Nonlinear and Discrete Optimization—Homework 8

1. Consider the discrete optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T Q x \\ \text{s.t.} \quad & x_i^2 = 1, \quad i = 1, ..., n \end{aligned}$$

   where Q is an $n \times n$ symmetric matrix. To study this problem, we take two approaches:

   - *Brute-force search:* We compute the objective function for all of the 2^n possible feasible points and denote the smallest objective value as p_*.
   - *Dual optimization:* We solve the convex dual problem and denote the optimal objective value as d_*.

   The goal of this example is to study the duality gap and to analyze the computational complexities of the above methods.

   i) Consider $n = 15$. Generate 150 random symmetric matrices Q such that each diagonal entry is equal to 25 and each off-diagonal entry is uniformly chosen from the interval $[0, 1]$. Write a code in CVX that finds the duality gap (in percentage) based on the formula $\frac{p_* - d_*}{|p_*|} \times 100$ for each random initialization of Q, and report the average of this gap over the 150 trials.

   ii) For $n = 1, 2, ..., 28$, generate a random $n \times n$ matrix Q (based on the procedure stated in Part (i)) and then find the corresponding numbers p_* and d_*. Let $t_p(n)$ and $t_d(n)$ denote the runtimes of your codes for finding p_* and d_*, respectively. Draw $t_p(n)$ and $t_d(n)$ as a function of n, and based on these curves discuss the benefits of using the dual optimization over the brute-force search.

2. Use the branch-and-bound method to solve the mixed-integer program:

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 2x_1 - x_2 \leq 6 \\ & x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \\ & x_1 : \text{integer} \end{aligned}$$

   (note: you can use CVX to solve the convex problems needed to form the tree).

3. Use the branch-and-bound method to solve the discrete optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^3} \quad & -4x_1 - 3x_2 - x_3 \\ \text{s.t.} \quad & 3x_1 + 2x_2 + x_3 \leq 7 \\ & 2x_1 + x_2 + 2x_3 \leq 11 \\ & x_1, x_2, x_3 \geq 0 \\ & x_1, x_2, x_3 : \text{integer} \end{aligned}$$

   (note: you can use CVX to solve the convex problems needed to form the tree).