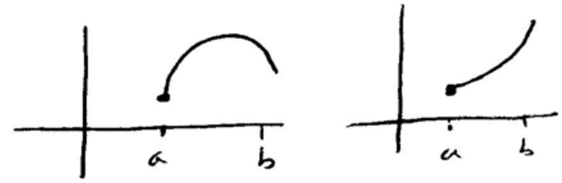
- Endpoints are always local solutions:

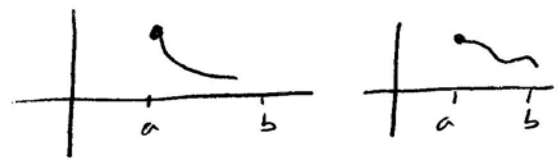
$$\min_{x \in \mathbb{R}} / \max \; f(x)$$

$$: \quad a \leq x \leq b$$

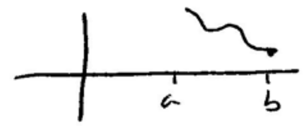
- $f'(a) > 0 \Rightarrow a$: local min



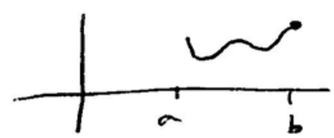
- $f'(a) < 0 \Rightarrow a$: local max



- $f'(b) < 0 \Rightarrow b$: local min



- $f'(b) > 0 \Rightarrow b$: local max



- $f'(a)$ or $f'(b) = 0 \Rightarrow$ $\underline{\underline{a}}$ or $\underline{\underline{b}}$ is a stationary point (type can be determined using the test given below).

Theorem: Consider a point x_* such that $f'(x_*) = 0$.

- If $f''(x_*) > 0 \Rightarrow x_* =$ local minimum
- If $f''(x_*) < 0 \Rightarrow x_0 =$ local maximum
- If $f''(x_*) = 0 \Rightarrow$ check higher-order derivatives:

$\rightarrow$ Find a number $\underline{\underline{k}}$ such that:

$$f'(x_*) = 0, \; f''(x_*) = 0, \; \ldots, \; f^{(k-1)}(x_*) = 0, \; f^{k}(x_*) \neq 0$$

- If k = even and $f^{k}(x_*) > 0 \Rightarrow x_* =$ Local min
- If k = even and $f^{k}(x_*) < 0 \Rightarrow x_* =$ local max
- If k = odd $\Rightarrow x_* =$ saddle point

Examples:

min $x^2 \Rightarrow f'(x_*) = 0 \;:\; x_* = 0$

$f'(x_*) = 0,\; f''(x_*) > 0 \underset{(k=2)}{\Rightarrow} x_* =$ local min

max $-x^2 \Rightarrow f'(x_*) = 0 \;:\; x_* = 0$

$f'(x_*) = 0,\; f''(x_*) < 0 \underset{(k=2)}{\Rightarrow} x_* =$ Local max

min $x^4 \Rightarrow f'(x_*) = 0 \;:\; x_* = 0$

$f'(x_*) = f''(x_*) = f'''(x_*) = 0,\; \underset{(k=4)}{f''''(x_*) > 0}$

$\Rightarrow x_* =$ Local min

min $x^3 \Rightarrow f'(x_*) = 0 \;:\; x_* = 0$

$f'(x_*) = f''(x_*) = 0,\; f'''(x_*) \neq 0$

$\Rightarrow k = 3 \Rightarrow x_* =$ saddle point

Example: It costs a monopolist \$5/unit to produce a product. If he produces $\underline{x}$ units of the product, then each can be sold for $10-x$ dollars $(0 \le x \le 10)$.

To maximize profit, how much should the monopolist produce?

profit: $p(x) = x(10-x) - 5x = 5x - x^2$

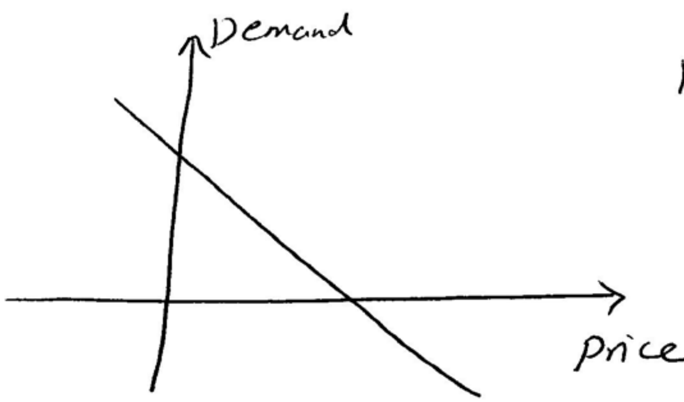
$$\Rightarrow \max_{\substack{x \\ 0 \le x \le 10}} p(x) \Rightarrow p'(x_*) = 0 : \quad x_* = 2.5$$

Candidate solutions:

$$\begin{cases} x_* = 2.5 : \ p''(2.5) = -2 < 0 \Rightarrow \text{Local max}, \ \text{profit: } p(2.5) = 6.25 \\ x_* = 0 \ (\text{endpoint}): \\ \quad p'(0) = 5 > 0 \Rightarrow \text{Local min} \\ x_* = 10 \ (\text{endpoint}): \\ \quad p'(10) = -15 < 0 \Rightarrow \text{Local min} \end{cases}$$

$$\Rightarrow x_* = 2.5 \quad \text{global max}$$

Example: profit-maximizing price:



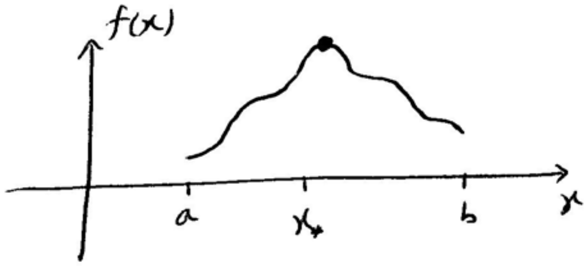
Demand = $a - b \times$ price

Cost $= c$

$$\Rightarrow \text{profit: } (\text{price} - c)(a - b \times \text{price}) \Rightarrow \max_{\substack{\text{price} \\ \text{s.t. } a \le \text{price} \le b}}$$

Numerical Algorithm:

Unimodal: $f(x)$ is unimodal on $[a,b]$ if for some point x_*, $f(x)$ is strictly increasing on $[a, x_*]$ and strictly decreasing on x_*.



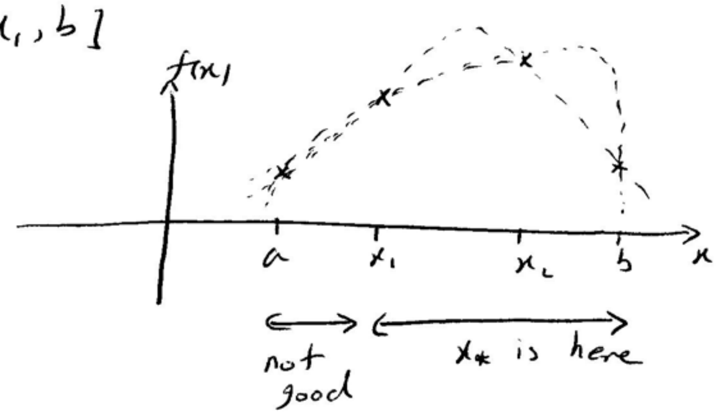
$\Rightarrow x_* =$ global solution

* strategy to solve $\max\limits_{x} f(x)$ s.t. $x \in [a,b]$?

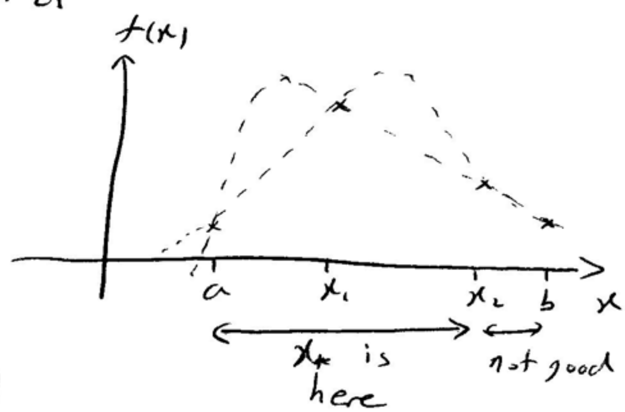
- pick two points x_1 and x_2 such that $a < x_1 < x_2 < b$.

- calculate $f(x_1)$ and $f(x_2)$.

- If $f(x_1) < f(x_2) \Rightarrow x_* \in [x_1, b]$



- If $f(x_1) \geq f(x_2) \Rightarrow x_* \in [a, x_2]$



$\Rightarrow$ The search area for x_* gets smaller after two evaluations ($f(x_1)$ and $f(x_2)$)