Math Programming II - Homework 7

Problem 1

Consider the problems

$$
\begin{aligned}
&\text{minimize } \sum_{i=1}^{p} \|F_i x + g_i\| \\
&\text{subject to } x \in \mathbb{R}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
&\text{minimize } \max_{i=1,\ldots,p} \|F_i x + g_i\| \\
&\text{subject to } x \in \mathbb{R}^n,
\end{aligned}
$$

where F_i and g_i are given matrices and vectors, respectively. Convert these problems to second-order cone programs and derive the corresponding dual problems.

Problem 2

Given $n \times n$ symmetric matrices $M_0, ..., M_r$ and $a_1, ..., a_r \in \mathbb{R}$, consider the problems

$$
\begin{aligned}
\min_{X \succeq 0} \quad & \text{trace}\{M_0 X\} \\
\text{s.t.} \quad & \text{trace}\{M_i X\} = a_i, \qquad i = 1, ..., r
\end{aligned}
\tag{1}
$$

and

$$
\min_{X \succeq 0} \quad \text{trace}\{M_0 X\} + \mu \sum_{i=1}^{r} |\text{trace}\{M_i X\} - a_i| \tag{2}
$$

where μ is a positive constant.

a) By introducing slack variables, reformulate (2) as an SDP.

b) Compute the dual of the SDP obtained in Part (a).

c) Assume that Slater's condition holds for (1) and let $(X^*, \mu_1^*, ..., \mu_r^*)$ denote a primal-dual solution for this problem. By comparing the dual of (1) with the dual obtained in Part (b), show that if $\mu > \max\{|\mu_1^*|, ..., |\mu_r^*|\}$, then X^* is a solution of (2).

Problem 3

Given $n \times n$ symmetric matrices $A_1, ..., A_m$, consider the problem

$$\min_{x \in \mathbb{R}^n} \lambda_{\max}\{(A_1 x_1 + A_2 x_2 + \cdots + A_m x_m) + I \times \left(1 + (x_1 + x_2 + \cdots + x_n)^2\right)\} \tag{3}$$

where $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue and I is the $n \times n$ identity matrix. Formulate the above problem as a linear conic.

Problem 4

Given $n \times n$ symmetric matrices $M_0, ..., M_p$ and $a_1, ..., a_p \in \mathbb{R}$, consider the problem

$$\begin{aligned} \min_{X \in \mathbb{S}^n} \quad & \text{trace}\{M_0 X\} & \\ \text{s.t.} \quad & \text{trace}\{M_i X\} = a_i, & i = 1, ..., p \\ & \begin{bmatrix} X_{i,i} & X_{i,i+1} \\ X_{i+1,i} & X_{i+1,i+1} \end{bmatrix} \succeq 0 & i = 1, ..., n-1 \end{aligned} \tag{4}$$

where $X_{i,j}$ denotes the $(i,j)^{\text{th}}$ entry of X and $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices.

a) By working through the cone of 2×2 positive semidefinite matrices and treating (4) as a conic optimization problem, find the dual of (4).

b) Obtain necessary and sufficient optimality conditions for (4) under Slater's condition.

c) Develop conditions under which both (4) and its conic dual attain their solutions.

Problem 5

Given $n \geq 3$, define $\mathcal{D}$ as the set of tridiagonal positive semidefinite matrices in $\mathbb{S}^n$, where $\mathbb{S}^n$ denotes the set of $n \times n$ symmetric matrices (a matrix is called tridiagonal if all entries not belonging to the main diagonal, the sub-diagonal below it and the sub-diagonal above it are zero).

a) Prove that $\mathcal{D}$ is a convex cone.

b) Prove that $\mathcal{D}$ is not self-dual.