Math Programming II - Homework 1

Problem 1

Let f: $\mathbb{R}^n \mapsto \mathbb{R}$ and g: $\mathbb{R}^n \mapsto \mathbb{R}$ be twice continuously differentiable functions, and let x^* be a local minimum of f satisfying the second-order sufficient condition. Show that there exist an $\bar{\epsilon} > 0$ and a $\delta > 0$ such that for all $\epsilon \in [0, \bar{\epsilon})$ the function

$$f(x) + \epsilon g(x) \tag{1}$$

has a unique local minimum x_e within the sphere $\{ x \mid \|x - x^*\| < \delta \}$, and we have

$$x_e = x^* - \epsilon(\nabla^2 f(x^*))^{-1} \nabla g(x^*) + o(\epsilon) \tag{2}$$

Hint: Use the implicit function theorem.

Problem 2

Suppose that f is quadratic and of the form $f(x) = \frac{1}{2}x'Qx - b'x$, where Q is positive definite and symmetric.

(a) Show that the Lipschitz condition $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$ is satisfied with L equal to the maximal eigenvalue of Q.

(b) Consider the gradient method $x^{k+1} = x^k - sD\nabla f(x^k)$, where D is positive definite and symmetric. Show that the method converges to $x^* = Q^{-1}b$ for every starting point x^0 if and only if $s \in (0, 2/\bar{L})$, where $\bar{L}$ is the max eigenvalue of $D^{1/2}QD^{1/2}$.

Problem 3

Consider the gradient method $x^{k+1} = x^k + \alpha^k d^k$, where α^k is chosen by the Armijo rule or the line minimization rule and

$$d^k = -\begin{bmatrix} 0 & \cdots & 0 & \frac{\partial f(x^k)}{\partial x_i} & 0 & \cdots & 0 \end{bmatrix}^T \tag{3}$$

where i is the index for which $|\partial f(x^k)/\partial x_j|$ is maximized over $j = 1, ..., n$. Show that every limit point of $\{x^k\}$ is stationary.