Math Programming II - Homework 5

Problem 1

Consider the problem

$$\begin{aligned}&\text{minimize } f(x)\\&\text{subject to } x \in X, \quad g_j(x) \le 0, \quad j = 1, \ldots, r,\end{aligned}$$

where f is continuously differentiable, and $X \subset \mathcal{R}^n$ is a closed convex set. Let x^* be a local minimum.

a) Assume that the functions g_j are convex over X, and that there exists a vector $\bar{x} \in X$ such that $g_j(\bar{x}) < 0$ for all $j = 1, \ldots, r$. Show that there exists a vector $\mu^* \ge 0$ such that

$$x^* \in \arg\min_{x \in X} \left\{ \nabla f(x^*)' x + \sum_{j=1}^{r} \mu_j^* g_j(x) \right\}, \quad \mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r.$$

b) Assume that the functions g_j are continuously differentiable, and that there exists a feasible direction d of X at x^* such that

$$\nabla g_j(x^*)' d < 0, \quad \forall j \in A(x^*).$$

Show that there exists a vector $\mu^* \ge 0$ such that

$$x^* \in \arg\min_{x \in X} \nabla L_x(x^*, \mu^*)' x, \quad \mu_j^* g_j(x^*) = 0, \quad j = 1, \ldots, r.$$

Problem 2

Consider the quadratic penalty method ($c^k \to \infty$) for the equality constrained problem of minimizing $f(x)$ subject to $h(x) = 0$, and assume that the generated sequence converges to a local minimun x^* that is also a regular point. Show that the condition number of the Hessian $\nabla_{xx}^2 L_{c^k}(x^k, \lambda^k)$ tends to ∞.

Problem 3

Let H be a positive definite symmetric matrix. Show that the pair (x^*, μ^*) satisfies the first-order necessary conditions for the problem

$$\begin{aligned}&\text{minimize } f(x)\\&\text{subject to } g_j(x) \le 0, \quad j = 1, \ldots, r,\end{aligned}$$

if and only if $(0, \mu^*)$ is a global minimum-Lagrange multiplier pair of the quadratic program

$$\begin{aligned}&\text{minimize } \nabla f(x^*)' d + \frac{1}{2} d' H d\\&\text{subject to } g_j(x^*) + \nabla g_j(x^*)' d \le 0, \quad j = 1, \ldots, r.\end{aligned}$$