Exactness of Semidefinite Relaxations for Nonlinear Optimization Problems with Underlying Graph Structure

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Polynomial Optimization

**Polynomial Optimization:**

\[
\text{min } x^T M x \\
\text{s.t. } x_i^2 = 1, \quad i = 1, 2, ..., n
\]

**Special case:** Combinatorial optimization and integer programming problems

**Very hard to solve**

**Different types of solutions:**

- **Point A:** Local solution
- **Point B:** Global solution
- **Point C:** Near-global solution

**Focus of our research**

**Approach:** Low-rank optimization, matrix completion, graph theory, convexification
Convexification

\[ \min_{x \in \mathbb{C}^n} \quad x^H M_0 x \]
\[ \text{s.t.} \quad x^H M_i x \leq a_i, \quad i = 1, 2, \ldots, m \]

\[ \text{SDP relaxation} \]

\[ \min_{W \in \mathbb{H}^n} \quad \text{trace}\{ M_0 W \} \]
\[ \text{s.t.} \quad \text{trace}\{ M_i W \} \leq a_i, \quad i = 1, 2, \ldots, m \]
\[ W \succeq 0 \]

\[ \text{Penalized SDP} \]

\[ \min_{W} \quad \text{trace}\{ M_0 W \} + \lambda g(W) \]
\[ \text{s.t.} \quad \text{trace}\{ M_i W \} \leq a_i, \quad i = 1, 2, \ldots, m \]
\[ W \succeq 0 \]

- **Transformation**: Replace \( xx^H \) with \( W \).

- \( W \) is positive semidefinite and rank 1

- **Rank-1 SDP**: Recovery of a global solution \( x \)

- **Rank-1 penalized SDP**: Recovery of a near-global solution \( x \)
Research Problems

Arbitrary Real/Complex Polynomial Optimization

How does structure make SDP relaxation exact?

Conversion

Connection between sparsity and rank?

SDP/ Penalized SDP

Complexity analysis based on generalized weighted graph

How to design penalized SDP?

Proof of existence of low-rank solution using OS and treewidth

Design scalable numerical algorithm?

Propose two methods to design penalty

Power optimization problems

Cheap iterations for large-scale problems

Finding near-global solutions using physics of power grids
**Structured Optimization**

- **Approach:** Map the structure into a *generalized weighted graph.*

![Diagram](image)

\[
\begin{align*}
\min_{x_{1,2}} & \quad x_1^4 + ax_2^2 + bx_1^2 x_2 + cx_1 x_2 \\
\min_{x \in \mathbb{R}^4} & \quad x_3^2 + ax_2^2 + bx_2 x_3 + cx_1 x_2 \\
\text{s.t.} & \quad x_1^2 - x_3 x_4 \leq 1 \\
& \quad x_4^2 - 1 = 0
\end{align*}
\]

Due to structure, SDP is always exact.

**Generalized weighted graph:**

\[
\begin{align*}
\min_{x_{1,2}} & \quad x_1^4 + a_0 x_2^2 + b_0 x_1^2 x_2 + c_0 x_1 x_2 \\
\text{s.t.} & \quad x_1^4 + a_i x_2^2 + b_i x_1^2 x_2 + c_i x_1 x_2 \leq \alpha_i, \quad i = 1, 2, \ldots, m
\end{align*}
\]
Real-Valued Optimization

```
\begin{align*}
\sigma_{ij} &\neq 0, & \forall (i,j) \in G \\
\prod_{(i,j) \in G} \sigma_{ij} &= (-1)^{|G_r|}, & \forall r \in \{1, \ldots, p\}
\end{align*}
```

- **Special cases:**
  - **Positive optimization:** Bipartite graph
  - **Negative optimization:** Arbitrary graph

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Complex-Valued Optimization

- **Real-valued case:** “$T$” is sign definite if $T$ and $-T$ are separable in $\mathbb{R}$:
- **Complex-valued case:** “$T$” is sign definite if $T$ and $-T$ are separable in $\mathbb{R}^2$:

Theorem: SDP is exact for acyclic graphs with sign definite sets and certain cyclic graphs.

- The proposed conditions include several existing ones ([Kim and Kojima, 2003], [Padberg, 1989], [Bose, Gayme, Chandy, and Low, 2012], etc.).

Complex-Valued Optimization

- **Purely imaginary weights** (lossless power grid):

- **Theorem**

  \[ \text{Exact relaxation for weakly cyclic graphs with homogeneous weight sets.} \]

- Consider a real matrix \( M \):

\[
\min_{x \in \mathbb{C}^n} \quad x^* M x \\
\text{s.t.} \quad |x_j| = 1, \quad j = 1, 2, ..., m
\]

- Polynomial-time solvable for weakly-cyclic bipartite graphs.

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Example: Physics of power grids reduces computational complexity.

\[ g_i + b_{ij} \]

Coefficients of \( x_i x_j \)

Sign definite due to passivity
Power System:

- A large-scale system consisting of generators, loads, lines, etc.
- Used for generating, transporting and distributing electricity.

ISO, RTO, TSO

1. Optimal power flow (OPF)
2. Security-constrained OPF
3. State estimation
4. Network reconfiguration
5. Unit commitment
6. Dynamic energy management

NP-hard
(real-time operation and market)
Optimal Power Flow: Optimally match supply with demand

- **Real-time operation**: OPF is solved every 5-15 minutes.
- **Market**: Security-constrained unit-commitment OPF
- **Complexity**: Strongly NP-complete with long history since 1962.
- **Common practice**: Linearization
- **FERC and NETSS Study**: Annual cost of approximation > $1 billion

A multi-billion critical system depends on optimization.

\[
\min_{x \in \mathbb{C}^n} x^H M_0 x
\]
\[\text{s.t. } x^H M_i x \leq a_i, \quad i = 1, 2, \ldots, m\]

Vector of complex voltages

OPF feasible set

(Ian Hisken et al. 2003)
Exactness of Relaxation

- SDP is exact for IEEE benchmark examples and several real data sets.

**Theorem:** Exact under positive LMPs with many transformers.

**Theorem:** Exact under positive LMPs.

Physics of power networks (e.g., passivity) reduces computational complexity for power optimization problems.

**Strategy:** Penalize reactive loss over problematic lines

- **Modified IEEE 118-bus:**
  - 3 local solutions
  - Costs: 129625, 177984, 195695

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