* Example $\min/\max_{x \in \mathbb{R}^2} f(x) = \min/\max f(x_1, x_2)$

- Denote a solution as (x_1^*, x_2^*)

- observation: x_1^* is a solution of $\min/\max_{x_1} f(x_1, x_2^*)$

$$\Downarrow \text{ optimality condition}$$

$$\frac{\partial f(x_1^*, x_2^*)}{\partial x_1^*} = 0$$

- similarly: x_2^* is a solution of $\min/\max f(x_1^*, x_2)$

$$\Rightarrow \frac{\partial f(x_1^*, x_2^*)}{\partial x_2^*} = 0$$

(Argument: (x_1^*, x_2^*) is the best among all points, so is the best among a subset of points in the form of (x_1, x_2^*) or (x_1^*, x_2)).

$$\Rightarrow \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = 0$$

- $\min/\max_{x \in \mathbb{R}^n} f(x)$

stationary points $\dashrightarrow$ $\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = 0$

- Example: $\min \ (x_1-1)^2 + (x_2-2)^2$

stationary points $\nabla f(x) = 0 \Rightarrow [2(x_1-1) \quad 2(x_2-2)] = 0$

$\Rightarrow (x_1^*, x_2^*) = (1, 2)$

- How to check the type (min, max, or saddle point) of a stationary point?

- A matrix H is positive definite (shown as $H \succ 0$) if all of its eigenvalues are positive.

- A matrix H is negative definite (shown as $H \prec 0$) if all of its eigenvalues are negative.

$\min_{x \in \mathbb{R}^n} f(x) \longrightarrow \nabla f(x) = 0$: stationary point x_*

1- All eigenvalues of $H(f(x_*))$ are strictly positive
$\Rightarrow x_*$ is a Local min

2- All eigenvalues of $H(f(x_*))$ are strictly negative
$\Rightarrow x_*$ is a Local max

3- $H(f(x_*))$ has no zero eigenvalues but has both positive and negative eigenvalues
$\Rightarrow x_*$ is a saddle point

4- $H(f(x_*))$ has at least one zero eigenvalues
$\Longrightarrow$ can't find type of x_*, needs higher order derivatives.

Examples: Consider different sets of values for eigenvalues of $H(f(x_*)) \in \mathbb{R}^{2\times 2}$:

1- $3, 5 \longrightarrow$ Local min

2- $-1, -5 \longrightarrow$ Local max

3- $-1, 5 \longrightarrow$ saddle point

4- $0, 5 \longrightarrow$ Don't know

- How to check if all eigs are strictly positive?

$$H = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} \end{bmatrix} \rightarrow \text{Find leading principal minors}$$

$$\det([a_{11}]),\ \det\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right),\ \det\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{23} & a_{33} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right), \dots$$

- $H \succ 0 \iff$ all determinants are strictly positive.

- How to check if all eigs are strictly negative?
Need to show $\underline{\underline{-H}} \succ 0$ so apply the above test to $\underline{\underline{-H}}$.

- How to check at least one eig is zer?

Need to show that $\det(H)=0$

Example:

- $H = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \Rightarrow \underset{=1>0}{\det([1])}, \underset{=1\times5-2\times2=1>0}{\det\left(\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\right)}$

$\Rightarrow H > 0$

- $H = \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix} \Rightarrow -H = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$\Rightarrow \underset{>0}{\det([1])}, \underset{>0}{\det\left(\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}\right)}$

$\Rightarrow -H > 0 \Rightarrow H < 0$

- $H = \begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix} \Rightarrow \underbrace{\det([0])}_{=0}, \underbrace{\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 4 \end{bmatrix}\right)}_{\neq 0}$

$\Rightarrow$ one positive eig and one negative eig.

Example: Find all local minima, local maxima and saddle points for $f(x_1,x_2) = x_1^2x_2 + x_2^3x_1 - x_1x_2$

$$\nabla f(x) = 0 \Rightarrow \begin{bmatrix} 2x_1x_2 + x_2^3 - x_2 & x_1^2 + 3x_2^2x_1 - x_1 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} x_2(2x_1 + x_2^2 - 1) = 0 \\ x_1(x_1 + 3x_2^2 - 1) = 0 \end{cases}$$

case 1: $x_2 = 0 \Rightarrow x_1 = 0$ or 1

case 2: $x_1 = 0 \Rightarrow x_2 = 0$ or ± 1

case 3: $2x_1 + x_2^2 - 1 = 0$ and $x_1 + 3x_2^2 - 1 = 0$

$$\Rightarrow (x_1, x_2) = \left(2/5, \frac{1}{\sqrt{5}}\right) \text{ or } \left(2/5, \frac{-1}{\sqrt{5}}\right)$$

$$H(f(x)) = \begin{bmatrix} 2x_2 & 2x_1 + 3x_2^2 - 1 \\ 2x_1 + 3x_2^2 - 1 & 6x_1x_2 \end{bmatrix}$$

check types of points:

- $(0,0) \Rightarrow H = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \longrightarrow$ saddle
- $(1,0) \Rightarrow H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow$ saddle
- $(0,1) \Rightarrow H = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} \longrightarrow$ saddle
- $(0,-1) \Rightarrow H = \begin{bmatrix} -2 & 2 \\ 2 & 0 \end{bmatrix} \longrightarrow$ saddle
- $\left(2/5, \frac{1}{\sqrt{5}}\right) \Rightarrow$ Local min
- $\left(2/5, \frac{-1}{\sqrt{5}}\right) \Rightarrow$ Local max

Example: A monopolist producing a single product has two types of customers. If q_1 units are produced for customer 1, then customer 1 is willing to pay $70-4q_1$ for each unit. If q_2 units are produced for customer 2, then customer 2 is willing to pay $150-15q_2$ for each each.
For $q>0$, the cost of manifacturing q units is $100+15q$. $\Rightarrow$ maximize profit ?

$$f(q_1,q_2) = q_1(70-4q_1) + q_2(150-15q_2) - 100 - 15(q_1+q_2)$$

$$\nabla f = 0 \Rightarrow \begin{cases} 70-8q_1-15=0 \\ 150-30q_2-15=0 \end{cases} \Rightarrow q_* = \left(\tfrac{55}{8}, \tfrac{9}{2}\right)$$

$$\Rightarrow H = \begin{bmatrix} -8 & 0 \\ 0 & -30 \end{bmatrix} < 0 \Rightarrow q_* : \text{global max}$$