262B-Lecture 21

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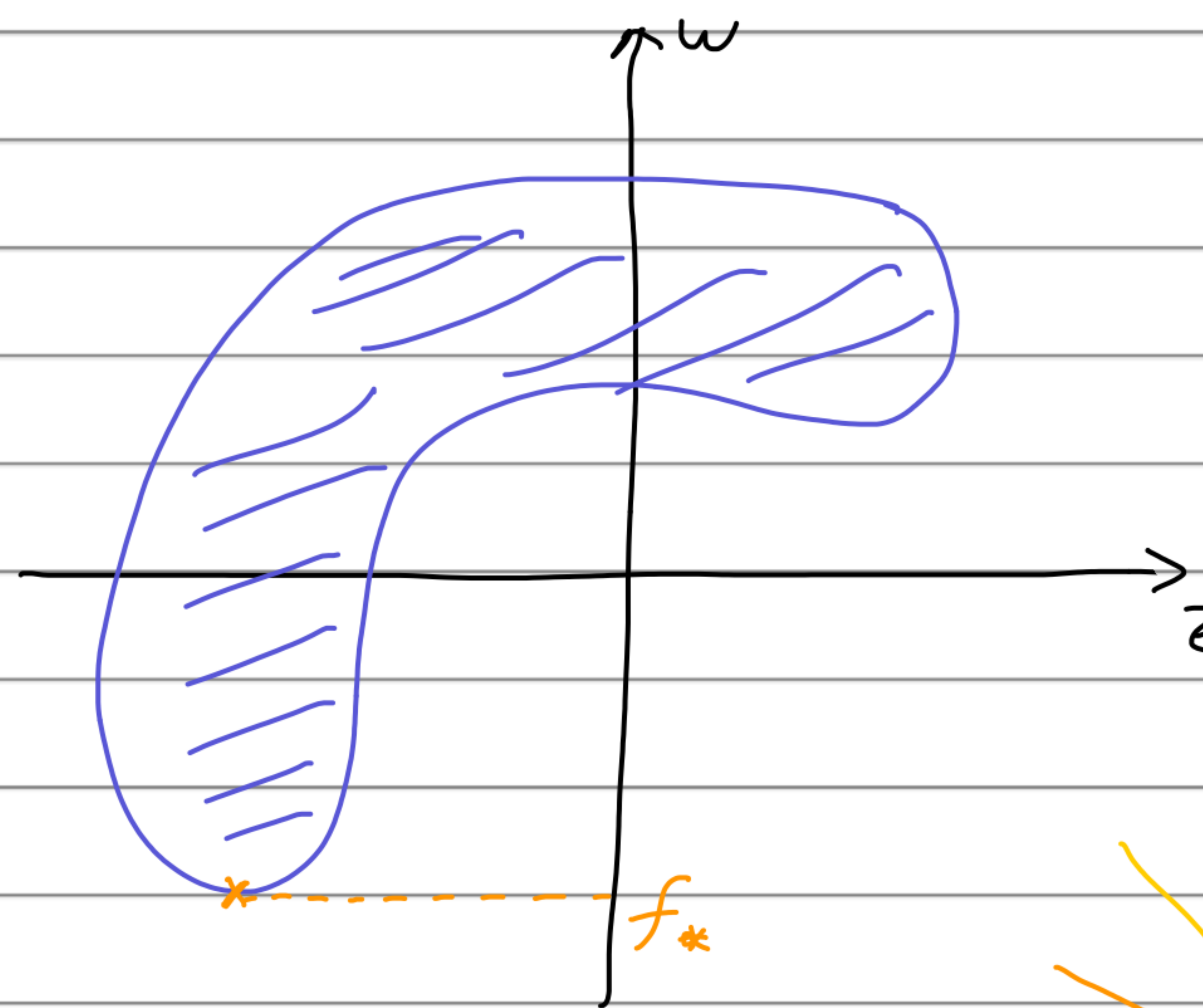
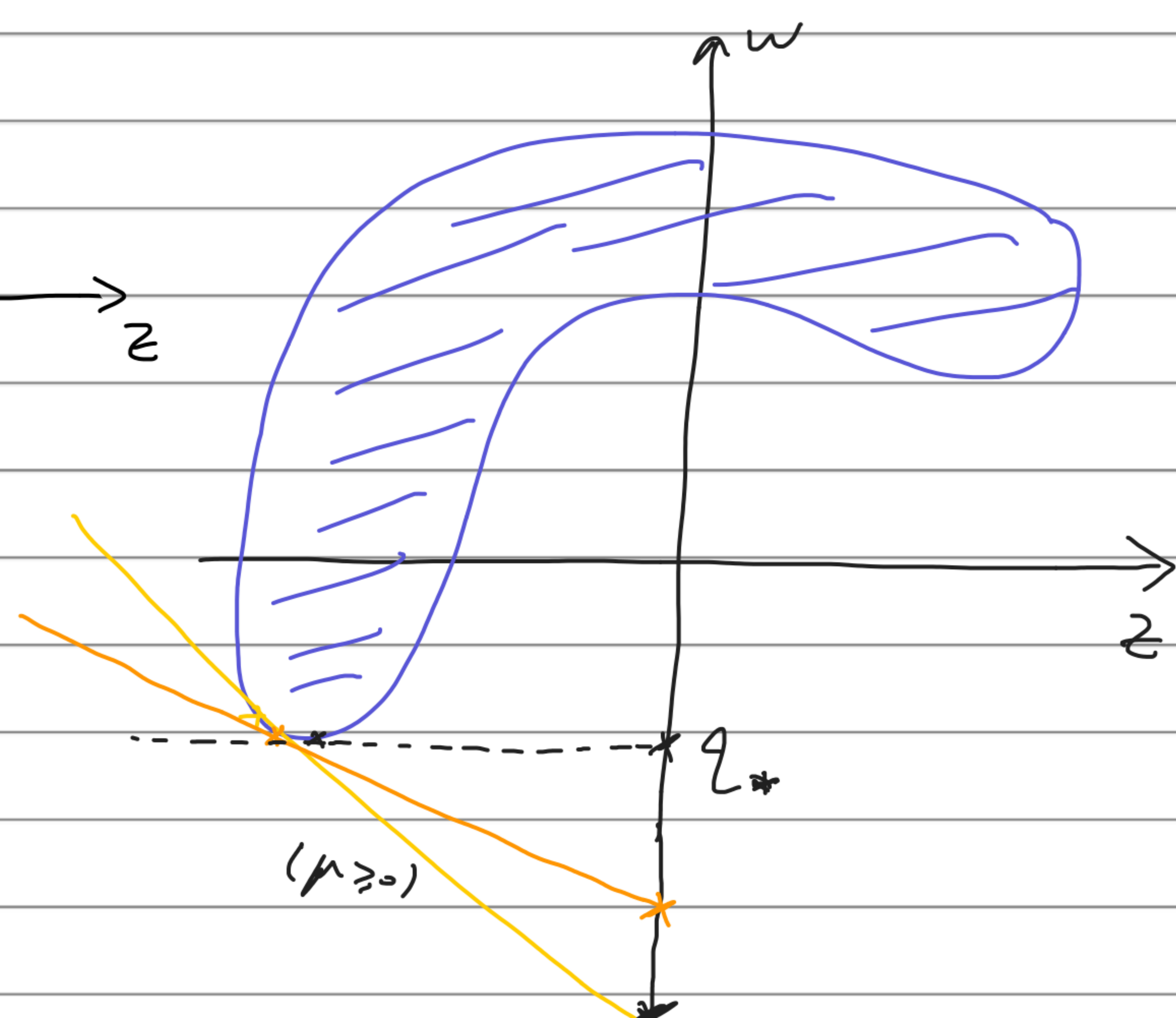
weak duality : $\inf_{x \in X} L(x, \mu) \leq f_*$

$\Rightarrow$ strong duality holds and μ_* is a geometric multiplier if and only if $\exists\, \mu_*^{\geq 0}$ s.t. the hyperplane with the normal vector $\begin{bmatrix} \mu_* \\ 1 \end{bmatrix}$ and with the highest interception with the w-axis and with S on its positive side gives the interception level f_*.

$\left(\min_{x \in X} f(x) \quad \text{s.t.} \quad g(x) \leq 0 \right)$

$\left(S = \{ (\underbrace{g(x)}_{z}, \underbrace{f(x)}_{w}) \mid x \in X \} \right)$

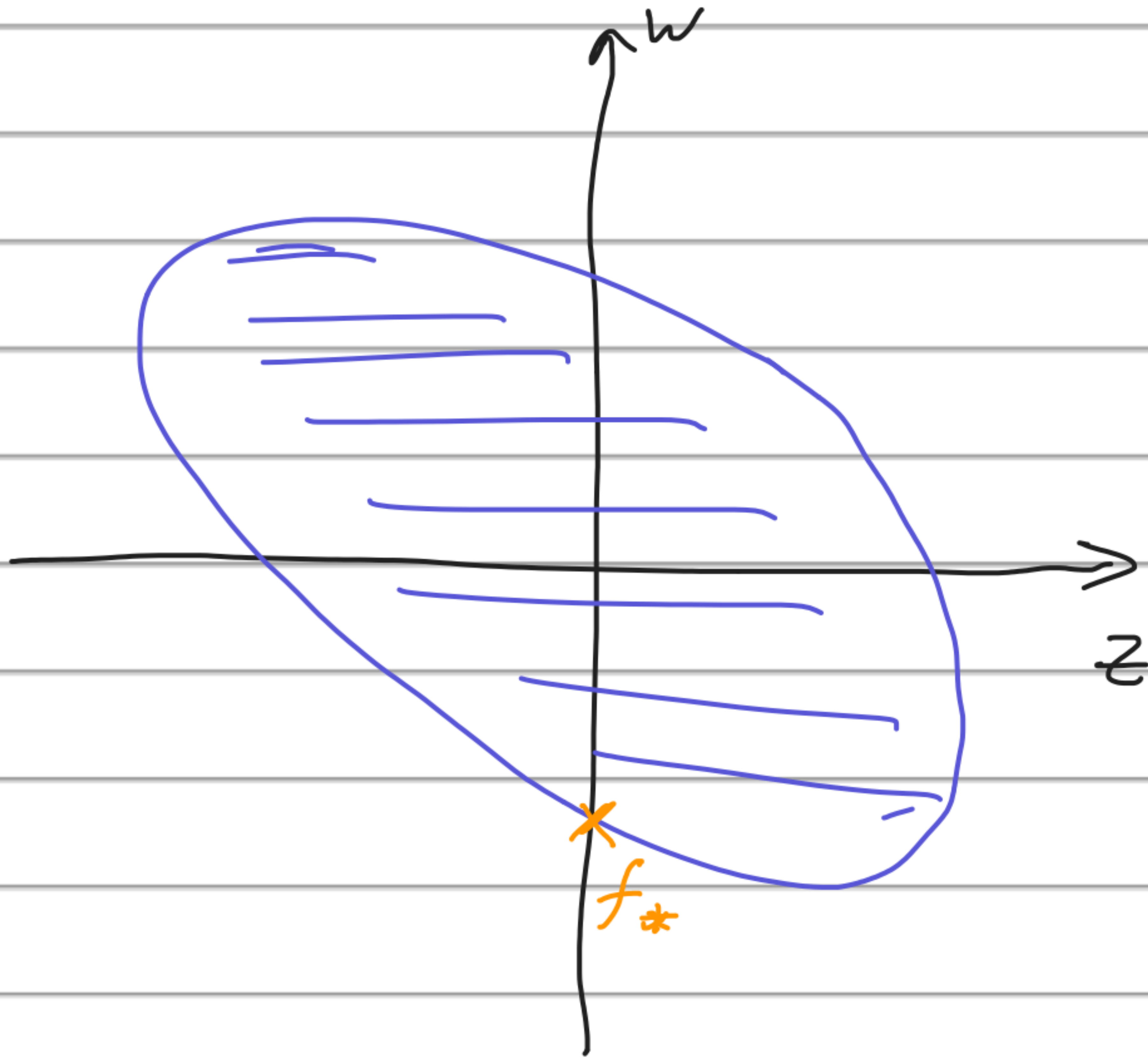


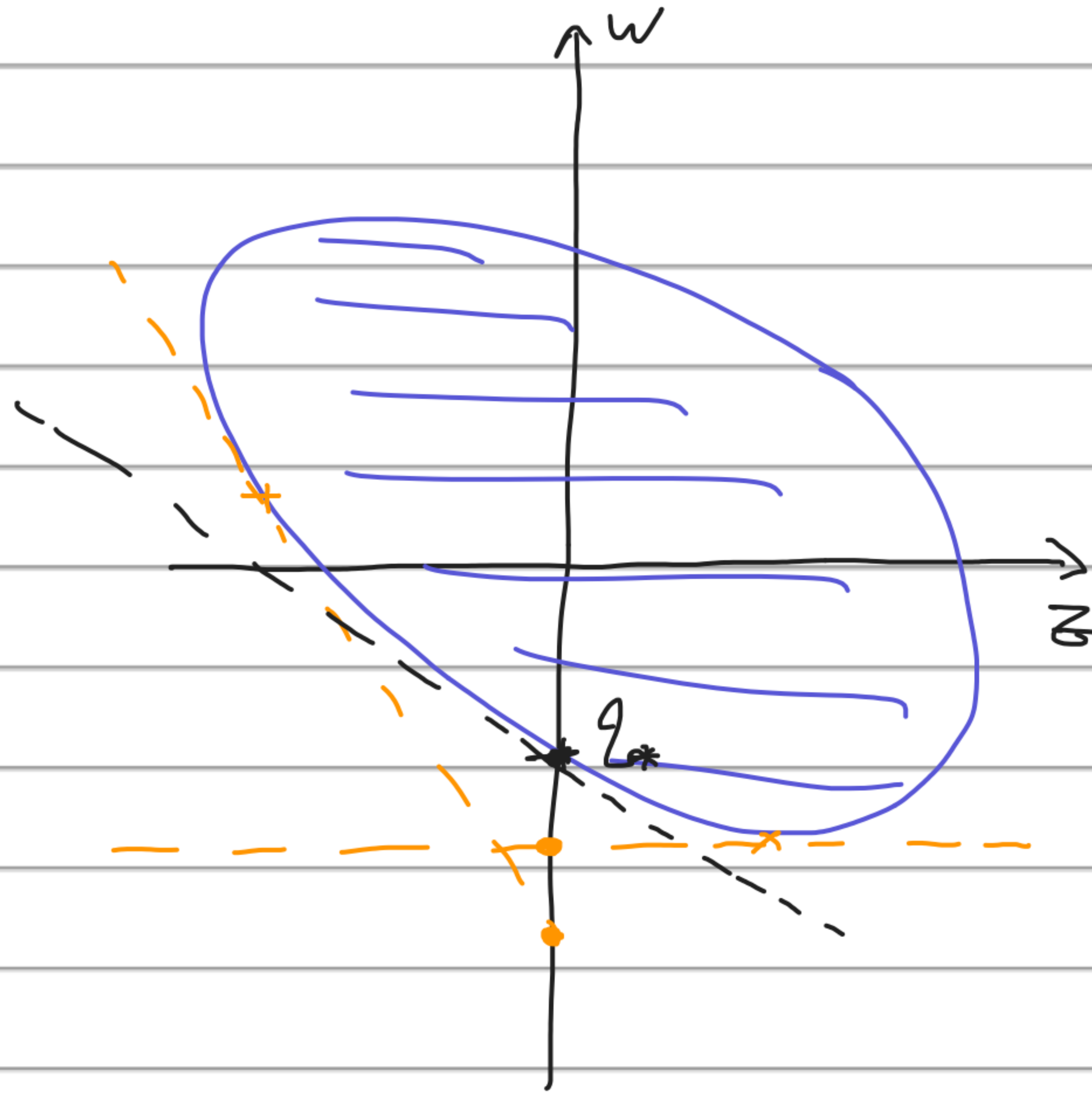


$\Rightarrow f_* = q_*$ (non-convex)

$z_* = g(x_*) < 0 \;, \; \mu_* = 0$

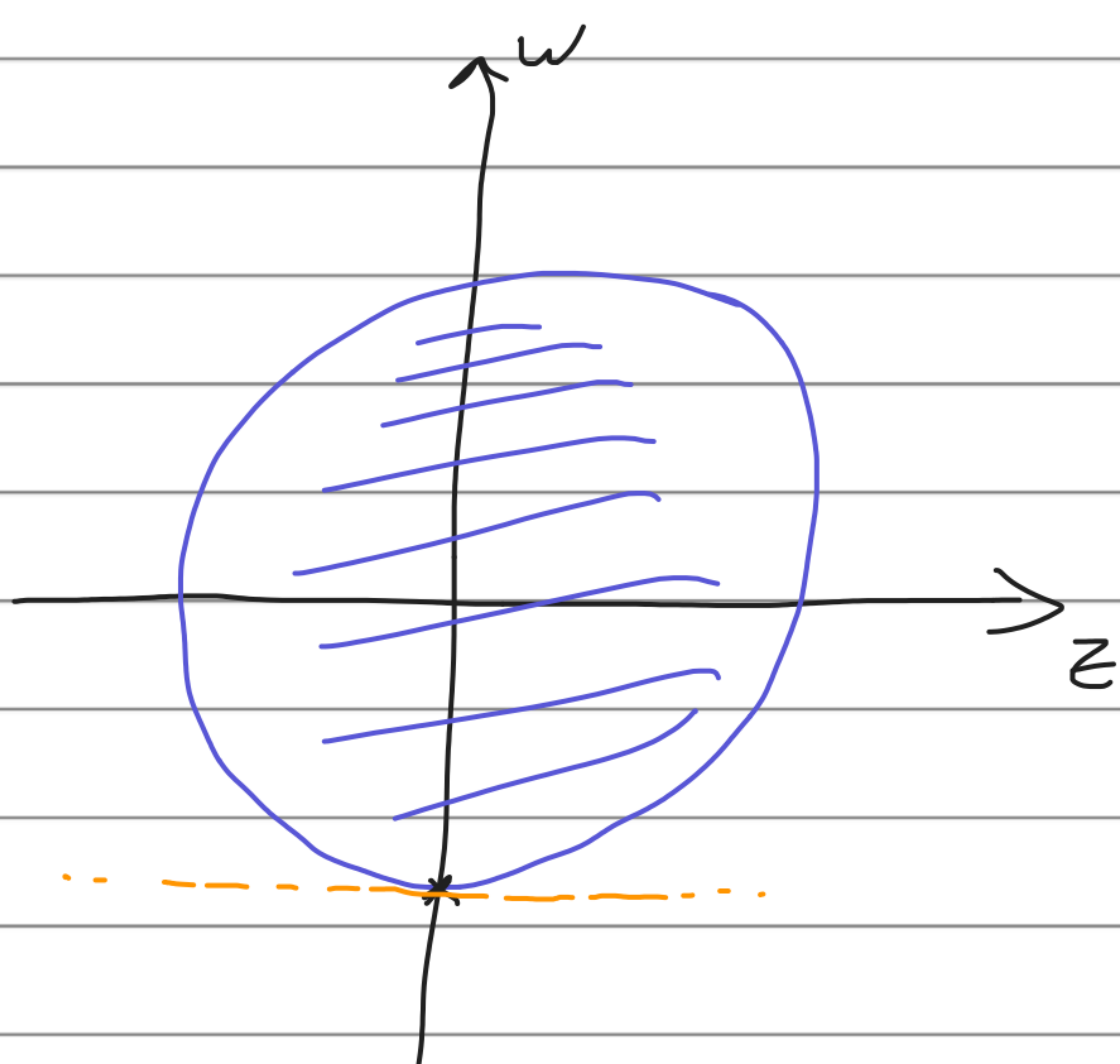
$\Rightarrow \quad \mu_* g(x_*) = 0$: complementary slackness





$\Rightarrow \quad f_* = q_*$

$z_* = g(x_*) = 0$, $\mu_* > 0$ $\Rightarrow$ $\mu_* g(x_*) = 0$
complementary slackness

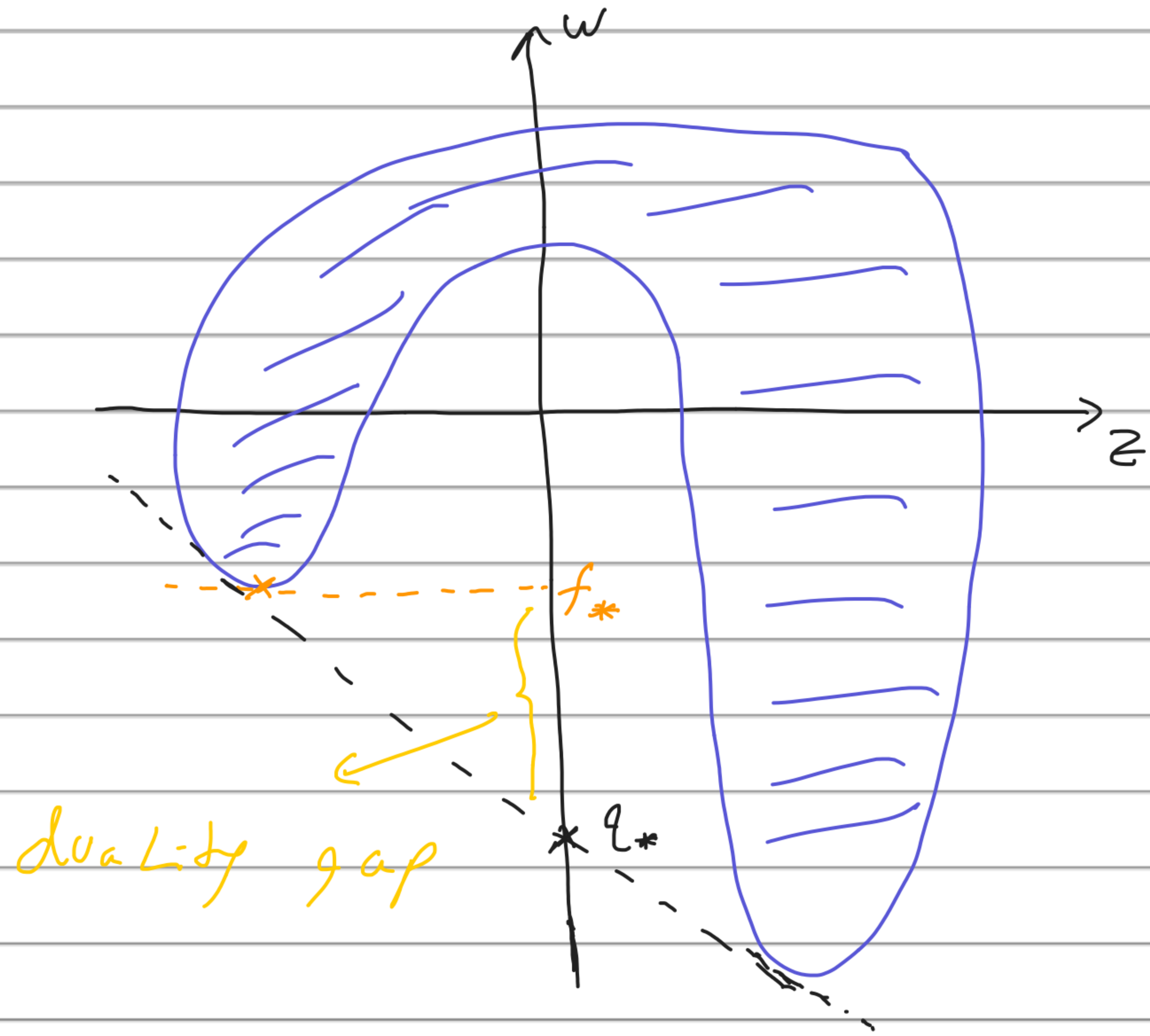


$f_* = q_*$

but $\mu_* = g(x_*) = 0$

degenerate case

Non-zero duality gap:



The issue with the definition of geometric multiplier is that: $f_* = \inf_{x \in X} L(x, \lambda_*, \mu_*)$

f_* is unknown. So, we should somehow drop f_*.

Thm: A tuple (x_*, λ_*, μ_*) is a global min - geometric multipliers if and only if

1 - primal feasibility: $x_* \in X$, $h(x_*)=0$, $g(x_*)\leq 0$

2 - dual feasibility: $\mu_* \geq 0$

3 - Lagrangian optimality $\quad \underbrace{x_* \in \arg\min_{x\in X} L(x,\lambda_*,\mu_*)}_{\text{global min}}$

(stronger than stationarity)

4 - Complementary slackness: $\mu_j^* g_j(x_*)=0$ $\quad j=1,\dots,r$

Saddle point theorem: A tuple (x_*,λ_*,μ_*) is an optimal (global) solution - geometric multipliers

$\underbrace{\text{if and only if}}$

1 - $x_* \in X$

2 - $\mu_* \geq 0$

3 - (x_*,λ_*,μ_*) is a saddle point for Lagrangian:

B $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ A

$$L(x_*,\lambda,\mu) \leq L(x_*,\lambda_*,\mu_*) \leq L(x,\lambda_*,\mu_*) \quad \forall x\in X, \lambda, \mu\geq 0$$

A: x_* is a min

B: (λ_*, μ_*) is a max

So, at the point (x_*, λ_*, μ_*) the Lagrangian increases in the direction of x and decreases in the direction of (λ, μ).

proof:

1- Assume (x_*, λ_*, μ_*) is min - geometric multipliers.

$$\Rightarrow \begin{cases} x_* : \text{primal solution} \\ (\lambda_*, \mu_*) : \text{dual solution} \end{cases} \Rightarrow A \text{ holds}$$

Also, $L(x_*, \lambda, \mu) = f(x_*) + \sum_{i=1}^{m} \lambda_i \underbrace{h_i(x_*)}_{=0}$

$+ \sum_{j=1}^{r} \underbrace{\mu_j}_{\geq 0} \underbrace{g_j(x_*)}_{\leq 0} \leq f(x_*) = L(x_*, \lambda_*, \mu_*)$

$\Rightarrow$ B holds.

2 - Assume A, B hold for some $x_* \in X$, λ_*, $\mu_* \geq 0$

$$\sup_{\lambda, \mu \geq 0} L(x_*, \lambda, \mu) = \sup_{\lambda, \mu \geq 0} \Big(f(x_*) + \sum_{i=1}^{m} \lambda_i h_i(x_*) + \sum_{j=1}^{r} \underbrace{\mu_j}_{\geq 0} g_j(x_*) \Big)$$

Linear in λ, μ

$h_i(x_*) \neq 0$ or $g_j(x_*) \cancel{\leq} 0$ since we just assumed $x_* \in X$

$$= \begin{cases} f(x_*) & \text{if } h(x_*) = 0,\ g(x_*) \leq 0 \\ +\infty & \text{otherwise} \end{cases}$$

B holds $\Rightarrow$ $\sup_{\lambda, \mu \geq 0} L(x_*, \lambda, \mu) \leq \underbrace{L(x_*, \lambda_*, \mu_*)}_{\text{finite}}$

$\Rightarrow h(x_*) = 0$, $g(x_*) \leq 0$

Then, solution of sup: $\Rightarrow \mu_j^* g_j(x_*) = 0$

Also, A implies $\Rightarrow x_* \in \operatorname{argmin}_{x \in X} L(x, \lambda_*, \mu_*)$

$\Rightarrow$ All 4 conditions are satisfied strong duality in special cases:

$$\begin{cases} \min & f(x) \longrightarrow \text{Convex} \\ \text{s.t.} & a_i^T x = b_i \quad i = 1, \dots, m \\ & c_j^T x \leq d_j \quad j = 1, \dots, r \\ & X = \text{polyhedral} \end{cases}$$

Case $\underline{1}$: finite f_*

$\Rightarrow$ no duality gap and there exists at least one geometric multiplier (dual solution)

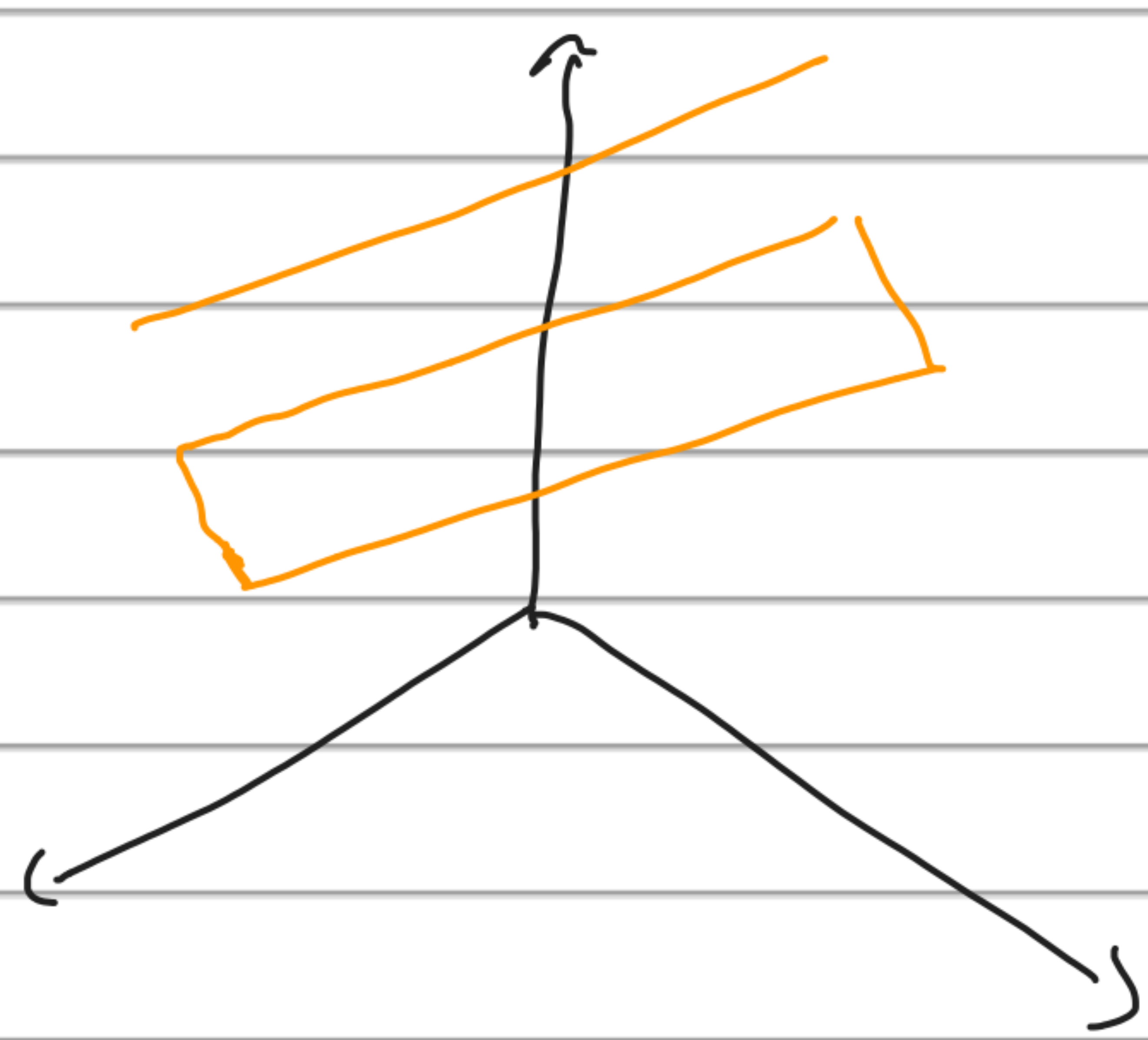
Case $\underline{2}$: finite f_*, $f(x) = \text{quadratic}$ (QP with X constraint)

$\Rightarrow$ no duality gap and primal & dual solutions exist.

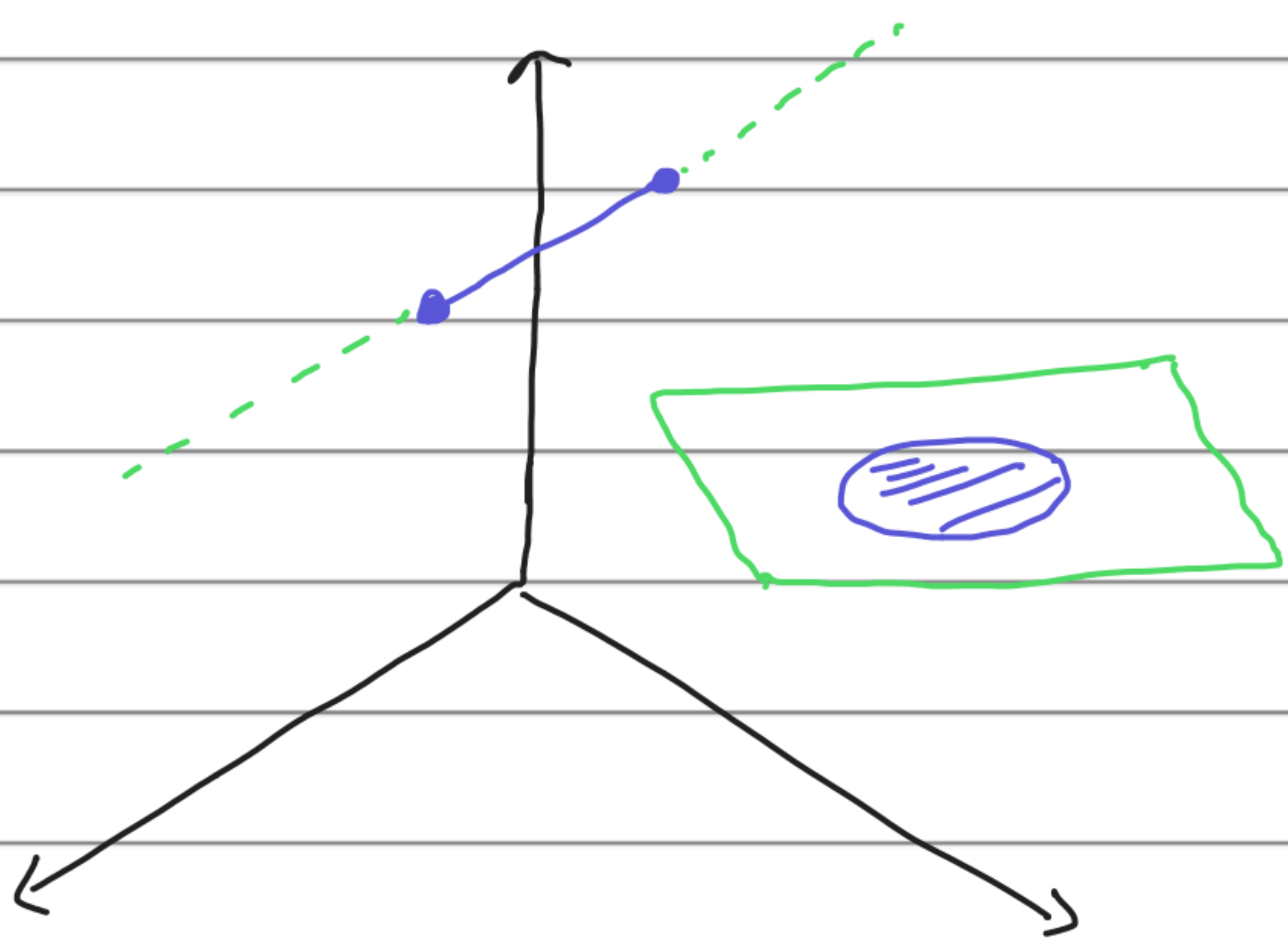
Relative interior:

Linear manifold: $b + S$

b ← Constant, S ← subspace

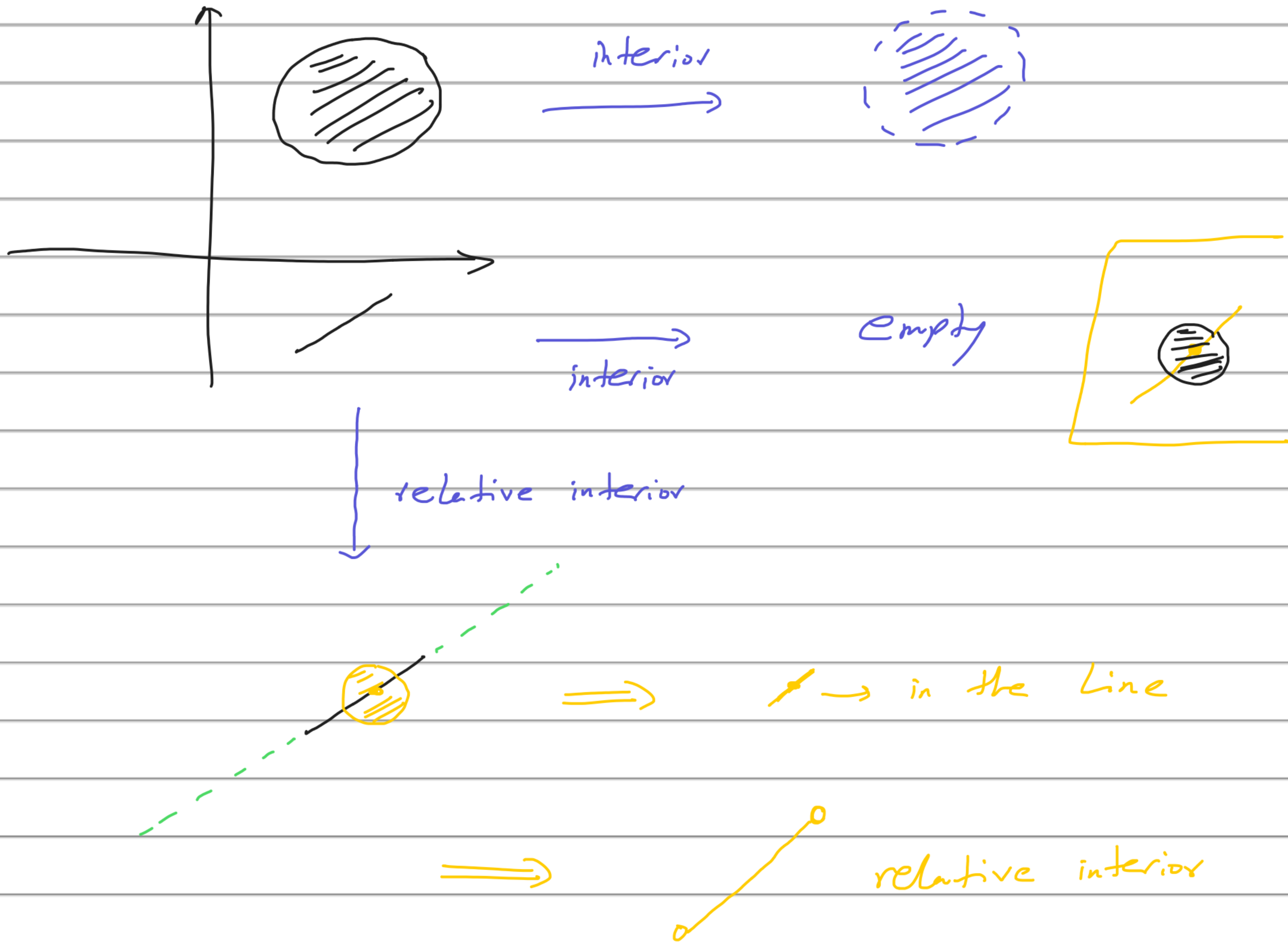


- Given a feasible set F, define its affine hull as the intersection of all linear manifolds containing F.



- Relative interior of F: $x \in ri(F)$ if $\exists$ a neighborhood (ball) of x called N s.t.

$N \cap affine(F) \subset F$



Two scenarios:

1- $\min f(x)$

$\text{s.t. } g_j(x) \leq 0 \quad j = 1, \dots, r$ (no equalities)

$x \in X$

Assume: - f_* finite

- Convex opt (f : convex, g_j : convex,
X : convex)

- $\exists\ \bar{x}$ s.t.

$$\underbrace{\bar{x} \in X}_{\text{no need to be in interior}}, \quad \underbrace{g_j(\bar{x}) < 0}_{\text{strictly negative}}$$

$\Rightarrow$ no duality gap & dual solution exists.

2 - $\min\ f(x)$

s.t. $g_j(x) \leq 0 \quad j = 1, \dots, r$ $\longrightarrow$ non-linear inequalities

$a_i^T x = b_i \quad i = 1, \dots, m$ $\longrightarrow$ Linear equalities

$c_k^T x \leq d_k \quad k = 1, \dots, l$ $\longrightarrow$ Linear inequalies

$x \in X$

Assume: - Convex opt (f, g_j, X = convex)

- f_* finite

- X = intersection of a polyhedral & convex set C.

(special case: C = cone)

- $\exists\, \bar{x}$: feasible, $g_j(\bar{x}) < 0 \quad j = 1, \dots, r$

, $\bar{x} \in ri(C)$

$\Rightarrow$ no duality gap & existence of dual solution.

(need to have a feasible point satisfying non-linear inequalities in a strict way and belonging to the relative interior of C.)

Fenchel duality:

$$\begin{cases} \min\limits_{x \in \mathbb{R}^n} & f_1(x) - f_2(x) \\ \text{s.t.} & x \in X_1 \cap X_2 \end{cases}$$

$$\Downarrow$$

(idea used extensively in distributed computation)

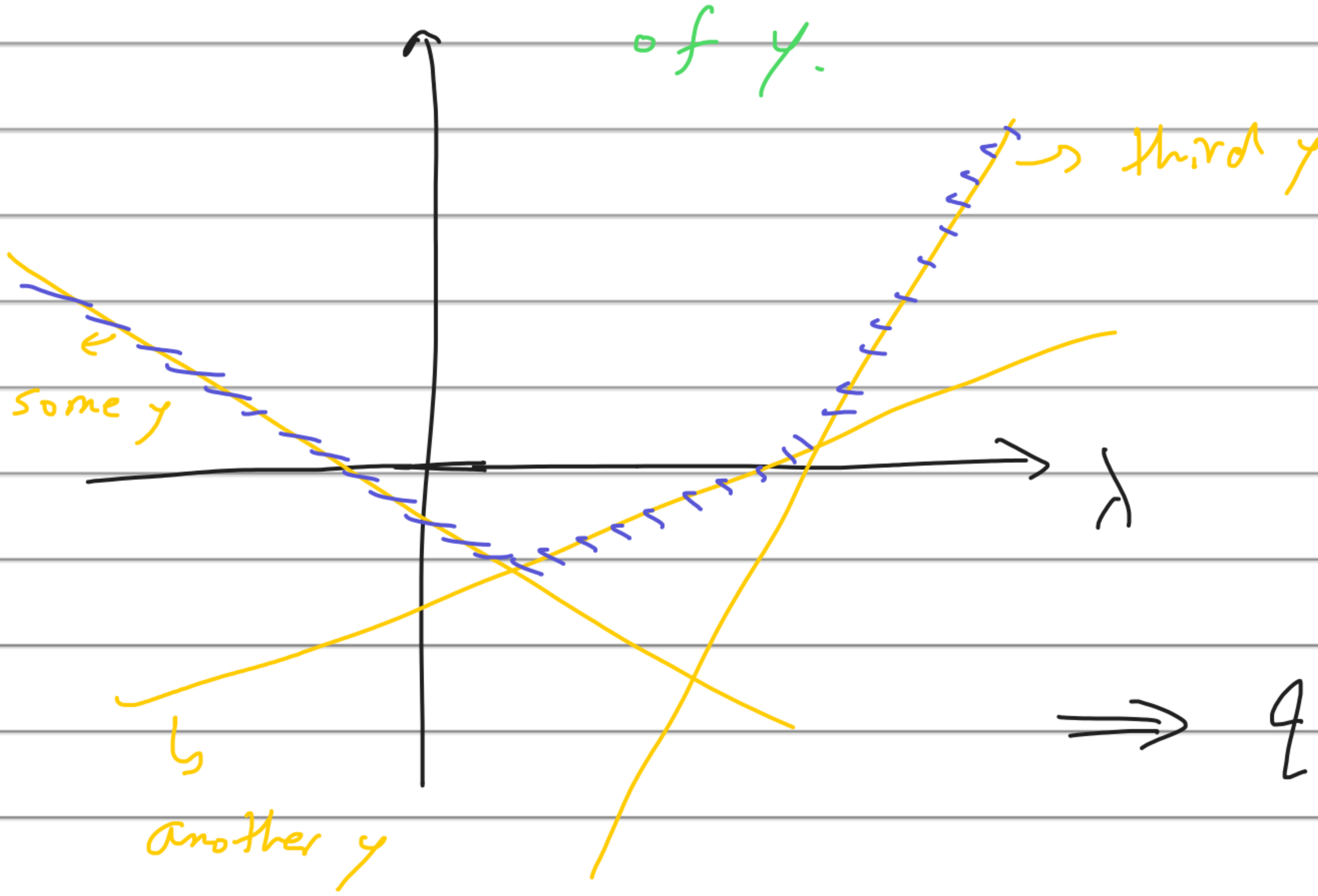
$$\begin{cases} \min\limits_{\substack{y \in \mathbb{R}^n \\ z \in \mathbb{R}^n}} & f_1(y) - f_2(z) \\ & \{z = y\} \longrightarrow \text{Lagrange multiplier } \lambda \in \mathbb{R}^n \\ & y \in X_1 , \ z \in X_2 \end{cases}$$

$$\Rightarrow q(\lambda) = \inf_{\substack{y \in X_1 \\ z \in X_2}} \left(f_1(y) - f_2(z) + \lambda^T (z - y) \right)$$

$$= \underbrace{\inf_{y \in X_1} \left(f_1(y) - \lambda^T y \right)}_{\underbrace{-\sup\limits_{y \in X_1} \left(-f_1(y) + \lambda^T y \right)}_{\underbrace{q_1(\lambda)}_{\text{Conjugate convex function}}}} + \underbrace{\inf_{z \in X_2} \left(-f_2(z) + \lambda^T z \right)}_{\underbrace{q_2(\lambda)}_{\text{Conjugate concave function}}}$$

$$q_1(\lambda) = \sup_{y \in X_1} \left(-f_1(y) + \lambda^T y \right)$$

maximum of infinitely many linear functions in λ for different values of y.



$$\Longrightarrow \quad q(\lambda) = \underbrace{q_2(\lambda)}_{\text{concave}} - \underbrace{q_1(\lambda)}_{\text{convex}}$$

(P)		(D)
$\min_{x} \; f_1(x) - f_2(x)$ s.t. $x \in X_1 \cap X_2$	$\longrightarrow$	$\max_{\lambda} \; q_2(\lambda) - q_1(\lambda)$ s.t. $\lambda \in \Lambda_1 \cap \Lambda_2$

$\Lambda_1 = \{ \lambda \mid q_1(\lambda) < +\infty \}$

$\Lambda_2 = \{ \lambda \mid q_2(\lambda) > -\infty \}$

(we want to make sure that the lower bound $q(\lambda) \neq -\infty$, and so $q_2(\lambda) > -\infty$, $-q_1(\lambda) > -\infty$)

Thm: A pair (x_*, λ_*) is global min - dual solution $\underbrace{\text{if and only if}}$

- Primal feasibility: $x_* \in X_1 \cap X_2$
- Dual feasibility: $\lambda_* \in \Lambda_1 \cap \Lambda_2$
- Lagrangian optimality:

$$x_* \in \underset{x \in X_1}{\arg\min} \left(f_1(x) - \lambda^T x \right)$$

$$x_* \in \underset{x \in X_2}{\arg\min} \left(\lambda^T x - f_2(x) \right)$$

Note: $y \in X_1$, $z \in X_2$

$\Rightarrow (y, z) \in X_1 \underset{\hookrightarrow \text{product}}{\times} X_2$

$\Rightarrow \quad ri(X_1 \times X_2) = ri(X_1) \times ri(X_2)$

Thm: Assume

- Convexity:

$$\begin{cases} X_1, X_2 = \text{convex} \\ f_1 = \text{convex on } X_1 \\ f_2 = \text{concave on } X_2 \end{cases}$$

(not saying f_1, f_2 are convex on $X_1 \cap X_2$)

- $ri(X_1) \cap ri(X_2) \neq \emptyset$

$\Rightarrow \exists$ dual solution & no duality gap

Does convexity of f_1 or concavity of f_2 matter?

No, we only care about epigraphs:

$$S_1 = \{(x, w_1) \mid f_1(x) \leq w_1,\ x \in X_1\}$$

$$S_2 = \{(x, w_2) \mid f_2(x) \geq w_2,\ x \in X_2\}$$

If S_1 and S_2 are convex & closed, then

$$f_1(x) \xrightarrow{\text{Conjugate}} q_1(\lambda) = \sup_{x \in X_1} \left(-f(x) + \lambda^T x\right)$$

$$\Big\downarrow \text{Conjugate}$$

$$\sup_{\lambda \in \Omega_1} \left(-q_1(\lambda) + x^T \lambda\right) = \begin{cases} f_1(x) & \text{if } x \in X_1 \\ +\infty & \text{if } x \notin X_1 \end{cases}$$

$$f_2(x) \xrightarrow{\text{Conjugate}} q_2(\lambda) = \inf_{x \in X_2} \left(-f(x) + \lambda^T x\right)$$

$$\Big\downarrow \text{Conjugate}$$

$$\inf_{\lambda \in \Omega_2} \left(-q_2(\lambda) + x^T \lambda\right) = \begin{cases} f_2(x) & \text{if } x \in X_2 \\ -\infty & \text{if } x \notin X_2 \end{cases}$$

$\Rightarrow$ Conjugate of Conjugate = itself

$\Rightarrow$ Thm: If - S_1, S_2 : Convex & closed

- $ri(\Omega_1) \cap ri(\Omega_2) \neq \emptyset$

$\Rightarrow$ no duality gap & existence of primal solution

(Convexity of X_1, X_2, f_1, f_2 are not directly related to zero duality gap).

LP $\subseteq$ QP $\subseteq$ SOCP $\subseteq$ SDP

$\subseteq$ Linear Conic $\subseteq$ Conic

$\subseteq$ Convex