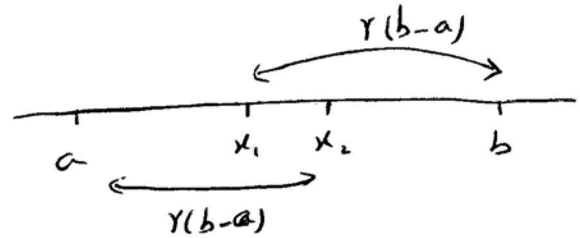
Golden section search:

- choose two points x_1 and x_2 with a ratio $\underline{\gamma}$ in a symmetric way:

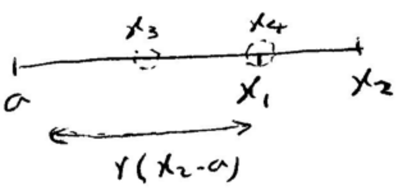


- By comparing $f(x_1)$ and $f(x_2)$, we reduce the search area to either $[a, x_2]$ or $[x_1, b]$.

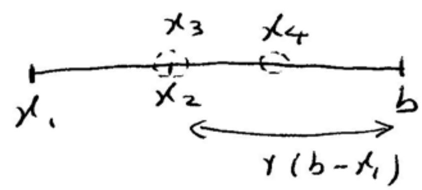
- Need to pick two new points x_3 and x_4 in the reduced search area.

- Is it possible to have one of the new points overlaps with one of the previous points?

Case I:



Case II:



- Then, one of the evaluations $f(x_3)$ and $f(x_4)$ are avaible from the previous iteration.

- This is possible if $\gamma^2 + \gamma - 1 = 0$ or $\gamma = 0.618\ldots$

- summary: At every iteration (after step $\underline{1}$), we need to add one point, do one evaluation, and then reduce the search interval from right or left.

* How many iterations do we need?

- Length of search area at iteration i : L_i

$\Rightarrow$ $L_0 = b-a$, $L_1 = \gamma(b-a)$, $L_2 = \gamma^2(b-a)$, ...

* Find $\underline{k}$ such that the search area is at most of length ε after $\underline{k}$ iterations: $\gamma^k(b-a) < \varepsilon$

given tolerance or interval of uncertainty

- Golden section search is only based on comparisons, without having to take derivates.

Example: We want to Use Golden section search to solve

$$\max_{x} \; -x^2-1$$
$$\text{s.t. } -1 \leq x \leq 0.75$$

with the final interval of uncertainty less than $1/4$. Find the minimum number of iterations.

$1.75 \times 0.618^k < 0.25 \implies k \geq 5$

Unconstrained multivariate optimization:

$$\min_{x \in \mathbb{R}^n} / \max \quad f(x) \quad , \quad x = [x_1, x_2, \dots, x_n]$$

- Assume: $f(x)$ is twice Continuously differentiable

- Example: $\min\limits_{x_1, x_2 \in \mathbb{R}} x_1^2 - x_1 x_2 + x_2^4$

$$\Updownarrow$$

$\min\limits_{x \in \mathbb{R}^2} f(x)$ where $f(x) = f(x_1, x_2) = x_1^2 - x_1 x_2 + x_2^4$ where $x = [x_1 \ x_2]$

- Recall univariate case:

$\min/\max\limits_{x} f(x) \longrightarrow f'(x_*) = 0$ stationary points

- $f''(x_*) > 0$ Local min
- $f''(x_*) < 0$ Local max
- $f''(x_*) = 0 \xrightarrow{\text{high-order derivatives}} f'''(x_*) = 0, \ldots, f^{(k-1)}(x_*) = 0, f^{k}(x_*) \neq 0$
  - $k =$ odd saddle point
  - $k =$ even and $f^{k}(x_*) > 0$ Local min
  - $k =$ even, $f^{k}(x_*) < 0$ Local max

- How to generalize the above result to multivariate case?

- Need to find first and second derivatives.

- First-order derivative ---> Gradient

- second-order derivative --> Hessian

- $f(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$

- Gradient: $\nabla f(x) = \left[\frac{\partial f(x)}{\partial x_1} \quad \frac{\partial f(x)}{\partial x_2} \quad \cdots \quad \frac{\partial f(x)}{\partial x_n} \right]$

- Hessian:

$$H(f(x)) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial f}{\partial x_n \partial x_n} \end{bmatrix}$$

= A symmetric $n \times n$ matrix whose (i,j) entry is $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x)}{\partial x_j \partial x_i}$

Example: $f(x) = e^{x_1 - x_2}$

$$\Rightarrow \nabla f(x) = \left[e^{x_1 - x_2} \quad -e^{x_1 - x_2} \right]$$

$$H(f(x)) = \begin{bmatrix} e^{x_1 - x_2} & -e^{x_1 - x_2} \\ -e^{x_1 - x_2} & e^{x_1 - x_2} \end{bmatrix}$$
