Conic Relaxation of the Unit Commitment Problem

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Abstract—The unit commitment (UC) problem aims to find an optimal schedule of generating units subject to demand and operating constraints for an electricity grid. The majority of existing algorithms for the UC problem rely on solving a series of convex relaxations by means of branch-and-bound and cutting-planning methods. The objective of this paper is to obtain a convex model of polynomial size for practical instances of the UC problem. To this end, we develop a convex conic relaxation of the UC problem, referred to as a strengthened semidefinite program (SDP) relaxation. This approach is based on first deriving certain valid quadratic constraints and then relaxing them to linear matrix inequalities. These valid inequalities are obtained by the multiplication of the linear constraints of the UC problem, such as the flow constraints of two different lines. The performance of the proposed convex relaxation is evaluated on several hard instances of the UC problem. For most of the instances, globally optimal integer solutions are obtained by solving a single convex problem. For the cases where the strengthened SDP does not give rise to a global integer solution, we incorporate other valid inequalities, including a set of boolean quadratic polytope constraints. We prove that the proposed convex relaxation correctly finds the optimal status of each generator that has a positive reliable lower bound. The proposed technique is extensively tested on various IEEE power systems in simulations.

Keywords: Unit commitment, semidefinite programming, valid inequalities, combinatorial optimization, convex optimization

I. INTRODUCTION

The unit commitment (UC) problem is concerned with finding an optimal schedule of generating units in a power system, by minimizing the operational cost of power generators subject to forecasted energy demand and operating constraints. The operating constraints include physical limits and security constraints. In a mixed-integer programming (MIP) formulation of the UC problem, discrete variables model the on/off status of each generator and the continuous variables account for the amount of production for each generator. The UC problem is hard and its large instances are computationally challenging to solve [1].

The UC problem has a vital role in the operation of electricity grids and been studied extensively [2–5]. The existing optimization techniques for UC include Lagrangian relaxation (LR) methods, branch-and-bound (BB) methods, dynamic programming (DP) methods, simulated-annealing (SA) methods, and cutting-plane methods [6]. The LR method provides an approximation for the optimal value of an intractable optimization problem by solving a simpler problem. Ongsakul et al. [7] propose an enhanced adaptive LR method by defining new decision variables. Dubost et al. [8] use the solution of a dual relaxation of the UC problem in a primal proximal-based heuristic method to attain a solution. Primal and dual solution methods for the UC problem in hydro-thermal power systems are studied by Gollmer et al. [9]. Moreover, there are several papers that propose unit decommitment and decomposition procedures for solving the UC problem [10–12]. Turgeon [13] designs an algorithm based on the BB method by recursively splitting the search space into smaller branches. Furthermore, Rajan et al. [14] propose a set of valid inequalities (turn on/off) instead of the simple minimum up- and down-time constraints to be able to solve hard cases of the UC problem by adopting a branch-and-cut technique.

A mixed-integer linear programming (MILP) UC reformulation was first proposed by Garver [15]. Morales-Espana et al. [16] provide new mixed-integer linear reformulations for start-up and shut-down constraints in the UC problem, which lead to tighter relaxations. O’Neill et al. [17] incorporate the transmission switching problem into the N-1 reliable UC problem and use a dual approach to solve the corresponding MILP. This method is extended by O’Neill et al. [18] to inter-regional planning and investment in a competitive environment. Furthermore, Ostrowski et al. [19] and Damc-Kurt et al. [20] propose classes of strong valid inequalities, including upper bounds for the generating powers as well as ramp-down and ramp-up constraints, to provide smaller feasible operating schedules for the generators. Muckstadt et al. [21] design a BB algorithm based on the LR method, which breaks down the UC problem into several simpler UC problems with one generator.

A two-stage stochastic program is introduced by Papavasiliiou et al. [22] that takes into account the high penetration of wind power and system component failures. Ji et al. [23] and Liao [24] use a scenario generation technique and the chaotic quantum genetic algorithm to incorporate uncertainties of wind power. Lorca et al. [25] propose a multi-stage robust optimization-based model that accounts for stochastic non-anticipative load profiles. Other stochastic schemas are introduced by Cerisola et al. [26] and Philpott et al. [27] that model the revenue of a power company in the UC problem posed in electricity markets. These papers consider uncertainties that stem from different possible outcomes of spot markets. Furthermore, Nikzad et al. [28] introduce a stochastic security-constrained model to model the time-of-use program.

Pang et al. [29] propose DP-based methods by decomposing
the problem into a set of smaller subproblems, which are then solved iteratively one at a time. More recently, a DP approach is used by Frangioni et al. [30] to solve a single-unit commitment problem with arbitrary convex cost functions. This work is an extension of the traditional MILP formulation of the UC problem but only considers one generator during the operating time horizon. Since the pure SA method would give an infeasible solution with a high probability, advanced SA-based methods aim to address this issue. For instance, Purushothama et al. [31] improve the rate of the feasible output by providing a heuristic local search in the neighborhood of the best solution for the UC problem. The work by Madrigal et al. [32] proposes an interior-point/cutting-plane method to solve the UC problem, which attempts to amend a proposed set repeatedly to ultimately find the optimal solution by solving the problem over a tighter feasible set.

In this paper, we adopt a semidefinite programming (SDP) relaxation scheme combined with valid inequalities based on the Sherali–Adams Reformulation-Linearization Technique (RLT) [33]. The SDP technique aims to find a strong convex model that returns a global minimum of the UC problem. This mathematical programming method has received significant attention due to numerous applications in many fields, including combinatorial and non-convex optimization, control theory and power systems [34]–[39].

In this paper, we provide a set of valid inequalities to attain a tighter description of the feasible operating schedules for the generators in the UC problem. In order to obtain the above-mentioned inequalities, we use RLT to generate valid non-convex quadratic inequalities and then relax them to valid convex inequalities in a lifted space. For instance, we multiply the flow constraints over two different lines to obtain a valid non-convex constraint and then resort to SDP for convexification. The proposed convex program is called a strengthened SDP, which contrasts with the traditional SDP relaxation without valid inequalities. The above procedure is used for producing valid inequalities and its impact on the feasible set of mixed-integer optimization problems is broadly studied in the literature [35], [40]–[43]. In this work, we will demonstrate that the strengthened SDP problem is able to find globally optimal discrete solutions for many trials of benchmark systems.

Since the strengthened SDP problem is computationally prohibitive for large power systems, its complexity is reduced through relaxing the high-order SDP constraint to lower-order conic constraints. As will be shown in simulations, the above step significantly reduce the complexity of the strengthened SDP problem without affecting its solution in the test systems. We also introduce the notion of reliable lower bound for generators and show that, independent of the objective function, the proposed strengthened convex model is able to recover the correct status of each generators that has a positive reliable lower bound. In the case where the SDP relaxation is not exact, we employ a number of other valid inequalities, including the triangle inequalities and a special case of the variable upper bound (VUB) ramping constraints [20], [41], [44]. Although the total number of the valid inequalities deployed in this paper is polynomial in the size of the problem, we further reduce it by identifying a subset of implied valid inequalities and removing them from the formulation.

The major benefit of our method compared to the existing techniques is that our convex relaxation can readily be used in convex hull pricing for electricity markets [45]. Because of the existence of discrete decision variables in the UC problem, it is often the case that there is no set of prices that supports the optimal solution of the UC problem. This is due to the fact that the prices are often determined by assuming that the decision variables are continuous whereas the actual decision variables do not respect this assumption. Due to this inconsistency, there may be no set of prices that satisfy the market equilibrium with “no arbitrage” property, which can incentivize the generators to change their commitments. The Independent System Operators (ISOs) overcome this issue by proposing the additional uplift payments (in the form of side-payments) to the generators. It has been shown by Gribik et al. [46] that convex hull pricing is one of the most consistent pricing methods with optimal quantities in a lifted space. However, the results can be applied to a nonlinear AC model of power systems by combining the proposed technique for handling discrete variables with the convexification method delineated by Lavaei et al. [37] for tackling the nonlinearity of continuous variables.

**Notations:** The symbol \( \operatorname{rank}\{\cdot\} \) denotes the rank of a matrix and the notation \( (\cdot)^\top \) represents the transpose operator. Vectors and matrices are shown by bold lower case and bold upper case letters, respectively. The notations \( W_{i,j} \) and \( V_{i,j} \) denote the \((i,j)\)th entry of a matrix \( W \). Likewise, the notations \( w^i \) and \( w_i \) show the ith entry of a vector \( w \). The symbols \( \mathbb{R} \) and \( S^n \) represent the sets of real numbers and \( n \times n \) real symmetric matrices, respectively. The relation \( u \geq v \) indicates that the vector \( v \) is less than or equal to the vector \( u \) entry-wise (the same relation is used for matrices). Given two sets of natural numbers \( V_1 \) and \( V_2 \) as well as a matrix \( W \), the notation \( W\{V_1, V_2\} \) denotes the submatrix of \( W \) that is obtained by keeping only those rows of \( W \) corresponding to the set \( V_1 \) and those columns of \( W \) corresponding to the set \( V_2 \). Given a vector \( w \), the notation \( w\{V_1\} \) denotes the subvector of \( w \) that is obtained by keeping only those elements of \( w \) corresponding to \( V_1 \). The notation \( W \geq 0 \) indicates that \( W \) is a symmetric positive-semidefinite matrix.

### II. Problem Formulation

Consider a power grid with \( n_b \) buses (nodes), \( n_g \) generators, and \( n_l \) lines. Assume that \( B = \{1, \ldots, n_b\} \), \( G = \{1, \ldots, n_g\} \) and \( L = \{1, \ldots, n_l\} \) denote the bus set, generator set and line set, respectively. Moreover, suppose that \( T = \{0, 1, \ldots, t_0, t_0 + 1\} \) is the set of time slots over which the UC problem should be solved. Let \( p_{i,t} \) and \( x_{i,t} \) denote the
amount of generation and the status of the generator $i$ at time $t$, respectively, for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$. Assume that the initial $(t = 0)$ and terminal $(t = t_0 + 1)$ statuses of all generators are off, implying that $p_{i;0} = x_{i;0} = p_{i;t_0+1} = x_{i;t_0+1} = 0$ for all $i \in \mathcal{G}$. The set of the decision variables consists of the continuous variables $p_{i;t}$ and the binary variables $x_{i;t}$ for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$. Let $f_{q;it}$ denote the flow of line $q \in \mathcal{L}$ (in an arbitrary direction) at time $t \in \mathcal{T}$. For the sake of notational simplicity, define $x_{t}$ as the vector of all commitment statuses and $p_{t}$ as the vector of all generator outputs at time $t \in \mathcal{T}$:

$$x_{t} \triangleq [x_{1;1}, \ldots, x_{n_{g};1}]^{\top}, \quad p_{t} \triangleq [p_{1;1}, \ldots, p_{n_{g};1}]^{\top}.$$  

The objective function of the UC problem is the sum of the operational costs of all generating units, which consist of the power generation, startup and shutdown costs. The power generation cost is commonly modeled as a quadratic function with respect to the amount of generation:

$$g_{i;it}(p_{i;it}, x_{i;it}) \triangleq a_i \times p_{i;it}^2 + b_i \times p_{i;it} + c_{i;fixed} \times x_{i;it}, \quad (1)$$

where $a_i$, $b_i$, and $c_{i;fixed}$ are constant coefficients for generator $i$. Note that the term $c_{i;fixed} \times x_{i;it}$ accounts for a fixed cost if the generator is on and becomes zero otherwise. The startup and shutdown costs are both assumed to be identical and modeled as

$$h_{i;it}(x_{i;it+1}, x_{i;it}) \triangleq c_{i;start} \times (x_{i;it+1} - x_{i;it})^2, \quad (2)$$

where $c_{i;start}$ is the amount of startup or shutdown cost. Note that since all generators are assumed to be off at the beginning and the end of the horizon (i.e., $t = 0$ and $t = t_0 + 1$), if the startup and shutdown costs have different values, we can precisely model the problem using the expression (2) after setting $c_{i;start}$ equal to the average of those two different costs.

The cost associated with turning on or off a generator induces a coupling between the decision variables at different times. There are some operational restrictions for the UC problem, such as physical limits and security constraints. Physical limits include unit capacity, line capacity, ramping, minimum up-time, and minimum down-time constraints. A unit capacity constraint ensures that the unit operates within certain limits. A line capacity constraint enforces the flow on each transmission line not to exceed its thermal limit. Due to the physical design of a generator, it may be impossible to significantly change the production level within a short time interval. These restrictions are referred to as ramping constraints. In addition, each generator may have minimum up-time and down-time constraints, which prohibit the status of a generator from changing over a short period of time.

In order to formulate the UC problem, we need to define several parameters below. Define the vector of demands at time $t$ as $d_{t}$, where its $j^{th}$ entry is equal to the demand at bus $j \in \mathcal{B}$ at time $t \in \mathcal{T}$ (shown as $d_{t,j}$). Let $f_{\text{max}}$ denote the the maximum flow vector for all transmission lines, where its $q^{th}$ entry is equal to the flow limit for the line $q \in \mathcal{L}$ (shown as $f_{\text{max},q}$). Assume that $p_{i;\text{max}}$ and $p_{i;\text{min}}$ represent the upper and lower bounds on the generation of unit $i \in \mathcal{G}$, respectively. Furthermore, define $s_{i}$ as the maximum amount of generation for the startup and shutdown of generator $i \in \mathcal{G}$. Moreover, $r_{i}$ denotes the maximum difference between the generations at two adjacent operating time slots for generator $i$. Furthermore, suppose that $U_{i}$ and $D_{i}$ denote the minimum up-time and down-time for generator $i$, respectively. Let $\mathbf{H}$ be the power transfer distribution factors (PTDF) or shift factor matrix and $\mathbf{C}_g \in \mathbb{R}^{n_{h} \times n_{y}}$ be the bus-to-generator incidence matrix. Note that $C_{g,j} = 1$ if and only if generator $i$ is connected to bus $j$, and $C_{g,j} = 0$ otherwise. Since we adopt the DC modeling of the UC problem, the flow of each line $q$ at time $t$ (shown as $f_{q;it}$) can be expressed as a linear combination of all generations at time $t$. Therefore, the UC problem can be formulated as follows:

$$\text{minimize} \sum_{i \in \mathcal{G}} g_{i;it}(p_{i;it}, x_{i;it}) + \sum_{t \in \mathcal{T}} h_{i;it}(x_{i;it+1}, x_{i;it}), \quad (3a)$$

subject to

$$x_{i;it} \in \{0, 1\}, \quad (3b)$$

$$p_{i;\text{min}} \times x_{i;it} \leq p_{i;it} \leq p_{i;\text{max}} \times x_{i;it}, \quad (3c)$$

$$\sum_{i=1}^{n_{g}} p_{i;it} = \sum_{j=1}^{n_{y}} d_{t,j}, \quad (3d)$$

$$|\mathbf{H}(d_{t} - \mathbf{C}_g \mathbf{p}_{t})| \leq f_{\text{max}}, \quad (3e)$$

$$|p_{i;it+1} - p_{i;it}| \leq (2s_{i} - r_{i}) + (r_{i} - s_{i}) |x_{i;it+1} + x_{i;it}|, \quad (3f)$$

$$x_{i;it+1} - x_{i;it} \leq x_{i;r}, \quad \forall \tau \in \{t + 1, \ldots, \min(t + U_{i}, t_0)\}, \quad (3g)$$

$$x_{i;it-1} - x_{i;it} \leq 1 - x_{i;r}, \quad \forall \tau \in \{t + 1, \ldots, \min(t + D_{i}, t_0)\}, \quad (3h)$$

where:

- $\mathcal{T} \triangleq \{1, 2, \ldots, t_0\}$ and $\mathcal{T}_{0} \triangleq \{0, 1, 2, \ldots, t_0\}$.

- $\mathcal{T}$ imposes that status of each generator to be binary and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$.

- $\mathcal{T}$ is the unit capacity constraint and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$.

- $\mathcal{T}_{0}$ represents the power balance equation and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}$.

- $\mathcal{T}$ indicates the line capacity constraint and holds for all $t \in \mathcal{T}$.

- $\mathcal{T}_{0}$ formulates the ramping constraint and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}_{0}$.

- $\mathcal{T}$ is the minimum up-time constraint and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}_{0}$.

- $\mathcal{T}_{0}$ is the minimum down-time constraint and holds for all $i \in \mathcal{G}$ and $t \in \mathcal{T}_{0}$.

Note that the security constraints have not been modeled explicitly in order to streamline the presentation. However, the results to be presented in this work are valid in presence of linear security constraints obtained using line outage distribution factors \([47]\).

**Remark 1.** Inequality (3f) encapsulates two types of ramping...
Therefore, the UC problem can be stated as

**Remark 3.** Constraints (3a), (3c), and (3b) can all be formulated linearly in terms of the decision variables.

### III. Convex Relaxation of UC Problem

In what follows, the main results of this paper will be developed. To streamline the presentation, the proofs are moved to the appendix.

**A. SDP Relaxations**

By relaxing the integrality condition (5b) to the linear constraint

\[ 0 \leq x_{i,t} \leq 1, \quad (4) \]

we obtain the basic (convex) quadratic programming (QP) relaxation of the UC problem. As will be shown in Section V the solution of this convex problem is almost always fractional for the test systems. Motivated by this observation, the objective is to design stronger relaxations. Consider the vector

\[ w \equiv [x_1^T, \ldots, x_n^T, p_1^T, \ldots, p_{t_0}^T]^T. \quad (5) \]

The constraint (4) together with the constraints of the UC problem except for (3b) can all be merged into a single linear vector constraint \( Mw \geq m \), for some constant matrix \( M \) and vector \( m \). Furthermore, the condition (3b) can be expressed as the quadratic equation

\[ x_{i,t}(x_{i,t} - 1) = 0. \quad (6) \]

Therefore, the UC problem can be stated as

\[
\begin{align*}
\text{minimize} & \quad c(w) \\
\text{subject to} & \quad Mw \geq m, \\
& \quad w_k(w_k - 1) = 0, \quad k = 1, 2, \ldots, n_{t_0},
\end{align*}
\]

where \( c(w) \) is equivalent to the total cost of the UC problem. It is straightforward to verify that \( c(w) \) is a convex function with respect to \( w \).

**Remark 3.** Let \( 0_{a \times b} \) and \( 1_{a \times b} \) denote \( a \times b \) matrices with all entries equal to 0’s and 1’s, respectively. Moreover, let \( I_n \) be the \( n \times n \) identity matrix. Given a vector \( p \), the notation \( \text{diag} \{ p \} \) represents a diagonal matrix such that the \((i, i)\)th entry equals \( p_i \). Assume that the \( i \)th entries of the vectors \( P_{\max} \) and \( P_{\min} \) represent the upper and lower bounds on the generation of unit \( i \in G \), respectively. In order to elaborate on the reformulation (7) and the structure of its parameters, note that

\[
M = \begin{bmatrix}
I_{n_g} & 0_{n_g \times n_g} \\
-I_{n_g} & 0_{n_g \times n_g} \\
-\text{diag} \{ P_{\min} \} & I_{n_g} \\
\text{diag} \{ P_{\max} \} & -I_{n_g} \\
0_{1 \times n_g} & 1_{1 \times n_g} \\
0_{1 \times n_g} & -1_{1 \times n_g} \\
0_{n_1 \times n_g} & H \cdot C_g \\
0_{n_1 \times n_g} & -H \cdot C_g \\
\end{bmatrix},
\quad m = \begin{bmatrix}
0_{n_g \times 1} \\
-1_{n_g \times 1} \\
0_{n_1 \times 1} \\
0_{n_1 \times 1} \\
\sum_{j=1}^{n_1} d_j \\
-\sum_{j=1}^{n_1} d_j \\
H \cdot d - f_{\max} \\
-H \cdot d - f_{\max}
\end{bmatrix},
\]

in the case \( t_0 = 1 \).

Consider a matrix variable \( \mathbf{W} \) and set it to \( \mathbf{w} \mathbf{w}^T \). The constraints of the UC problem can all be written as inequalities in terms of \( \mathbf{W} \) and \( \mathbf{w} \). This leads to a reformulation of the UC problem, where \( \mathbf{W} = \mathbf{w} \mathbf{w}^T \) is the only non-convex constraint. An SDP relaxation of the UC problem can be obtained by relaxing \( \mathbf{W} = \mathbf{w} \mathbf{w}^T \) to the conic constraint \( \mathbf{W} \succeq \mathbf{w} \mathbf{w}^T \). This yields the convex optimization problem

\[
\begin{align*}
\text{minimize} & \quad c_r(\mathbf{w}, \mathbf{W}) \\
\text{subject to} & \quad \mathbf{Mw} \geq \mathbf{m}, \quad \mathbf{W} \succeq \mathbf{w} \mathbf{w}^T,
\end{align*}
\]

where

\[
\begin{align*}
c_r(\mathbf{w}, \mathbf{W}) &= \sum_{i \in G} \sum_{t \in T} (a_i W_{n_g t_0 + n_g (t-1) + i, n_g t_0 + n_g (t-1) + i} \\
&\quad + b_i W_{n_g t_0 + n_g (t-1) + i} + c_i; \text{fixed} w_{n_g (t-1) + i}) \\
&\quad + \sum_{i \in G} \sum_{t \in T_0} (\text{start} \cdot W_{n_g t + i, n_g t + i} + W_{n_g (t-1) + i, n_g (t-1) + i} \\
&\quad - W_{n_g t + i, n_g (t-1) + i} - W_{n_g (t-1) + i, n_g t + i}).
\end{align*}
\]

Note that (10) can be written as a linear matrix inequality with respect to \( \mathbf{w} \) and \( \mathbf{W} \). This problem is called the SDP relaxation of the UC problem.

**Theorem 1.** The optimal objective values of the SDP relaxation (9) and the basic QP relaxation of the UC problem are the same if \( t_0 = 1 \).

**Remark 4.** Note that (9) is indeed a relaxation of the UC problem. This is due to the fact that if \( \mathbf{w} \), defined in (5), is an optimal solution of the UC problem, then \( (\mathbf{w}, \mathbf{w} \mathbf{w}^T) \) is feasible for (9) and has the same objective value as the optimal cost of the UC problem. Furthermore, the proposed SDP relaxation solves the UC problem if and only if it has an optimal solution \( (\mathbf{w}^*, \mathbf{W}^*) \) for which the matrix

\[
\begin{bmatrix}
1 \\
\mathbf{w}^* \\
\mathbf{W}^*
\end{bmatrix}
\]

has rank 1. From a different perspective, in the case where \( x_{i,t}^* \)‘s are all binary numbers at an optimal solution of (9), the relaxation is exact.
As will be demonstrated in Section IV the solution of the convex problem (9) is almost always fractional for the test systems.

B. Valid Inequalities

Let \( S \) denote the set of feasible points of the UC problem (3). An inequality is said to be valid if it is satisfied by all points in \( S \). The SDP relaxation (9) can be strengthened by adding valid inequalities to the problem. Consider two scalar inequalities of the UC problem, namely

\[
\mathbf{u}^T \mathbf{w} - z_1 \geq 0, \quad \mathbf{v}^T \mathbf{w} - z_2 \geq 0,
\]

for fixed coefficients \( \mathbf{u}, \mathbf{v}, z_1 \) and \( z_2 \). Since both of these inequalities hold for all points \( \mathbf{w} \in S \), the quadratic inequality

\[
\mathbf{u}^T \mathbf{w} \mathbf{w}^T \mathbf{v} - (\mathbf{v}^T z_1 + \mathbf{u}^T z_2)\mathbf{w} + z_1 z_2 \geq 0,
\]

is also satisfied for every \( \mathbf{w} \in S \). The above quadratic inequality can be relaxed to the linear inequality

\[
\mathbf{u}^T \mathbf{W} \mathbf{v} - (\mathbf{v}^T z_1 + \mathbf{u}^T z_2)\mathbf{w} + z_1 z_2 \geq 0.
\]

C. Strengthened SDP Relaxation

In this part, we construct a set of valid inequalities via the multiplication of all linear inequalities of the UC problem, using the strategy delineated in Section II-B. The resulting quadratic inequalities obtained from (9) can be expressed as the matrix constraint \((\mathbf{Mw} - \mathbf{m})(\mathbf{Mw} - \mathbf{m})^T \geq 0\), or equivalently,

\[
\mathbf{Mw} \mathbf{w}^T \mathbf{M}^T - \mathbf{m} \mathbf{w} \mathbf{w}^T \mathbf{M}^T - \mathbf{Mwm}^T + \mathbf{mm}^T \geq 0.
\]

Replacing the non-convex constraint \((7c)\) in the UC formulation (7) with the linear constraint (11) leads to a Reformulation-Linearization Technique (RLT) relaxation of the UC problem. Although it has been proven in [33] that this relaxation outperforms the basic QP relaxation, it will be shown in Section IV that this method often fails to generate feasible solutions for the UC problem.

The relaxation of this non-convex inequality yields the linear matrix inequality

\[
\mathbf{MWM}^T - \mathbf{mw} \mathbf{w}^T \mathbf{M}^T - \mathbf{Mwm}^T + \mathbf{mm}^T \geq 0.
\]

D. Exactness of RLT and Strengthened SDP Relaxations

In this section, we will show that load factors and small line ratings can both make the RLT and strengthened SDP relaxations exact. The exactness is due to the added valid inequalities and the SDP relaxation without these valid inequalities is not exact in general. To streamline the presentation, we assume that \( t_0 = 1 \) and then drop the subscript \( t \) from the formulation. The results can easily be extended to the case \( t_0 \geq 1 \). Define \( \mathbf{W} \) as

\[
\mathbf{W} = \begin{bmatrix} 1 & \mathbf{w}^T \\ \mathbf{w} & \mathbf{W} \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{w}_{21}^T & \mathbf{w}_{31}^T \\ \mathbf{w}_{21} & \mathbf{W}_{22} & \mathbf{W}_{23} \\ \mathbf{w}_{31} & \mathbf{W}_{32} & \mathbf{W}_{33} \end{bmatrix}.
\]

Real-world UC problems are large-scale due to the size of power grids and the number of time slots. Hence, the strengthened SDP relaxation (12) would be computationally expensive for practical systems. Later in this paper, constraint (12e) will be replaced by a number of lower-order conic constraints without affecting the solution.

### Definition 1

For every \( i \in \{1, 2, \ldots, n_g\} \), define the reliable lower bound \( l^i \) and the reliable upper bound \( u^i \) of generator \( i \) as

\[
\begin{align*}
l^i &= \min_{\mathbf{w}_{31} \in P_i} w_{i31}^i, \\
u^i &= \max_{\mathbf{w}_{31} \in P_i} w_{i31}^i.
\end{align*}
\]

Moreover, define 1 and \( \mathbf{u} \) as the vectors \([l^1, l^2, \ldots, l^{n_g}]\) and \([u^1, u^2, \ldots, u^{n_g}]\), respectively.

Define \( \mathcal{G}^+ \) as the index set of generators with strictly positive reliable lower bounds.

### Theorem 2

Let \( (\mathbf{w}, \mathbf{W}) \) denote an arbitrary feasible solution of the RLT or strengthened SDP relaxation, and \( \mathbf{x}^{opt} \) denote an arbitrary globally optimal commitment of generators in the UC problem. Furthermore, let \( (\mathbf{w}, \mathbf{W}) \) denote an optimal solution of the SDP relaxation. The following statements hold for every \( i \in \mathcal{G}^+ \):

(i) \( w_{i21}^i = x_{i}^{opt} = 1 \).

(ii) \( w_{i21}^i \neq x_{i}^{opt} \) if \( u^i < p_{i; max} \).
Corollary 1. The SDP relaxation (9) is not exact if there does not exist a globally optimal solution \((x^{\text{opt}}, p^{\text{opt}})\) of the UC problem such that \(p_{i}^{\text{opt}} \in \{0, p_{i:\text{max}}\}\) for every \(i \in \mathcal{G}\).

Remark 5. Theorem 2 states that, regardless of the objective functions of the RLT and strengthened SDP relaxations, the added valid inequalities ensure that the relaxation correctly finds the optimal statuses of those generators whose reliable lower bounds are strictly positive. Furthermore, it unveils that a global minimum of the UC problem might be recoverable by the SDP relaxation (without valid inequalities) only in the scenario where each generator is turned off or operates at its maximum capacity at an optimal solution of the UC problem. By fixing the limits \(p_{i: \text{max}}\) for every \(i \in \mathcal{G}\), the previous statement implies that although \(\mathbf{a}\) could take infinitely many values, only a finite number of them would make the SDP relaxation exact (because \(\sum_{j \in \mathcal{B}} d_{j}\) is equal to the summation of a subset of the limits \(p_{i: \text{max}}\)’s in the exact SDP case). This shows the clear difference between the SDP and strengthened SDP relaxations.

One may speculate that the performance of the strengthened SDP and RLT relaxations could be increased by first identifying generators with nontrivial positive lower bounds on their productions (via bound tightening on \(m_{j} \geq m_{0}\)) and then setting their corresponding binary variables to 1. Theorem 2 shows that this is indeed not the case since this process is automatically incorporated in the above relaxations. For every \(i \in \mathcal{B}\), let \(\mathcal{N}_{f}(i)\) denote the set of lines that are connected to bus \(i\). Furthermore, for every \(j \in \mathcal{G}\), define \(\mathcal{G}_{j}\) as the index set of those generators that are connected to the same bus as generator \(j\).

Theorem 3. Suppose that the UC problem is feasible, and that either of the following conditions is satisfied:

Condition 1: For every generator \(j \in \mathcal{G}\), the relation

\[
d_{j} - \min \left\{ \sum_{i \notin \mathcal{B} \cup \mathcal{G}} p_{i: \text{max}} \right\} > \sum_{k \in \mathcal{N}_{f}(j)} f_{k: \text{max}} = (15)\]

holds, where \(b_{j}\) denotes the bus adjacent to generator \(j\).

Condition 2: The relation

\[
\sum_{j \in \mathcal{B}} d_{j} > \min_{k \in \mathcal{G}} \left\{ \sum_{i \notin \mathcal{B} \cup \mathcal{G}} p_{i: \text{max}} \right\} = (16)\]

holds.

Then, the RLT and strengthened SDP relaxations of the UC problem are both exact. However, the SDP relaxation is exact only when

\[
\sum_{j \in \mathcal{B}} d_{j} = \sum_{i \in \mathcal{G}} p_{i: \text{max}} = (17)\]

Consider the case where there are not any two generators connected to the same bus. It can be inferred from Theorem 3 that large load factors and/or small line ratings both result in the exactness of the RLT and strengthened SDP relaxations.

E. Reduced-Strengthened SDP Relaxation

In this subsection, we design a reduced-strengthened SDP relaxation with conic constraints smaller than that of the strengthened SDP relaxation. Define the sets

\[
\mathcal{V}_{x_{t}} \triangleq \{ n_{g}(t-1) + 1, n_{g}(t-1) + 2, \ldots, n_{g}(t+1) \},
\]

\[
\mathcal{V}_{p_{t}} \triangleq \{ n_{g}(t_{0} + t-1) + 1, n_{g}(t_{0} + t-1) + 2, \ldots, n_{g}(t_{0} + t+1) \},
\]

\[
\mathcal{V}_{t} \triangleq \mathcal{V}_{x_{t}} \cup \mathcal{V}_{p_{t}}
\]

for every \(t \in \{1, \ldots, t_{0} - 1\}\). Observe that \(\mathcal{V}_{x_{t}}\) and \(\mathcal{V}_{p_{t}}\) are the index sets of those elements of \(\mathbf{w}\) that correspond to \(\{x_{1:t}, x_{n_{g}+1:t}, x_{n_{g}+2:t}, \ldots, x_{n_{g}+n_{g}+1:t}\}\) and \(\{p_{1:t}, p_{1:t+1}, \ldots, p_{1:t+n_{g}-1}\}\), respectively. There are constant matrices \(\mathbf{Y}_{1}, \ldots, \mathbf{Y}_{t_{0}-1}\) and vectors \(\mathbf{y}_{1}, \ldots, \mathbf{y}_{t_{0}-1}\) such that, for every \(t \in \{1, \ldots, t_{0} - 1\}\), the inequality

\[
\mathbf{Y}_{t}^{\mathbf{w}} \{ \mathcal{V}_{t} \} \geq \mathbf{y}_{t} = (18)
\]

is equivalent to the collection of those inequalities in (9b) that only include the decision variables \(x_{1:t}, p_{1:t}, x_{t+1}, p_{t+1}\), and \(p_{t+1}\) for all \(i \in \mathcal{G}\). Note that the inequalities given in (18) for \(t \in \{1, \ldots, t_{0} - 1\}\) cover all inequalities in (9b) except for the minimum up-time and down-time constraints.

To handle the minimum up- and down-time constraints, define the set \(\mathcal{V}_{t_{0}} \triangleq \{1, \ldots, n_{g}t_{0}\}\). Note that \(\mathcal{V}_{t_{0}}\) is the index set of those elements of \(\mathbf{w}\) that correspond to the statuses of the generators over different time slots. There are a matrix \(\mathbf{Y}_{t_{0}}\) and a vector \(\mathbf{y}_{t_{0}}\) such that the inequality

\[
\mathbf{Y}_{t_{0}}^{\mathbf{w}} \{ \mathcal{V}_{t_{0}} \} \geq \mathbf{y}_{t_{0}} = (19)
\]

is equivalent to the minimum up- and down-time constraints (3g) and (3h). Note that these constraints are inherently linear functions of the variables \(x_{1:t}\)’s.

So far, it has been shown that the condition (9b) can be replaced by (18) and (19) for \(t = 1, \ldots, t_{0}\). Based on this fact, we introduce a relaxation of the strengthened SDP problem as follows:

\[
\begin{align*}
\min_{\mathbf{w} \in \mathbb{R}^{n_{g}t_{0}}} & \quad \mathbf{c}_{r}(\mathbf{w}, \mathbf{W}) = (20a) \\
\text{subject to} & \quad \mathbf{Y}_{t}^{\mathbf{w}} \{ \mathcal{V}_{t} \} \geq \mathbf{y}_{t}, \quad t = 1, 2, \ldots, t_{0}, \quad (20b) \\
& \quad \mathbf{Y}_{t}^{\mathbf{w}} \{ \mathcal{V}_{t} \} \mathbf{Y}_{t}^{\mathbf{w}} - \mathbf{y}_{t}^{\mathbf{w}} \{ \mathcal{V}_{t} \}^{\mathbf{w}} \mathbf{Y}_{t}^{\mathbf{w}} \quad (20c) \\
& \quad W_{kk} - w_{k} = 0, \quad k = 1, 2, \ldots, n_{g}t_{0}, \quad (20d) \\
& \quad \mathbf{W} \succeq \mathbf{w} \mathbf{w}^{\mathbf{T}}. \quad (20e)
\end{align*}
\]

After this relaxation, the exactness of the proposed relaxation can be certified if and only if the variables \(x_{1:t}\)’s take binary values at optimality. The next theorem shows that the large conic constraint (20e) can be broken down into smaller conic constraints.
Theorem 4. The conic constraint $W \succeq ww^T$ in the relaxation of the strengthened SDP problem, i.e., \((20)\), is equivalent to the following set of smaller conic constraints:

$$W \{v_i, v_j\} \succeq w \{v_i\} w \{v_j\}^T, \quad t = 1, 2, \ldots, t_0$$ \((21)\)

in the absence of minimum up- and down-time constraints.

Substituting \((20e)\) with \((21)\) gives rise to the reduced-strengthened SDP relaxation of the UC problem.

F. Triangle and VUB Constraints

It will be shown in simulations that the proposed SDP relaxations are able to find a global solution of the UC problem for many test systems under various conditions. However, there are cases for which the relaxations are not exact. To further improve the relaxations for such systems, the so-called triangle inequalities are incorporated in the UC problem.

$$x_{i,t}x_{j,t} + x_{k,t} \geq x_{i,t}x_{k,t} + x_{j,t}x_{k,t},$$
$$x_{i,t}x_{j,t} + x_{i,t}x_{k,t} + x_{j,t}x_{k,t} + 1 \geq x_{i,t} + x_{j,t} + x_{k,t},$$

for every $i, j, k \in G$ and $t \in T$. The efficacy of these valid inequalities has been studied by Burer et al. \([44]\) and Anstreicher et al. \([41]\). Moreover, the proposed method is reinforced by adding the VUB ramp constraints

$$p_{i,t} \leq p_{i; \text{max}} - x_{i,t} - (p_{i; \text{max}} - s_i) \cdot (x_{i,t} - x_{i,t-1}),$$
$$p_{i,t} \leq p_{i; \text{max}} - x_{i,t} - (p_{i; \text{max}} - s_i) \cdot (x_{i,t} - x_{i,t+1}),$$
developed by Damcı-Kurt et al. \([20]\). Note that the above valid inequalities are a subclass of VUB ramp constraints for only two adjacent time slots. Although the number of all VUB ramp constraints is exponential in the size of the UC problem, the number of the inequalities considered above (for adjacent time slots) is linear.

IV. Numerical Results

In this section, numerical results for evaluating the proposed relaxations on IEEE case systems are provided. To generate multiple UC problems for each test case, we multiply all loads of each IEEE system by a load factor $\alpha$ chosen from a discrete set $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. For each IEEE system, we plot four curves for $k$ load profiles: (i) the optimal cost of the (reduced) strengthened SDP, (ii) the optimality gaps for three different relaxations (SDP, strengthened SDP and RLT). As the load factor changes from $\alpha_1$ to $\alpha_k$, the optimal statuses of the generators may change multiple times. Whenever the statuses of the generators for a load/rating scenario varies from those of the previous one, the corresponding scenario is marked on the curve by a red cross. Hence, if there is no mark on the SDP cost curve for a particular load/rating scenario, it means that the statuses of the generators are the same as those in the previous load scenario. Each red cross is accompanied by an integer number, which can be interpreted as follows: if this number is converted from base $10$ to $2$, it is the concatenation of the globally optimal status of all generators. For example, for a case with $3$ generators, the number $5$ on the SDP cost curve indicates that the first and third generators are active while the second generator is off at a globally optimal solution of UC (note that $5 = (101)_2$). Moreover, for every scenario that at least one of generator statuses found by the strengthened SDP is neither $0$ nor $1$, we write “Not Rank-1” on the curve instead of the an integer number encoding the optimal generator statuses. To further assess the performance of the proposed relaxations, we redo the above experiment for each test case and draw curves with respect to line ratings as opposed to load factors. More precisely, we impose a constant limit on the flows of all lines and solve various relaxations of the UC problem for different values of this limit.

Figure 3 shows the solutions found by the strengthened SDP for $20$ load scenarios for the IEEE $9$-bus system with $3$ generators over one time slot ($t_0 = 1$). The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, \ldots, 20$. It can be observed that the proposed convex relaxation has found a global solution of the UC problem for $19$ out of $20$ scenarios. The load profile associated with the factor $\alpha_2$ is the only unsuccessful case, for which we have: After adding triangle constraints to the formulation, the relaxation becomes exact and it retrieves the optimal solution of UC problem.

We define the optimality gap for any relaxation of the UC problem as

$$\text{Optimality gap} \equiv \frac{\text{upper bound} - \text{lower bound}}{\text{upper bound}} \times 100,$$

where “upper bound” and “lower bound” denote the globally optimal cost of the UC problem (found using Gurobi solver) and the optimal cost of the relaxation, respectively. The optimality gaps for the SDP, RLT and strengthened SDP relaxations are compared in Figure 4(b). Notice that the SDP and RLT relaxations perform very poorly and the proposed valid inequalities are essential for obtaining rank-1 (integer) solutions.

Figure 2 shows the performance of the proposed relaxations for different line ratings for the IEEE $9$-bus system over one time slot ($t_0 = 1$) with the load factor equal to $0.3$. The uniform ratings of the lines are chosen as $\alpha_i = 30 + 5 \times i$ for $i = 1, 2, \ldots, 10$. It can be observed that the strengthened SDP relaxation is exact in all scenarios. As rating of the lines decreases, the RLT relaxation becomes exact. This is due to the fact that the reliable lower bounds of the generators become strictly positive for small line ratings, which leads to the exactness of both RLT and strengthened SDP relaxations due to Theorem 2.

Figure 3 shows the solutions found by the strengthened SDP for $20$ load scenarios for the IEEE $14$-bus system with $5$ generators over one time slot. The load factors are $\alpha_i = 0.1 \times i$ for $i = 1, 2, \ldots, 20$. The relaxation is exact in $19$ load scenarios. More precisely, the load scenario $\alpha_{14}$ is the only unsuccessful trial, for which we have: As before, the proposed relaxation can retrieve the exact solution for this case after adding the triangle inequalities to the formulation.

The performance of the strengthened SDP relaxation with respect to different line ratings for the IEEE $14$-bus system is reported in Figure 3. The uniform line ratings are $\alpha_i =$
15 + 5 × i for i = 1, 2, ..., 17, where the load factor is equal to 0.8. Except for α3, the proposed relaxation is exact for all line ratings. For α3, we have:

Figure 5 illustrates the results of the strengthened SDP for 13 load scenarios for the IEEE 30-bus system with 6 generators over one time slot. The load factors are αi = 0.1 × i for i = 1, 2, ..., 13. It can be observed that the proposed convex relaxation is exact and finds the globally optimal solution of the problem for all scenarios. If the load factor is greater than or equal to 1.4, the UC problem becomes infeasible since the total load exceeds the total capacity of the generators.

Figure 6 depicts the performance of the strengthened SDP with respect to different line ratings for the IEEE 30-bus system. The load factor is equal to 0.7 and the uniform line ratings are chosen as αi = 15 + 5 × i for i = 1, 2, ..., 10. There is only one case (corresponding to α2) for which the proposed relaxation is not exact. For α2, we have:

Figure 7 shows the output of the strengthened SDP for 15 load scenarios for the IEEE 57-bus system with 7 generators over one time slot. The load factors are αi = 0.1 × i for i = 1, 2, ..., 15. The proposed convex relaxation obtains the globally optimal solution of the problem for all scenarios. Furthermore, the UC problem becomes infeasible if the load factor is greater than or equal to 1.6 since the total load exceeds the total generation capacity. Furthermore, Figure 8 illustrates the performance of our proposed method for the IEEE 57-bus system with the uniform line ratings αi = 30 + 5 × i for i = 1, 2, ..., 10. The load factor is set to 0.5. As before, the proposed relaxation successfully recovers the exact solution for all tested cases.

Consider 10 load scenarios for the IEEE 118-bus system with 54 generators over one time slot. The load factors are αi = 0.1 × i for i = 1, 2, ..., 10. The results are plotted for the reduced-strengthened SDP problems in Figure 9.
Figure 10 illustrates the results of the reduced-strengthened SDP \((20)\) with low-order conic constraints for 10 load scenarios for the IEEE 30-bus system with 6 generators over \(t_0 = 5\) time slots. The load factors are \(\alpha_i = 0.8 + 0.02 \times i\) for \(i = 1, 2, \ldots, 10\). Observe that reduced-strengthened SDP relaxation fails in only two cases. For these two cases, the optimality gap is close to zero. Note that each red cross in Figure 10a is accompanied by a vertical array of 5 numbers, each showing the commitment parameters (in base 10) for different time instances. Figure 11 shows the solutions of the relaxed strengthened SDP \((20)\) for 10 load scenarios for the IEEE 57-bus system with 7 generators over 6 time slots. The load factors are \(\alpha_i = 0.1 \times i\) for \(i = 1, 2, \ldots, 10\). The proposed relaxation is exact for all load scenarios. Consider the IEEE 300-bus system with 69 generators over one time slot and for the single load factor of 1. The strengthened SDP relaxation achieves the global minimum of the UC problem. The number natural 1833848176079218650 encodes the optimal statuses of all generators in base 10. After converting this number to a binary vector, it can be observed that 53 generators are on and 16 generators are off at optimality.

Next, consider the IEEE 14-bus system with 5 generators over 24 time slots. As before, the proposed convex model \((20)\) achieves the globally optimal solution of the UC problem for this scenario. Figure 12 displays the total load distribution over this horizon. Furthermore, the integer number on top of each column represents the optimal configuration of the generators at the corresponding time slot. The optimal cost associated with the reduced-strengthened SDP relaxation \(205838\). However, the optimal cost for the SDP relaxation without the proposed valid inequalities is equal to \(162600\).

Table I presents the running time of different test cases using the proposed relaxations. The simulations are run on a laptop computer with an Intel Core i7 quad-core 2.50 GHz CPU and 16GB RAM. The computation times reported in this section are for a serial implementation in MATLAB using the CVX framework and MOSEK solver.

Finally, we aim to show that even if two lines are far from each other in the network, they may still generate a valid inequality that is crucial in finding a globally optimal solution of...
Finding a global solution to the unit commitment (UC) problem is a daunting challenge. The objective of this paper is to design a convex model (of polynomial size) for practical instances of the UC problem. Our approach is based on developing a convex conic relaxation for the UC problem. This is achieved by generating valid constraints and then relaxing them to linear matrix inequalities. These valid inequalities are obtained by the multiplication of the linear constraints of the UC problem, such as the flow constraints of two different lines.

The proposed technique is extensively tested on benchmark instances of the UC problem. Our approach is based on developing a convex conic relaxation for the UC problem. This is achieved by generating valid constraints and then relaxing them to linear matrix inequalities. These valid inequalities are obtained by the multiplication of the linear constraints of the UC problem, such as the flow constraints of two different lines.

The weights are normalized with respect to the largest magnitude of the Lagrange multipliers. Recall that Lagrange multipliers show the sensitivity of the optimal objective value of the strengthened SDP problem to infinitesimal perturbations. It can be observed that the largest magnitudes of the optimal Lagrange multipliers corresponding to the constraints in (22) are visualized as a weighted graph in Figure 13. The thickness (weight) of each blue line is proportional to the magnitude of the optimal Lagrange multiplier for the valid inequality obtained by multiplying the flow constraints of that line and the red line (8, 9). The weights are normalized with respect to the largest magnitude of the Lagrange multipliers.

V. CONCLUSIONS

The thickness of each blue line of the IEEE 57-bus system is proportional to the Lagrange multiplier for the valid inequality in the strengthened SDP problem that is obtained by multiplying the flow constraints of that line and the red line (8, 9).

To this end, consider the IEEE 57-bus system with the load factor 0.5 and the uniform line rating equal to 35 over one time slot (for convenience, we drop the subscript $t_0$). At optimality, the lines (8, 9), (1, 15), (7, 8), and (12, 13) are congested. We solve the strengthened SDP relaxation and consider the Lagrange multipliers corresponding to the valid inequalities

$$\mathbf{H}_i \mathbf{C}_g \mathbf{W}_{35} \mathbf{C}_g^\top \mathbf{H}_i^\top - (\mathbf{H}_i \mathbf{d} + f_{i,\text{max}}) \mathbf{H}_i \mathbf{C}_g \mathbf{w}_{31}$$

$$-(\mathbf{H}_i \mathbf{d} + f_{i,\text{max}}) \mathbf{H}_i \mathbf{C}_g \mathbf{w}_{31} + (\mathbf{H}_i \mathbf{d} + f_{i,\text{max}}) (\mathbf{H}_i \mathbf{d} + f_{i,\text{max}}) \mathbf{H}_i \mathbf{C}_g \mathbf{w}_{31} \geq 0$$

for $i = 1, 2, ..., n_t$. The above inequalities correspond to the multiplication of the flow constraints of the line (8, 9) and every line of the network, i.e., $\mathbf{H}_i (\mathbf{d} - \mathbf{C}_g \mathbf{p}) + f_{i,\text{max}} \geq 0$ and $\mathbf{H}_i (\mathbf{d} - \mathbf{C}_g \mathbf{p}) + f_{i,\text{max}} \geq 0$. Note that the number 8 is the index of those rows of $\mathbf{H}$ and $f_{\text{max}}$ that are associated with the line (8, 9). The magnitudes of the optimal Lagrange multipliers corresponding to the constraints in (22) are visualized as a weighted graph in Figure 13. The thickness (weight) of each blue line is proportional to the magnitude of the optimal Lagrange multiplier for the valid inequality obtained by multiplying the flow constraints of that line and the red line (8, 9). The weights are normalized with respect to the largest magnitude of the Lagrange multipliers.

TABLE I: Running times and objective values of different test cases with given load factors (LF) and time slots ($t_0$) using the SDP, RLT, Strengthened SDP (SSDP) relaxations.

<table>
<thead>
<tr>
<th>Case</th>
<th>SDP</th>
<th>RLT</th>
<th>SSDP</th>
</tr>
</thead>
<tbody>
<tr>
<td>IEEE 9-bus, LF = 0.5, $t_0 = 1$</td>
<td>Obj. value 1421.5</td>
<td>Time (s) 1.0979</td>
<td>Obj. value 1598.1</td>
</tr>
<tr>
<td>IEEE 14-bus, LF = 0.5, $t_0 = 1$</td>
<td>Obj. value 3265.0</td>
<td>Time (s) 1.1229</td>
<td>Obj. value 3319.8</td>
</tr>
<tr>
<td>IEEE 30-bus, LF = 0.5, $t_0 = 1$</td>
<td>Obj. value 268.5</td>
<td>Time (s) 1.2782</td>
<td>Obj. value 268.5</td>
</tr>
<tr>
<td>IEEE 57-bus, LF = 0.5, $t_0 = 1$</td>
<td>Obj. value 16820.1</td>
<td>Time (s) 1.1665</td>
<td>Obj. value 16822.2</td>
</tr>
<tr>
<td>IEEE 118-bus, LF = 0.5, $t_0 = 1$</td>
<td>Obj. value 54845.8</td>
<td>Time (s) 54.8311</td>
<td>Obj. value 54845.9</td>
</tr>
<tr>
<td>IEEE 30-bus, LF = 0.9, $t_0 = 5$</td>
<td>Obj. value 3167.1</td>
<td>Time (s) 1.7100</td>
<td>Obj. value 3266.2</td>
</tr>
<tr>
<td>IEEE 57-bus, LF = 0.5, $t_0 = 6$</td>
<td>Obj. value 105691.4</td>
<td>Time (s) 2.3609</td>
<td>Obj. value 105735.6</td>
</tr>
<tr>
<td>IEEE 14-bus, LF = 0.95, $t_0 = 24$</td>
<td>Obj. value 162600.1</td>
<td>Time (s) 421.3616</td>
<td>Obj. value 207362.2</td>
</tr>
</tbody>
</table>

Fig. 13: The thickness of each blue line of the IEEE 57-bus system is proportional to the Lagrange multiplier for the valid inequality in the strengthened SDP problem that is obtained by multiplying the flow constraints of that line and the red line (8, 9).
REFERENCES


VI. APPENDIX

A. Elimination of Redundant Valid Inequalities

In this section, we aim to reduce the number of added valid inequalities by identifying a subset of redundant (implied) constraints and removing them from the formulation. We assume that \( t_0 = 1 \) and drop the subscript \( t \) from the formulation. However, the results can easily be extended to the case \( t_0 = 1 \). According to the definition of the matrix \( M \) in \([33]\), the number of linear inequalities in the strengthened SDP problem is equal to

\[
\frac{4n_g + 2n_l + 2 + (4n_g + 2n_l + 2)^2 + n_g}{2} = 4n_l^2 + 16n_g^2 + 16n_gn_l + 21n_g + 10n_l + 6. \tag{23}
\]

On the other hand, since \([12c]\) is symmetric, the constraints corresponding to the lower triangular part of \([12c]\) are redundant and can be removed. The number of remaining constraints amounts to

\[
\frac{(4n_g + 2n_l + 2)^2 + (4n_g + 2n_l + 2) + 4n_g + 2n_l + 2 + n_g}{2} = 2n_l^2 + 8n_g^2 + 8n_gn_l + 15n_g + 7n_l + 5. \tag{24}
\]

Lemma 1. The constraint \( x_i \geq 0 \) is implied by the inequalities \( p_{i;\min} \times x_i \leq p_i \leq p_{i;\max} \times x_i \), for every \( i \in \{1, ..., n_g\} \).

Notice that Lemma 1 immediately follows from the relation \( p_{i;\min} < p_{i;\max} \).

Lemma 2. \([12b]\) is implied by \([12c]\).

Proof. The proof can be found in \([33]\). \(\square\)

Using Lemmas 1 and 2, the number of potentially required linear inequalities will be reduced to \( 2n_l^2 + 4.5n_g^2 + 6n_gn_l + 8.5n_g + 5n_l + 3 \). Furthermore, instead of writing the constraint \( \sum_{i=1}^{n_g} p_i = \sum_{j=1}^{n_g} \delta_j \) as two inequalities and incorporating them into the matrix \( M \), we can consider it as a separate linear equality. Therefore, the constraint \( Mw \geq m \) can be written as \( M'w \geq m' \) and \( c^Tw = b \) where \( c = [0_{1 \times n_g}, 1_{1 \times n_g}]^T \), \( b = \sum_{j=1}^{n_g} d_j \), and

\[
M' = \begin{bmatrix}
M'_{1} \\
M'_{2} \\
M'_{3} \\
M'_{4} \\
M'_{5}
\end{bmatrix},
\]

\[
m' = \begin{bmatrix}
m'_{1} \\
m'_{2} \\
m'_{3} \\
m'_{4} \\
m'_{5}
\end{bmatrix},
\]

Define

\[
T_i = M_i'Wc - m'_i w^T c - bM_i'w + bm'_i = 0, \tag{27}
\]

for \( i \in \{1, ..., 5\} \).

Theorem 5. The set of equalities \( \{T_i = 0 \mid i = 1, ..., 5\} \) is equivalent to the union of the following constraints:

(i) \( \sum_{i=1}^{n_g} w_{31}^i = b \)
(ii) \( \sum_{i=1}^{n_g} W_{32}^{ij} = b \times w_{31}^j \) for \( j = 1, ..., n_g \).
(iii) \( \sum_{i=1}^{n_g} W_{33}^{ij} = b \times w_{31}^j \) for \( j = 1, ..., n_g \).
(iv) \( HC_g W_{33} \times 1_{n_g \times 1} = b \times HC_g w_{31} \).

Proof. To prove the theorem, it is enough to show that (i), (ii), and (iv) are necessary and sufficient to describe the set of equalities \( \{T_i = 0 \mid i = 1, ..., 5\} \). First, consider \( T_2 = 0 \). This equality is generated by multiplying \( p_j - p_{j;\min} \times x_j \geq 0 \) with \( \sum_{i=1}^{n_g} p_i = b \). After multiplication, it can be obtained that

\[
p_{j;\min} \sum_{i=1}^{n_g} W_{32}^{ij} - \sum_{i=1}^{n_g} W_{33}^{ij} - p_{j;\max} b w_{32}^j + b w_{31}^j = 0. \tag{28}
\]

Similarly, it results from \( T_3 = 0 \) that

\[
p_{j;\max} \sum_{i=1}^{n_g} W_{32}^{ij} - \sum_{i=1}^{n_g} W_{33}^{ij} - p_{j;\max} b w_{32}^j + b w_{31}^j = 0. \tag{29}
\]

for \( j = 1, ..., n_g \). Since \( p_{j;\min} < p_{j;\max} \), it is straightforward to verify that (28) and (29) are equivalent to (ii) and (iii). Moreover, the equality \( T_1 = 0 \) is generated by multiplying \( 1 - x_j \geq 0 \) with \( \sum_{i=1}^{n_g} p_i = b \). Performing this multiplication yields \( \sum_{i=1}^{n_g} w_{31}^i - \sum_{i=1}^{n_g} W_{32}^{ij} = b - b \times w_{32}^j \). This equality can be simplified to (i) using (ii).

Finally, we will show that (iv) together with (i) implies \( T_4 = 0 \) and \( T_5 = 0 \). Notice that the equality \( T_4 = 0 \) is generated by multiplying \( f_{\max} - Hd + HC_g p \geq 0 \) with \( \sum_{i=1}^{n_g} p_i = b \). This multiplication yields that

\[
\sum_{i=1}^{n_g} w_{31}^i \times (Hf_{\max} - d) + HC_g W_{33} \times 1_{n_g \times 1}
= b \times (Hf_{\max} - d) + b \times HC_g w_{31}. \tag{30}
\]

Using (i), one can simplify (30) to (iv). Similarly, it can be shown that \( T_5 = 0 \) is implied by (i) and (iv). This completes the proof. \(\square\)
Remark 6. Theorem 5 shows the set of equalities \( \{ T_i = 0 : i = 1, \ldots, 5 \} \) can be replaced with another set with a smaller number of constraints. In particular, the number of equalities in \( \{ T_i = 0 : i = 1, \ldots, 5 \} \) is \( 3n_g + 2n_l \) whereas (i)-(iv) have only \( 2n_g + n_l + 1 \) equalities. Hence, the number of valid constraints in the strengthened SDP problem can be reduced to \( 2n_l^2 + 4.5n_g^2 + 6n_gn_l + 4.5n_g + 2n_l + 2 \).

One may speculate that more valid inequalities in the strengthened SDP problem can be declared redundant and eliminated by analyzing the geographical locations of generators and lines. In particular, a question arises as to whether it is necessary to incorporate those valid inequalities that are obtained by multiplying the constraints of two devices (lines or generators) that are geographically far from each other. As will be shown in simulations on a test system, such valid inequalities may be crucial for the exactness of the strengthened SDP relaxation.

B. Proof of Theorem 7

Assume that \( (w^*, W^*) \) denotes an optimal solution of the SDP relaxation (9). First, we aim to show that \( w^* \) is a feasible point of the basic QP relaxation. Consider an index \( k \) corresponding to an element of \( w \) associated with a generator status. The constraint (9b) is the same as (7b). Moreover, (9d) implies that \( W_{kk}^* \geq w_k^* \), which together with the constraint (9c) leads to the relation \( 0 \leq w_k^* \leq 1 \). As a result, \( w^* \) is a point feasible for the basic QP problem. Due to the definitions of \( c_r(w, W) \) and \( c(w) \) as well as the inequality \( W_{kk}^* \geq w_k^* \), one can verify that \( c_r(w^*, W^*) \geq c(w^*) \). Therefore, the optimal cost of the SDP relaxation is greater than or equal to the cost of the QP relaxation.

In order to complete the proof, it suffices to show that the optimal cost of the QP relaxation is greater than or equal to the optimal cost of the SDP relaxation. Suppose that \( \tilde{w} \) denotes an optimal solution of the SDP relaxation of the UC problem. One can build a matrix \( \tilde{W} \) such that \( (\tilde{w}, \tilde{W}) \) is a feasible point for the SDP relaxation with a cost equal to the optimal cost of the QP relaxation. The constraint (9b) is a reformulation of the linear constraints and therefore it holds true. Furthermore, the constraint \( 0 \leq w_k \leq 1 \) implies that \( w_k^* \leq \tilde{w}_k \). Therefore, we can construct a non-negative diagonal matrix \( W_0 \) such that \( (\tilde{W}_0 + w_k^\top) - \tilde{w}_k = 0 \). As a result, \( (\tilde{w}, \tilde{W}) \) is feasible for SDP relaxation, where \( \tilde{W} = \tilde{w}\tilde{w}^\top + W_0 \). Note that the only possibly required positive elements of \( W_0 \) are the diagonal elements corresponding to the statuses of generators. Furthermore, notice that these diagonal elements do not appear in the objective whenever \( t_0 = 1 \). Therefore, one can verify that \( c_r(\tilde{w}, \tilde{W}) = c(\tilde{w}) \). This completes the proof.

C. Proof of Theorem 2

The set \( \mathcal{P}_l \) can be described as \( M_l w_{31} \geq m_l \), where

\[
M_l = \begin{bmatrix}
I_{n_g} & 0_{n_g \times n_g} \\
-I_{n_g} & 0_{n_g \times n_g} \\
0_{2n_l + 2n_g + 2 \times n_g} & M_l
\end{bmatrix}, \quad m_l = \begin{bmatrix}
0_{n_g \times n_g} \\
0_{n_g \times n_g} \\
H_d - f_{\max}
\end{bmatrix}.
\]

Therefore, one can rewrite \( M \) and \( m \) as

\[
M = \begin{bmatrix}
M_1 \\
M_2 \\
M_3 \\
M_4 \\
M_5
\end{bmatrix}, \quad m = \begin{bmatrix}
m_1 \\
m_2 \\
m_3 \\
m_4 \\
m_5
\end{bmatrix}
\]

For notational simplicity, define

\[
L_{(i,j)} = M_i W_j^\top - m_i w^\top M_j^\top - M_j w_i^\top + m_i m_j^\top,
\]

for every \( i, j \in \{1, \ldots, 5\} \). Furthermore, \( L_{i,j}^{m,n} \) will be used to refer to the \((m, n)\)th element of the matrix \( L_{(i,j)} \).

First, we prove Part (i) using a particular set of valid inequalities introduced by (12a). It follows from \( L_{(3,2)} \geq 0 \) that

\[
-p_{i;\min} u_{21}^i + w_{31}^i + p_{i;\min} W_{22}^i - W_{32}^i \geq 0,
\]

or equivalently,

\[
p_{i;\min} (u_{21}^i - W_{22}^i) \leq w_{31}^i - W_{32}^i,
\]

for every \( i, j \in \{1, 2, \ldots, n_g\} \). Likewise, the inequality \( L_{(4,2)} \geq 0 \) leads to

\[
w_{31}^i - W_{32}^i \leq p_{i;\max} (u_{21}^i - W_{22}^i),
\]

for every \( i, j \in \{1, 2, \ldots, n_g\} \). If \( i = j \), combining (35) and (36) with the constraint \( u_{21}^i = W_{22}^i \) yields that

\[
w_{31}^i = W_{32}^i.
\]

Consider the constraints \( w_{31}^i \geq 1 \) and \( u_i \geq w_{31} \). Moreover, consider the following inequalities for every \( i \in \{1, 2, \ldots, n_g\} \):

\[
-u_i^1 - w_{31}^i - u_i^1 w_{21}^i + W_{32}^i \geq 0, \tag{38a}
\]

\[
-l_i^1 + w_{31}^i + u_i^1 w_{21}^i - W_{32}^i \geq 0. \tag{38b}
\]

These valid inequalities are generated by multiplying \( w_{31}^i - l_i \geq 0 \) and \( u_i - w_{31} \geq 0 \) with \( 1 - w_{21}^i \geq 0 \). According to (33), one can show that adding (38a) and (38b) to the formulation does not change the feasible region of the RLT relaxation (and as a result, the strengthened SDP) since they are implied by
other added valid inequalities. However, one can combine (37) with (38a) and (38b) to arrive at

\[ u^i (1 - w_{21}^i) \geq 0, \]
\[ l^i (w_{21}^i - 1) \geq 0. \]

Since \( i \in G^+ \), we have \( 0 < l^i \leq u^i \). Therefore, (39) implies that \( w_{21}^i = 1 \). Furthermore, it can be inferred from \( 0 < l^i \) that \( x_{i}^{\text{opt}} = 1 \). This completes the proof of Part (i).

Next, we prove Part (ii). Notice that according to Theorem 1, the SDP relaxation is equivalent to the QP relaxation whenever \( t_0 = 1 \). Since \( w_{21}^i \) appears in the objective function of the SDP relaxation with a positive coefficient, it can be deduced that

\[ \bar{w}_{21}^i = \frac{\bar{w}_{31}^i}{p_{i;\text{max}}}, \]

for every \( i \in G \). This implies that \( \bar{w}_{21}^i \in \{0, 1\} \) if and only if \( \bar{w}_{31}^i \in \{0, p_{i;\text{max}}\} \). However, it is easy to verify that since \( u^i < p_{i;\text{max}} \) by assumption, the inequalities \( 0 < l^i \leq \bar{w}_{31}^i \leq u^i < p_{i;\text{max}} \) hold. The proof follows from (40).

D. Proof of Corollary 1

Let \((\bar{w}, \bar{W})\) denote an arbitrary solution of the SDP relaxation. Assume that \( p_{i;\text{opt}} \not\in \{0, p_{i;\text{max}}\} \) for some index \( i \in G \). Due to the proof of Theorem 2, this means that \( \bar{w}_{21}^i \not\in \{0, 1\} \). As a result, \((x^{\text{opt}}, p^{\text{opt}})\) does not correspond to a global minimum of the UC problem. This implies that the SDP relaxation is not exact.

E. Proof of Theorem 3

Assume that either of Conditions 1 and 2 is satisfied. It can be verified that \( l^i > 0 \) for every \( i \in G \), which yields that \( G^+ = G \). Now, it follows from Theorem 2 that the RLT and strengthened SDP relaxations are both exact. Furthermore, if \((x^{\text{opt}}, p^{\text{opt}})\) denotes a globally optimal solution of the UC problem, then \( p_{i;\text{opt}} \geq l^i > 0 \) for every \( i \in G \). Therefore, it results from Corollary 1 that the SDP relaxation could possibly be exact only when \( p_{i;\text{opt}} = p_{i;\text{max}} \) for all \( i \in G \), which lead to the equation (17).

F. Proof of Theorem 4

Assume that the minimum up- and down-time constraints do not exist. It can be observed that in (20b), (20c), and (20d), the decision variables at each time instance are coupled only with the decision variables of the next and previous time slots. Using the chordal extension technique (see [48]), it is easy to verify that relaxing the constraint (20e) to (21) does not affect the optimal cost. This is due to the fact that the tree decomposition of the above problem is a path. The details are omitted for brevity.