262B-Lecture 22

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Conic programming:

$$\min_{x} \quad f(x) \qquad \text{s.t.} \qquad x \in X \cap C$$

- X : convex set
- f : convex on X (and not just $X \cap C$)
- C : closed, convex cone

- epigraph $\{(x, w) \mid x \in X,\ f(x) \leq w\} \rightarrow$ closed

Fenchel duality: $\min_{x} \ f_1(x) - f_2(x)$

$\text{s.t.} \quad x \in X_1 \cap X_2$

$f_1(x) = f(x), \quad f_2(x) = 0, \quad X_1 = X, \quad X_2 = C$

$q_1(\lambda) = \sup_{x \in X} \left(-f_1(x) + \lambda^T x \right), \quad \Lambda_1 = \{ \lambda \mid q_1(\lambda) < +\infty \}$

$q_2(\lambda) = \inf_{x \in C} \left(0 + \lambda^T x \right), \quad \Lambda_2 = \{ \lambda \mid q_2(\lambda) > -\infty \}$

If $\exists\ \bar{\lambda}, \bar{x} \in C$ s.t. $\bar{\lambda}^T \bar{x} < 0$

$\Rightarrow \bar{\lambda}^T (\alpha \bar{x}) \rightarrow -\infty$ as $\alpha \rightarrow \infty$, $\alpha \bar{x} \in C$ due to cone

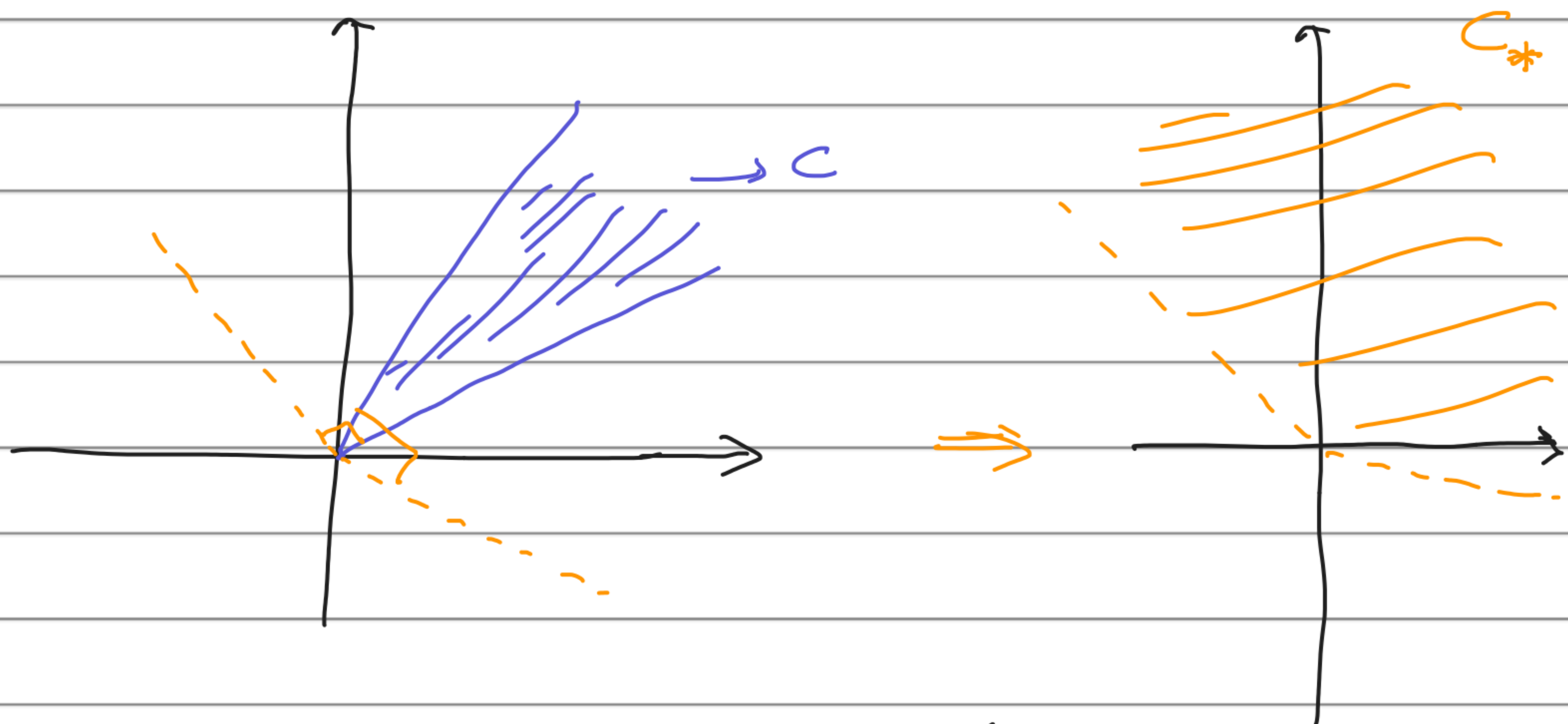
$\Rightarrow \quad \Lambda_2 = \underbrace{C_*}_{\text{dual cone}} = \{ \lambda \mid \lambda^T x \geq 0 \quad \forall x \in C \}$

$$\Rightarrow \quad q_2(\lambda) = \begin{cases} 0 & \text{if } \lambda \in C_* \\ -\infty & \text{if } \lambda \notin C_* \end{cases}$$

(note 0 belongs to cone due to closed-ness)

$\Rightarrow$ Dual opt: $\max_{\lambda} \; q_1(\lambda)$
s.t. $\lambda \in \Lambda_1 \cap C_*$

example:



If $C = C_*$, then C is called self-dual.

Primal		Dual
$\min_{x} \; f(x)$ s.t. $x \in X \cap C$	$\longrightarrow$	$\max_{\lambda} \; -q_1(\lambda)$ s.t. $\lambda \in \Lambda_1 \cap C_*$

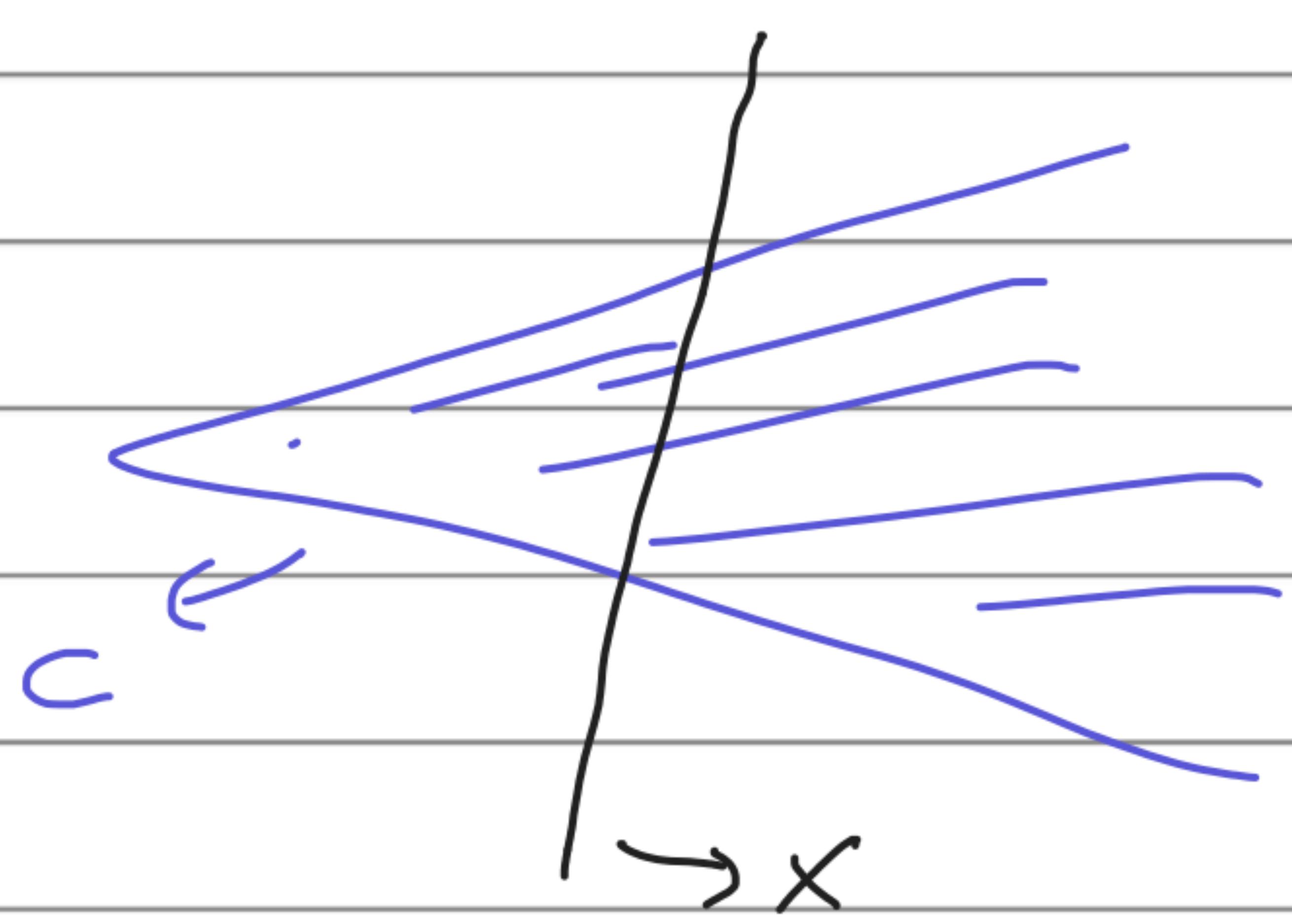
Thm: If primal problem has a finite objective value and if $ri(X) \cap ri(C) \neq \emptyset \Longrightarrow$

1- no duality gap

2- existence of dual solution

special case: Linear - Conic

- X: affine $\Longrightarrow X = \underbrace{b}_{\text{vector}} + \underbrace{S}_{\text{subspace}}$
- f: Linear $\Longrightarrow f(x) = a^T x \quad \forall x \in b + S$



$\Longrightarrow X \cap C$:

feasible set

Linear - Conic programming : minimize a Linear function over the intersection of an affine set and a convex cone.

Canonical: $\begin{cases} \min & a^T x \\ \text{s.t.} & x - b \in S \;,\quad x \in C \end{cases}$

$$\Rightarrow \quad q_1(\lambda) = \sup_{x \in S + b} \left(-a^T x + \lambda^T x \right)$$

$$= \sup_{\underbrace{x-b}_{y} \in S} (\lambda - a)^T x$$

$$= \sup_{y \in S} (\lambda - a)^T (y + b)$$

$$= \sup_{y \in S} \underbrace{(\lambda - a)^T y}_{\text{Linear}} + \underbrace{(\lambda - a)^T b}_{\text{no } \underline{y} \text{ in it}}$$

$$= \begin{cases} (\lambda - a)^T b & \text{if } (\lambda - a) \in S^\perp \\ +\infty & \text{if } (\lambda - a) \notin S^\perp \end{cases}$$

orthogonal space

Dual optimization :

$$\max_{\lambda} \; -b^T\lambda + a^T b \rightarrow \text{Constant}$$

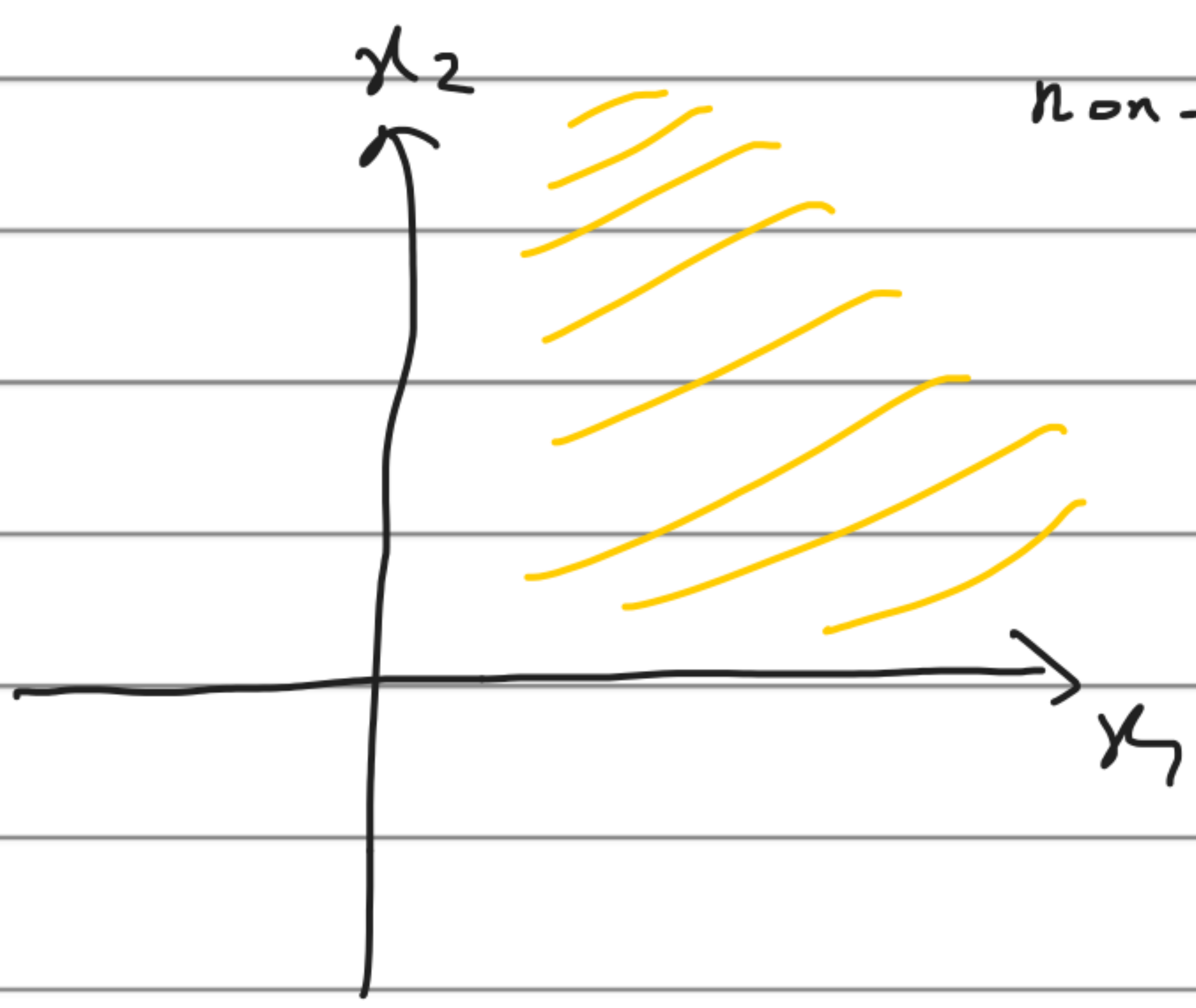
$$\text{s.t.} \quad \lambda - a \in S^{\perp}, \quad \lambda \in C_*$$

Note : $ri(b+S) = b+S$

So, If primal has optimal objective value & $(b+S) \cap ri(C) \neq \emptyset \Rightarrow$ no duality gap + existence of solution for dual.

special case: $C = \{x \mid x_i \geq 0 \quad i=1,\dots,n\}$

non-negative orthant



$$\Rightarrow \quad C = C_*$$

Canonical form of LP:

$$\begin{cases} \min & a^T x \\ \text{s.t.} & Ax = b \longrightarrow \text{affine constraint} \\ & x \geq 0 \longrightarrow x \in C \end{cases}$$

LP: Linear - conic

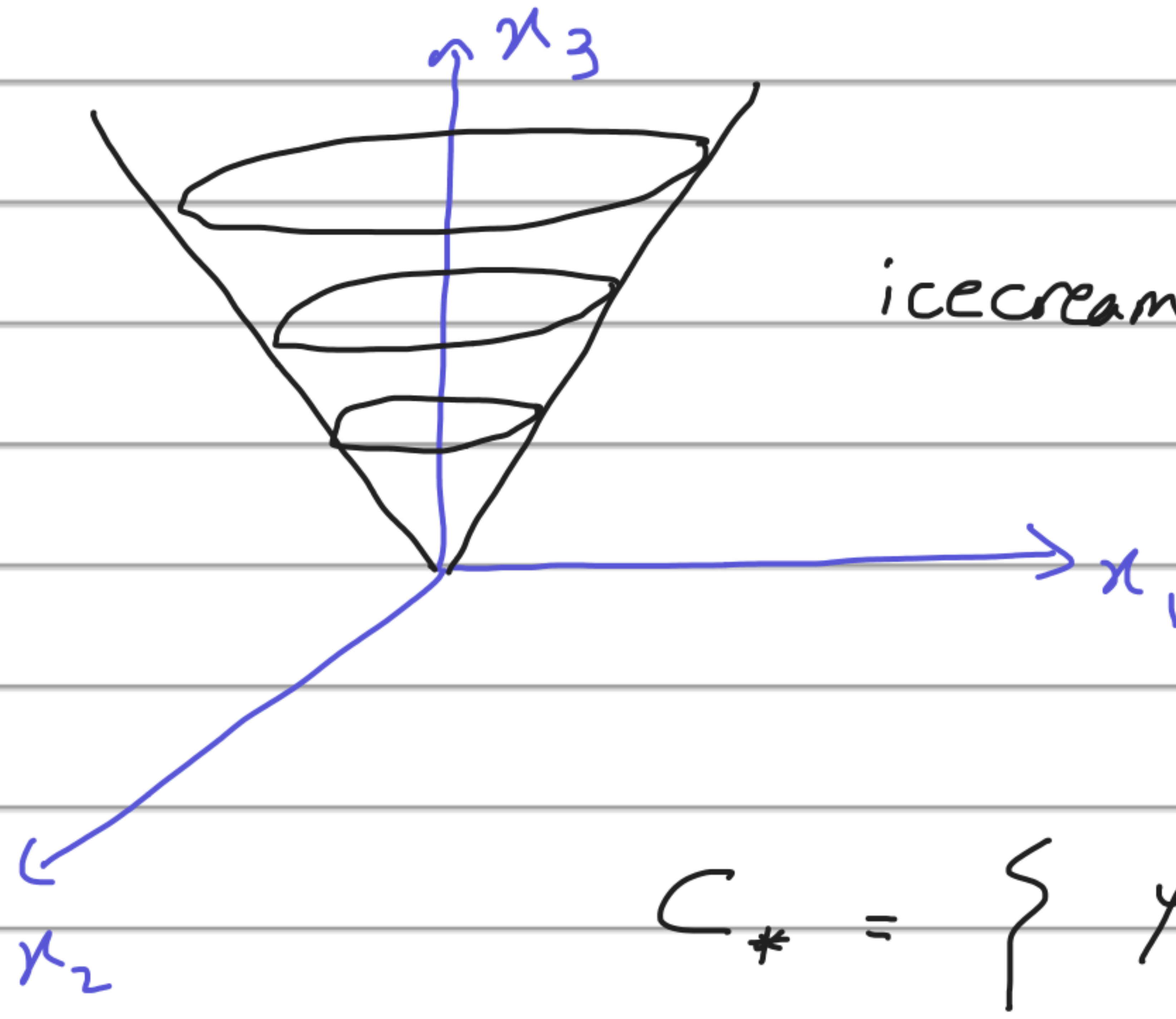
Primal		Dual
$\min_x \; a^T x$ s.t. $Ax = b$ ← μ: Lagrange multiplier $x \in C$	⟶	$\max_\mu \; b^T \mu$ $a - A^T \mu \in C$

SOCP: second-order cone programming

$LP \leq QP \leq SOCP$

second-order cone in $\mathbb{R}^n$:

$$C = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n \;\middle|\; x_n \geq \underbrace{\sqrt{x_1^2 + x_2^2 + \dots + x_{n-1}^2}}_{\left\| \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \right\|} \right\}$$



icecream cone for $n = 3$

$$C_* = \left\{ y \in \mathbb{R}^n \;\middle|\; y^T x \geq 0 \quad \forall x \in C \right\}$$

$$\min_x \; y^T x \qquad \text{s.t.} \quad x \in C$$

$$\downarrow$$

$$\min_x \; y_1 x_1 + \dots + y_n x_n \qquad \text{s.t.} \quad x_n \geq \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \right\|$$

$$\downarrow$$

$$\min_x \; \underbrace{\begin{bmatrix} y_1 & \cdots & y_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}}_{\text{minimize this linear term}} + y_n x_n \qquad \text{s.t.} \quad x_n \geq \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \right\|$$

$$\downarrow$$

$$\min_x \; -\left\| \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \right\| x_n + y_n x_n \longrightarrow \min_{x_n \geq 0} \left(y_n - \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \right\| \right) x_n$$

(note: $x_n \geq \left\| \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \right\| \geq 0$)

$$\begin{array}{ll} \min\limits_{x} & y^T x \\ \text{s.t.} & x \in C \end{array} = \begin{cases} 0 & \text{if } \underbrace{y_n - \left\| \begin{bmatrix} y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} \right\| \geq 0}_{y \in C} \\ -\infty & \text{otherwise} \end{cases}$$

$\Rightarrow$ $C_* = C$ = self-dual

SOCP: $$\begin{cases} \min\limits_{x} & a^T x \\ \text{s.t.} & \underbrace{A_i}_{n_i \times n} x - \underbrace{b_i}_{n_i \times 1} \in \underbrace{C_i}_{\text{second-order cone in } \mathbb{R}^{n_i}} \qquad i = 1, \dots, m \end{cases}$$

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \quad \underbrace{C = C_1 \times C_2 \times \dots \times C_m}_{\text{Product of second-order cones}}$$

So, $\min\limits_{x} \; a^T x \quad \text{s.t.} \quad Ax - b \in C$

LP $\subseteq$ SOCP :

$$\begin{cases} \text{equality} \longleftrightarrow \text{inequalities} \\ e^T x = f \longleftrightarrow e^T x - f \geq 0 \ , \ -(e^T x - f) \geq 0 \end{cases}$$

inequalites $\longrightarrow$ Conic Constraints

example: $e^T x - f \geq 0 \Rightarrow e^T x - f \geq \sqrt{0^2} = 0$

$$\Rightarrow \begin{bmatrix} 0 \\ e^T \end{bmatrix} x - \begin{bmatrix} 0 \\ f \end{bmatrix} \in \text{second-order cone in } \mathbb{R}^2$$

Dual:
$$\begin{cases} \max\limits_{\lambda_1,\dots,\lambda_m} & \sum\limits_{i=1}^{m} b_i^T \lambda_i \\ \text{s.t.} & \sum\limits_{i=1}^{m} A_i^T \lambda_i = a \\ & \lambda_i \in C_i \quad i = 1,\dots,m \end{cases}$$

Thm: - If primal has optimal objective value and $\exists\ \bar{x}$: $A_i \bar{x} - b_i \in \text{int}(C_i)$ $\quad i = 1,\dots,m$

$\Rightarrow$ no duality gap + solution for dual

- If dual has optimal objective value and

$\exists \ \bar{\lambda}_1, \dots, \bar{\lambda}_m$ s.t. $\bar{\lambda}_i \in \text{int}(C_i)$ $i = 1, \dots, m$

and $\sum_{i=1}^{m} A_i^T \bar{\lambda}_i = a$ $\implies$

no duality gap + solution for primal

Algebraic model of SOCP:

$$\min_x \quad a^T x$$

$$\text{s.t.} \quad \| \underbrace{A_i}_{n_i \times n} x + \underbrace{b_i}_{n_i \times 1} \|_2 \leq \underbrace{e_i^T}_{n \times 1} x + \underbrace{f_i}_{1 \times 1}$$

left-side: norm of a Linear function

right-side Linear function

$$QP \overset{?}{\subseteq} SOCP$$

$$QP: \begin{cases} \min \ x^T \underset{\succeq 0}{P_0} x + q_0^T x + r_0 \\ \text{s.t.} \ Ax = b \\ \quad\quad Cx \leq d \end{cases}$$

epigraph

$$\min_{x,\alpha} \quad \alpha$$

$$\text{s.t.} \quad Ax = b$$

$$Cx \leq d$$

$$\underbrace{x^T P_0 x + q_0^T x + r_0}_{\| P_0^{1/2} x + P_0^{-1/2} q_0 \|^2 + \text{constant}} \leq \alpha$$

$$\min \quad a^T x$$

$$\text{s.t.} \quad \underbrace{\begin{bmatrix} A_i \\ e_i^T \end{bmatrix} x - \begin{bmatrix} -b_i \\ -f_i \end{bmatrix}}_{\begin{bmatrix} A_i x + b_i \\ e_i^T x + f_i \end{bmatrix}} \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

Dual:

$$\max_{\lambda_1, \dots, \lambda_m} \quad \sum_{i=1}^{m} [-b_i^T \quad -f_i] \lambda_i$$

$$\text{s.t.} \quad \sum_{i=1}^{m} [A_i^T \quad e_i] \lambda_i = a$$

$$\lambda_i \in \text{second-order cone in } \mathbb{R}^{n_i+1}$$

$$i = 1, \dots, m$$

$$\lambda_i \in \mathbb{R}^{n_i+1} \longrightarrow \begin{bmatrix} \underbrace{v_i}_{\in \mathbb{R}^{n_i}} \\ \underbrace{\mu_i}_{\in \mathbb{R}} \end{bmatrix}$$

Dual:

$$\begin{aligned} \max_{v_i, \mu_i} \quad & -\sum_{i=1}^{m} b_i^T v_i - \sum_{i=1}^{m} f_i \mu_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} A_i^T v_i + \sum_{i=1}^{m} e_i \mu_i = a \\ & \|v_i\|_2 \le \mu_i \qquad i = 1, \dots, m \end{aligned}$$

algebraic dual (SOCP)

alternative approach:

$$\begin{aligned} \min \quad & a^T x \\ \text{s.t.} \quad & \|A_i x + b_i\| \le e_i^T x + f_i \qquad i = 1, \dots, m \end{aligned}$$

raise both sides to power 2 loses convexity

↓

Lagrangian ⟶ non-differentiable ⟶ hard to minimize

↓

dual function ⟶ dual optimization ⟹ same solution

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP

SDP: semidefinite programming

positive semidefinite (PSD) Cone:

$$C = \{ X \mid X \in \mathbb{R}^{n \times n},\ X = X^T \succeq 0 \}$$

$\downarrow$ vectorize

$$\mathrm{vec}(C) = \{ \mathrm{vec}(X) \in \mathbb{R}^{n^2} \mid X \in C \}$$

So, C can be regarded as $n \times n$ matrices

or $n^2 \times 1$ vectors.

Dual Cone: $(x^T y) \geq 0$

If $X, Y \in C \implies \underbrace{\mathrm{vec}(X)^T \mathrm{vec}(Y)}_{\mathrm{trace}(XY) = \langle X, Y \rangle} \geq 0$

(inner product)

$$\implies C_* = \{ Y \mid \mathrm{trace}(XY) \geq 0 \quad \forall X \in C \}$$

symmetric

Thm: $C = C_*$ (self-dual)

proof: 1- If $Y \in C \Rightarrow \exists M : Y = MM^T$

$\Rightarrow \forall \underbrace{X}_{\geq 0} \in C : \langle X, Y \rangle = M^T \underbrace{X}_{\geq 0} M \geq 0$

$\Rightarrow Y \in C_*$

2- If $Y \notin C \Rightarrow \exists x \in \mathbb{R}^n$ s.t.

$x^T Y x < 0 \Rightarrow \text{trace}(Y \underbrace{x x^T}_{X}) < 0$

$\Rightarrow \text{trace}(YX) < 0$ for some $X \in C$

$\Rightarrow Y \notin C_*$

SDP: minimize a linear objective over intersection of an affine set and PSD cone.

$$\text{SDP:} \quad \min_{X} \ \langle \overset{\text{symmetric}}{D}, X \rangle \longrightarrow \text{Linear}: \ \operatorname{vec}(D)^T \operatorname{vec}(X)$$

$$\text{s.t.} \quad \langle \underbrace{A_i}_{\text{symmetric}}, X \rangle = b_i \qquad i=1,\dots,m$$

$$X \in C \qquad (X \succeq 0)$$

Compare with canonical LP:

$$\min_{x} \ a^T x$$

$$\text{s.t.} \quad Ax = b$$

$$x \geq 0$$

SDP is a major generalization of L

by replacing $\{x \mid x \geq 0\}$ with $\{X \mid X \succeq 0\}$

(vector optimization $\longrightarrow$ matrix optimization)

$$X = \begin{bmatrix} x_1 & & & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & & & x_n \end{bmatrix} \Rightarrow \boxed{x \geq 0 \iff X \succeq 0}$$

$$LP \subseteq SDP$$