262B-Lecture 25

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$$\max_{X} \ \operatorname{trace}(QX)$$

$$\text{s.t.} \quad X \succeq 0$$

$$X_{ii} = 1 \quad i = 1, \dots, n$$

with no loss of generality, assume $Q \succeq 0$

Nestrov: $\frac{2}{\pi} f_{SDP} \leq f_{original} \leq f_{SDP}$

Proof:

original problem $x \in \{-1, 1\}^n$ $\longrightarrow$ SDP relaxation $X \succeq 0$

$\xrightarrow{\text{solution } X_*}$

(rounding technique)

Consider a probability distribution in $\mathbb{R}^n$ (Gaussian) with mean $\underline{0}$ and covariance X_*

$\xrightarrow{\text{pick a vector}}$ $x_* \in \mathbb{R}^n$

$\xrightarrow{\text{truncate it}}$ $\hat{x} \in \{-1, 1\}^n$

$$\hat{x}_i = \begin{cases} +1 & \text{if } x_i^* \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

$$\mathcal{E}(\hat{x}_i \hat{x}_j) = \frac{2}{\pi} \arcsin(X^*_{ij}) \qquad i,j = 1, \dots, n$$

expected value

$$\mathcal{E}(\hat{x}\hat{x}^T) = \frac{2}{\pi} \arcsin(X_*) \quad , \quad (1)$$

elementwise

- easy to show: $\arcsin(X_*) \succeq X_* \quad , \quad (2)$

- $\hat{x}$ is feasible for original problem, but not optimal

$$f_{original} \geq \mathcal{E}(\hat{x}^T Q \hat{x}) = \mathcal{E}(\mathrm{trace}(Q^{\frac{1}{2}} \hat{x}\hat{x}^T Q^{\frac{1}{2}}))$$

$$= \mathrm{trace}(Q^{\frac{1}{2}} \mathcal{E}(\hat{x}\hat{x}^T) Q^{\frac{1}{2}})$$

(1), (2) $\Rightarrow$ $\geq \mathrm{trace}(Q^{\frac{1}{2}} X_* Q^{\frac{1}{2}}) \times \frac{2}{\pi}$

$$= \mathrm{trace}(Q X_*) \times \frac{2}{\pi}$$

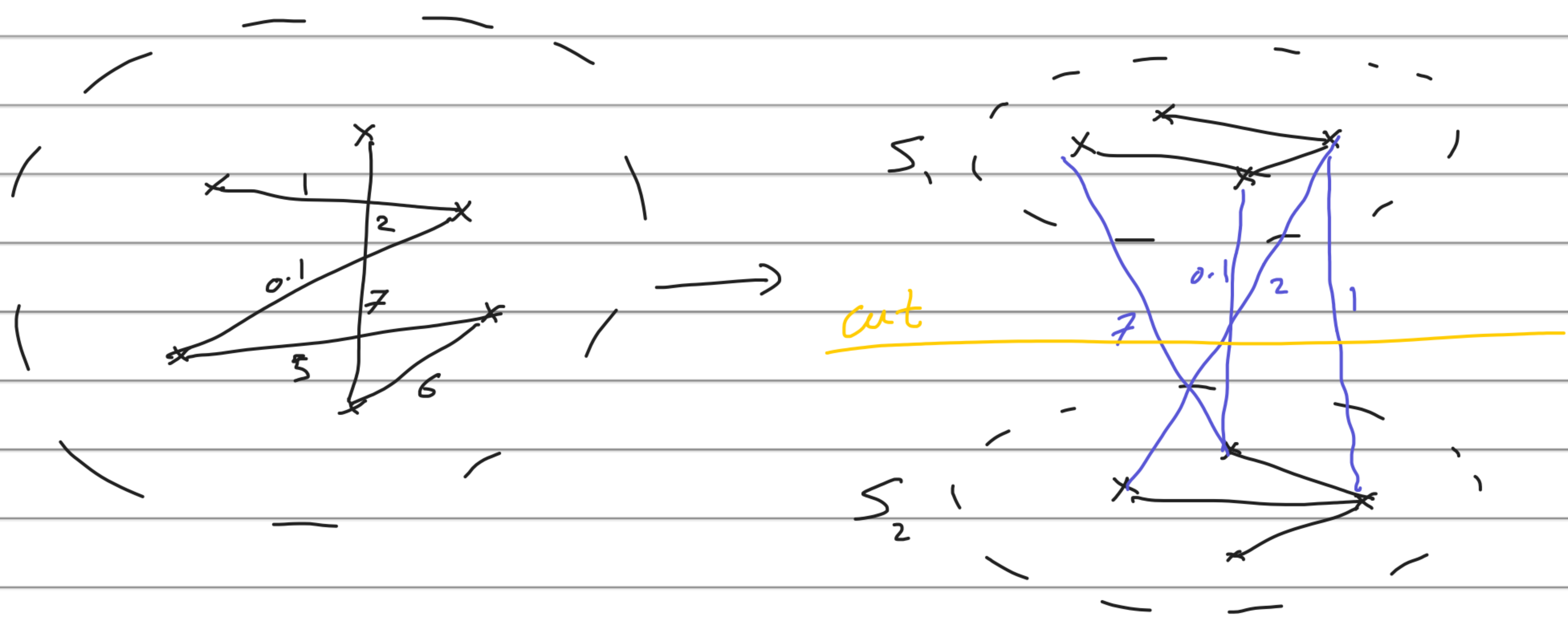
$$= f_{SDP} \times \frac{2}{\pi}$$

1- SDP relaxation gives good bound

2- generate $\hat{x}$ to find a solution.

special case : max cut

Given a graph with non-negative edge weights, partition the vertices into two sets s.t. the sum of weights for edges between two sets is maximum.



NP-hard $\longrightarrow$ many graph problems are similar.

W_{ij} = weight for edge (i,j)

x_i = variable for vertex i $\to +1$ if $i \in S_1$; $\to -1$ if $i \in S_2$

cut value: $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}(1-x_i x_j)}{4}$

- $(i,j) \in S_1 \Rightarrow 1 - x_i x_j = 0$
- $(i,j) \in S_2 \Rightarrow 1 - x_i x_j = 0$
- $i,j \in$ different sets $\Rightarrow 1 - x_i x_j = 1 - (+1)(-1) = 2$

$$1 - x_j x_i = 2$$

$$\Rightarrow \frac{w_{ij}(1 - x_i x_j)}{4} + \frac{w_{ji}(1 - x_j x_i)}{4} = w_{ij}$$

max cut: $\displaystyle\max_{x} \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{w_{ij}(\overset{x_i^2}{1} - x_i x_j)}{4}$

s.t. $\quad x_i^2 = 1 \quad i = 1, \dots, n$

$\Rightarrow$ $\max\ x^T Q x$ s.t. $x_i^2 = 1 \quad i = 1, \dots, n$

Q: non-positive off-diagonal entries

strengthen previous inequality:

$$\overset{0.87}{\alpha \times \frac{2}{\pi}} \times f_{SDP} \le \underbrace{f_{max-cut}}_{\text{original}} \le f_{SDP}$$

$$\min_{0 \le \theta \le \pi} \frac{\theta}{1-\cos\theta}$$

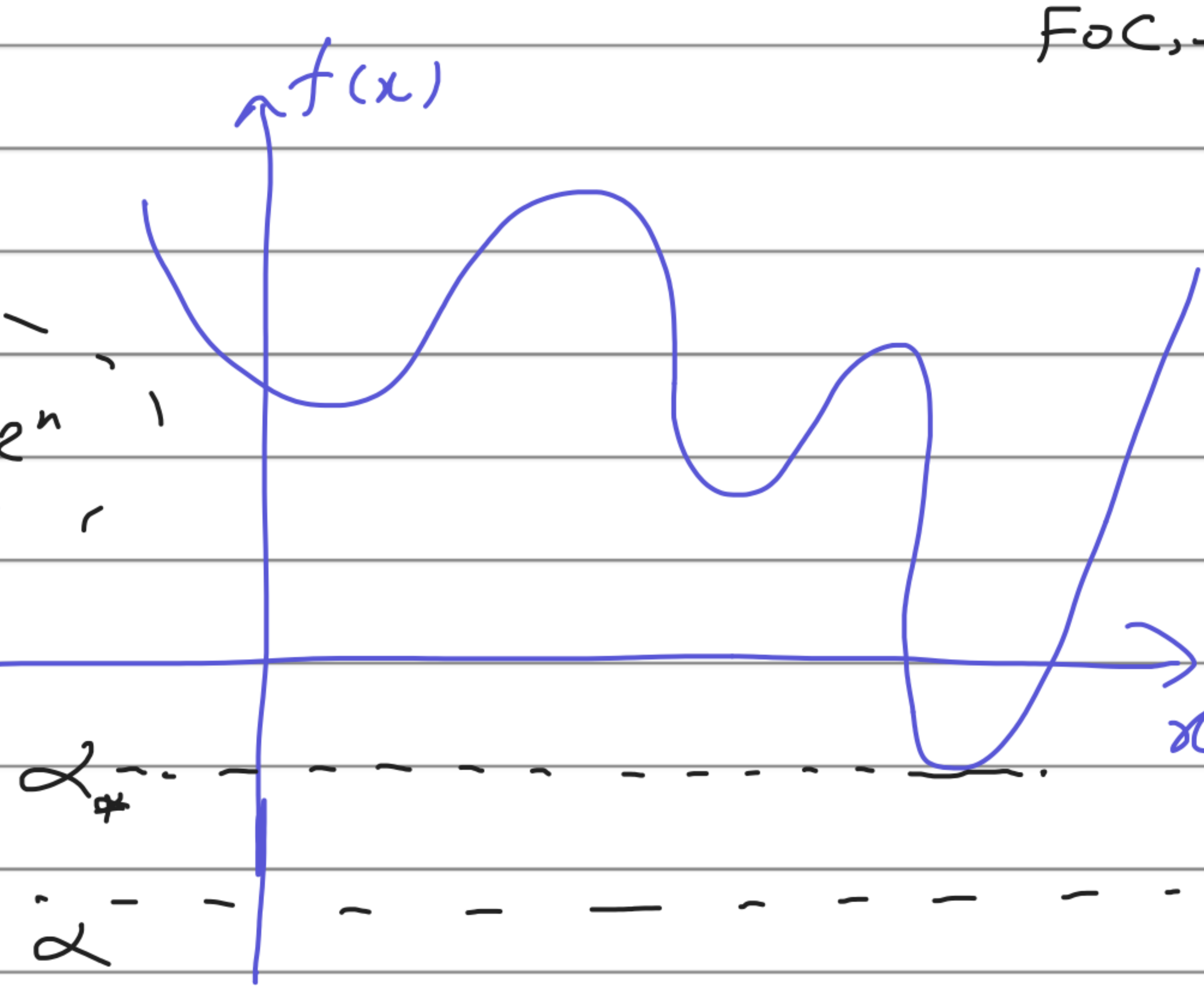
$\Longrightarrow$ optimality gap using SDP $\le 13\%$

$$\min_{x} f(x) \xrightarrow{\text{epigraph}} \begin{aligned} &\min_{x,\alpha} \ \alpha \\ &\text{s.t.} \ f(x) \le \alpha \end{aligned}$$

find global min

study local/global via FOC, SOC

$$\begin{aligned} &\max_{\alpha} \ \alpha \\ &\text{s.t.} \ \underbrace{f(x) - \alpha \ge 0 \quad \forall x \in \mathbb{R}^n}_{\text{infinitely many inequalities}} \end{aligned}$$



General case: $\min \; f(x)$

s.t. $h_i(x) = 0 \quad i = 1, \dots, m$

$g_j(x) \leq 0 \quad j = 1, \dots, r$

global min

$$\max_{\alpha} \; \alpha \quad \text{s.t.} \quad \underbrace{f(x) - \alpha}_{f_\alpha(x)} \geq 0 \quad \forall x \in S$$

semi-algebraic set $S =$

$$\{ x \mid h_i(x) = 0 \;\; i = 1, \dots, m, \;\; g_j(x) \leq 0 \;\; j = 1, \dots, r \}$$

- How to check whether a polynomial $f_\alpha(x)$ is non-negative over a set S?

- If $f_\beta(x) \geq 0$ over S, then β is a lower bound on a global min.

Def: A polynomial $p(x)$ is called non-negative if $p(x) \geq 0 \quad \forall x \in \mathbb{R}^n$

Def: A polynomial $p(x)$ is called a sum of squares (SOS) if $\exists$ polynomials $q_1(x), \ldots, q_k(x)$ s.t.

$$p(x) = q_1(x)^2 + q_2(x)^2 + \cdots + q_k(x)^2$$

(By polynomial: real-valued coefficients)

Obvious:

$$\begin{tikzcd} \text{SOS} \arrow[r] & \text{non-negative} \arrow[l, bend left, dashed, "?"] \end{tikzcd}$$

(No)

$\exists\ p(x)$: $p(x) \geq 0$ and $p(x) \neq$ SOS

Motzkin Counterexample:

$$p(x) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3 x_1^2 x_2^2 + 1$$

SOS = non-negative polynomials only

special cases:

1 - $n = 1$ (univariate)

2 - quadratic (degree = 2)

3 - $n = 2$ and degree = 4

$$p(x) = \sum_{i=1}^{k} q_i(x)^2$$

$$q_i(x) = \underbrace{q_i^T}_{\text{vector of coefficients}} \times [x]_d \rightarrow \text{vector of monomials of degree at most } d$$

Ex: $x_1^2 x_2^2 + x_1^4 - x_2^4 + x_1 + 5$

$$= \begin{bmatrix} 1 & 1 & -1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1^2 x_2^2 \\ x_1^4 \times x_2^0 \\ x_1^0 \times x_2^4 \\ x_1^1 \times x_2^0 \\ x_1^0 \times x_2^0 \end{bmatrix}$$

(define d: max degree of $q_i(x)$)

$$\Rightarrow p(x) = \sum_{i=1}^{k} q_i(x)^2 = \sum_{i=1}^{k} [x]_d^T q_i q_i^T [x]_d$$

$$= [x]_d^T \left(\sum_{i=1}^{k} q_i q_i^T \right) [x]_d \rightarrow \succeq 0$$

Thm: $\underbrace{p(x)}_{\text{degree } 2d} = SOS$ if and only if

$\exists \; Q \in \mathbb{R}^{\binom{n+d}{d} \times \binom{n+d}{d}}$ s.t.

$$p(x) = [x]^T_{\binom{n+d}{d}} \; Q \; [x]_{\binom{n+d}{d}} \quad , \quad Q \succeq 0$$

($\binom{n+d}{d}$: # of monomials in a polynomial of degree d)

example:

$$p(x) = 5x_1^2 x_2^2 + x_1^2 + x_2^2 + x_1 \overset{?}{=} SOS$$

$$= [1 \;\; x_1 \;\; x_2 \;\; x_1x_2 \;\; x_1^2 \;\; x_2^2] \underset{(6\times 6)}{Q} \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ x_1x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$$

match coefficients:

$$x_1^2 x_2^2 : \quad 5 = \underset{(x_1x_2)(x_1x_2)}{Q_{44}} + \underset{(x_1^2)(x_2^2)}{Q_{56}} + \underset{(x_2^2)(x_1^2)}{Q_{65}}$$

checking $p(x) \stackrel{?}{=}$ SOS

$$\Updownarrow$$

whether $\exists Q$: $\underbrace{Q \succeq 0 \ , \ \underbrace{p(x) = [x]^T Q [x]}_{\text{a bunch of Linear equation in } Q}}_{\text{(feasibility) SDP}}$

Farkas Lemma:

$$h_i(x) = a_i^T x + b_i \qquad i = 1, \dots, m$$

$$-g_j(x) = c_j^T x + d_j \qquad j = 1, \dots, r$$

$$f(x) = e^T x + f$$

- $f(x)$ is non-negative over S if and only if $\exists$ $(\lambda_1, \dots, \lambda_m)$, $(\mu_1, \dots, \mu_j) \geq 0$, $\mu_0 \geq 0$ s.t. $f(x) = \mu_0 + \sum_{i=1}^{m} \lambda_i h_i(x) - \sum_{j=1}^{r} \mu_j g_j(x)$:

1

$\Updownarrow$

$$e = \sum_{i=1}^{m} \lambda_i a_i + \sum_{j=1}^{r} \mu_j c_j ,$$

$$f = \mu_0 + \sum_{i=1}^{m} \lambda_i b_i + \sum_{j=1}^{r} \mu_j d_j ,$$ 2

- $f(x)$ is non-negative over S if and only if

$$\nexists x \quad \text{s.t} \quad f(x) < 0 , \; g_j(x) \le 0 , \; h_i(x) = 0 ;$$ 3

2 $\longleftrightarrow$ 3 Farkas Lemma

If f, h_i, g_j are nonlinear, then

$\lambda_i \longrightarrow \lambda_i(x)$

$\mu_j \longrightarrow \mu_j(x) = SOS$

$$\begin{cases} \min\limits_{x} \ f(x) \\ \text{s.t.} \ \ h_i(x) = 0 \quad i = 1, \dots, m \\ \qquad (g_j(x) \le 0) \quad j = 1, \dots, r \end{cases}$$

$\longrightarrow$ feasible set $= S$

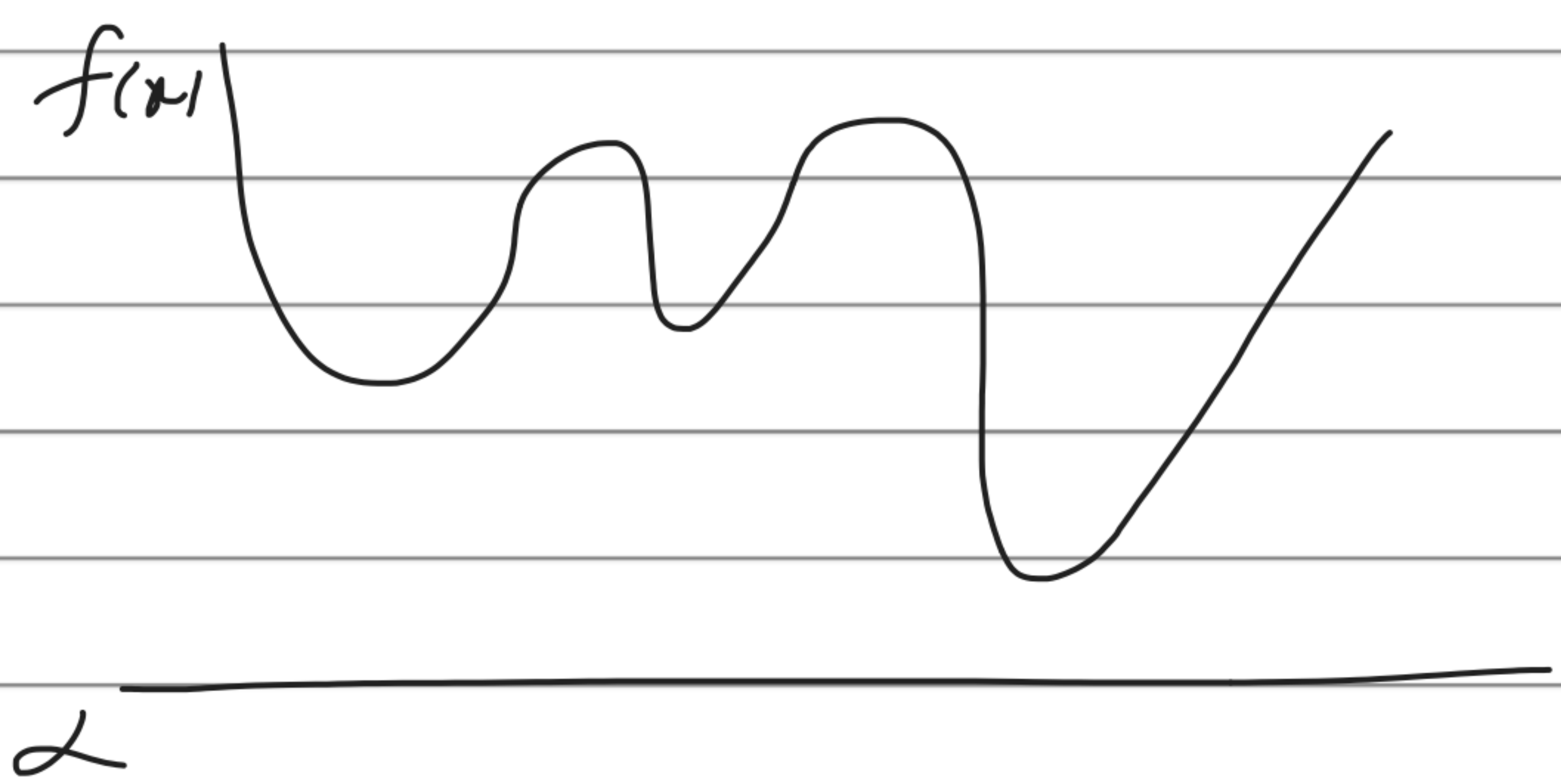
$(-g_j(x) \ge 0)$

$$\underbrace{(-g_1(x))(-g_2(x)) \ge 0}_{\text{valid inequality}}$$

valid inequalities:

$$(-g_1(x))^{\tau_1} (-g_2(x))^{\tau_2} \cdots (-g_r(x))^{\tau_r} \ge 0$$

$\tau_1, \dots, \tau_r \in \{0, 1\} \implies 2^r - 1$ valid inequalities



$f(x) - \alpha$ is non-negative over

$$\tilde{S} = \{ h_i(x) = 0, \ i = 1, \dots, m, \ (-g_1(x))^{\tau_1} \cdots (-g_r(x))^{\tau_r} \ge 0, \ \tau_i \in \{0,1\} \}$$

Define:

$$F(x) = \{ k(x) \mid$$

set of polynomials

$$k(x) = \underbrace{\mu_0(x)}_{SOS} + \sum_{i=1}^{m} \lambda_i(x)\, h_i(x) + \sum_{j=1}^{2^r-1} (-g_1(x))^{\tau_1} \cdots (-g_r(x))^{\tau_r} \times \underbrace{\mu_{(\tau_1,\dots,\tau_r)}(x)}_{SOS}$$

Psatz (Positivstellensatz):

$$\underbrace{p(x)}_{f(x)-\alpha} > 0 \text{ on } S \text{ if and only if}$$

$$\exists\ k_1(x), k_2(x) \in F(x) \quad \text{s.t.}$$

$$\underbrace{k_1(x)\, p(x) = 1 + k_2(x)}_{*}$$

How to check satisfaction of $*$?

Coefficients of polynomials $\lambda_i(x)$, $\mu_I(x)$ in $K_1(x)$, $K_2(x)$

$\longrightarrow$ pick the degrees (d)

$$\begin{array}{ccccccc} d=1 & \longrightarrow & d=2 & \longrightarrow & d=3 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ SDP & & SDP & & SDP & & \end{array}$$

$\Longrightarrow$ a hierarchy of SDPs.

Issues: $2^r - 1$ terms (valid inequalities)

Putinar: under some constraint qualification, if $p(x) > 0$ on S, then

$$p(x) = \underbrace{\mu_0(x)}_{SOS} + \sum_{i=1}^{m} \lambda_i(x) h_i(x) + \sum_{j=1}^{r} \underbrace{\mu_j(x)}_{SOS} (-g_j(x))$$

To summarize:

$$\min \ f(x)$$
$$\text{s.t.} \ h_i(x) = 0 \qquad i = 1, \dots, m$$
$$g_j(x) \leq 0 \qquad j = 1, \dots, r$$

Find a global min

$$\max \ \alpha$$
$$\text{s.t.} \ f(x) - \alpha = \underbrace{\mu_0(x)}_{SOS} + \sum_{i=1}^{m} \lambda_i(x) h_i(x)$$
$$- \sum_{j=1}^{r} \underbrace{\mu_j(x)}_{SOS} g_j(x)$$

Assume: $\mu_0, \lambda_i, \mu_j = d$

$$d = 1, \quad d = 2, \quad d = 3, \quad \dots$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$SDP \qquad SDP \qquad SDP$$

$$\alpha_1^* \leq \alpha_2^* \leq \alpha_3^* \leq \dots \longrightarrow \text{global min}$$