred to in the paper.

Viscosity solution

Let

\[
G(t, x, u, p, P) = \frac{1}{2} \sigma(t, x, u)'P\sigma(t, x, u) + p'h(t, x, u) - L(t, x, u),
\]

where \( \sigma : [0, T) \times \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R}^n \), \( h : [0, T) \times \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R} \) and \( L : [0, T) \times \mathbb{R}^n \times \mathbb{R}^{n_u} \rightarrow \mathbb{R} \). Consider the second-order partial differential equation (PDE)

\[
\begin{cases}
  v_t + \inf_{u \geq 0} G(t, x, u, v_x, v_{xx}) = 0, & (t, x) \in [0, T) \times \mathbb{R}^n, \\
  v(T, x) = g(x), &
\end{cases}
\]  

where \( g : \mathbb{R}^n \rightarrow \mathbb{R} \).

Clearly the HJB equation (11) is a special case of (39). It is well-known that (39) does not in general have classical (smooth) solutions. A generalized concept of solution, called a viscosity solution, is introduced in [5]. The main result in [33] is that under certain mild conditions, there exists a unique viscosity solution in the first order case. In the second-order case, uniqueness is proven in [15]. See also [9, 33] for more details about viscosity solution and application in stochastic control.
**Definition 6.1** Let $v \in C([0,T] \times \mathbb{R}^n)$ and $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Then the second-order super-differential of $v$ at $(t_0, x_0)$ is defined by

$$D^{1,2,+}_{t,x}v(t_0, x_0) = \left\{ (\varphi_t(t_0, x_0), \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \mid \varphi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \varphi \text{ has a local maximum at } (t_0, x_0) \right\},$$

and the second order sub-differential of $v$ is defined by

$$D^{1,2,-}_{t,x}v(t_0, x_0) = \left\{ (\varphi_t(t_0, x_0), \varphi_x(t_0, x_0), \varphi_{xx}(t_0, x_0)) \mid \varphi \in C^\infty((0, T) \times \mathbb{R}^n) \text{ and } v - \varphi \text{ has a local minimum at } (t_0, x_0) \right\}.$$ 

Moreover, $v$ is a viscosity solution of (39) if

$$v(T, x) = g(x), \quad \forall x \in \mathbb{R}^n,$$ 

and

$$q + \inf_{u \in U} G(t, x, u, p, P) \geq 0, \quad \forall (q, p, P) \in D^{1,2,+}_{t,x}v(t, x),$$

$$q + \inf_{u \in U} G(t, x, u, p, P) \leq 0, \quad \forall (q, p, P) \in D^{1,2,-}_{t,x}v(t, x),$$

for all $(t, x) \in [0, T) \times \mathbb{R}^n$.

In particular, $v$ is called a viscosity sub-solution if it satisfies (42)-(43), and a viscosity super-solution if it satisfies (42) and (44).

**Convex analysis**

The following results from convex analysis are used in deriving the optimal control (21).

**Lemma 6.1** Let $s$ be a continuous, strictly convex quadratic function

$$s(z) \triangleq \frac{1}{2}\| (\mathcal{D}')^{-1}z + (\mathcal{D}')^{-1}\mathcal{B}' \|^2$$

over $z \in [0, \infty)^m$, where $\mathcal{B}' \in \mathbb{R}^{m\times m}$, $\mathcal{D} \in \mathbb{R}^{m\times m}$ and $\mathcal{D}'\mathcal{D} > 0$. Then $s$ has a unique minimizer $\bar{z} \in [0, \infty)^m$, i.e.,

$$\| (\mathcal{D}')^{-1}\bar{z} + (\mathcal{D}')^{-1}\mathcal{B}' \|^2 \leq \| (\mathcal{D}')^{-1}z + (\mathcal{D}')^{-1}\mathcal{B}' \|^2, \quad \forall z \in [0, \infty)^m.$$ 

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The Kuhn-Tucker conditions for the minimization of $s$ in (45) over $[0, \infty)^m$ lead to the Lagrange multiplier vector $\tilde{\nu} \in [0, \infty)^m$ such that $\tilde{\nu} = \nabla s(\tilde{z}) = (D'D)^{-1}\tilde{z} + (D'D)^{-1}B'$ and $\tilde{\nu}' \tilde{z} = 0$.

**Lemma 6.2** Let $h$ be a continuous, strictly convex quadratic function

$$h(z) \triangleq \frac{1}{2} z' D' D z - \alpha B z$$

over $z \in [0, \infty)^m$, where $B' \in \mathbb{R}_+^m$, $D \in \mathbb{R}^{m \times m}$ and $D'D > 0$.

(i) For every $\alpha \geq 0$, $h$ has the unique minimizer $\alpha D^{-1} \bar{\xi} \in [0, \infty)^m$, where $\bar{\xi} = (D')^{-1} \tilde{z} + (D')^{-1} B'$. Here $\bar{z}$ is the minimizer of $s(z)$ specified in Lemma 6.1. Furthermore, $\bar{z}' D^{-1} \bar{\xi} = 0$ and

$$h(\alpha \tilde{\nu}) = h(\alpha D^{-1} \bar{\xi}) = -\frac{1}{2} \alpha^2 \| \bar{\xi} \|^2.$$  

(ii) For every $\alpha < 0$, $h$ has the unique minimizer $0$.

Lemma 6.1 and Lemma 6.2-(i) are proved in Section 5.2 and Lemma 3.2 of [31], while Lemma 6.2-(ii) is obvious.

**Remark 6.1** It should be noted that the vector $\bar{\xi}$ is independent of the parameter $\alpha$.

**References**


