Math Programming II - Homework 2

Problem 1

Consider the steepest descent method

$$x^{k+1} = x^k - \alpha(\nabla f(x^k) + e^k), \tag{1}$$

where α is a constant stepsize, e^k is an error satisfying $\|e^k\| \leq \delta$ for all k, and $f(x)$ is a positive definite quadratic function:

$$f(x) = \frac{1}{2}(x - x^*)'Q(x - x^*). \tag{2}$$

Let

$$q = \max\{|1 - \alpha m|, |1 - \alpha M|\}, \tag{3}$$

where m and M are the smallest and largest eigenvalue of Q. Assuming $q < 1$, show that the following holds for all values of k:

$$\|x^k - x^*\| \leq \frac{\alpha\delta}{1 - q} + q^k\|x^0 - x^*\| \tag{4}$$

Problem 2

Consider the following variant of the steepest descent method:

$$x^{k+1} = x^k - \alpha\nabla f(x^k) + \beta(x^k - x^{k-1}), \quad k = 1, 2, ... \tag{5}$$

where α is a constant positive step size and $\beta \in (0, 1)$. Consider the quadratic function $f(x) = (1/2)x'Qx + c'x$, where Q is positive definite and symmetric, and m and M denote the minimum and maximum eigenvalues of Q. Show that the method converges linearly to the unique solution if $0 < \alpha < 2(1 + \beta)/M$. Prove that with optimal choices of α and β, the ratio of linear convergence for the sequence $\|x^k - x^*\|$ is

$$\frac{\sqrt{M} - \sqrt{m}}{\sqrt{M} + \sqrt{m}} \tag{6}$$

Hint: Write the iterations as

$$\begin{pmatrix} x^{k+1} \\ x^k \end{pmatrix} = \begin{pmatrix} (1+\beta)I - \alpha Q & -\beta I \\ I & 0 \end{pmatrix} \begin{pmatrix} x^k \\ x^{k-1} \end{pmatrix} \tag{7}$$