# Example: work-scheduling problem:

Post-office problem: A post office requires different number of employees on different days of the week. The minimum number of employees required on each day is listed below:

| Day | # of employees |
|---|---|
| 1- Monday | 17 |
| 2- Tuesday | 13 |
| 3- Wednesday | 15 |
| 4- Thursday | 19 |
| 5- Friday | 14 |
| 6- Saturday | 16 |
| 7- Sunday | 11 |

Each employee works for 5 consecutive days and then takes 2 days off. Find the minimum number of employees that need to be hired.

Decision variables: $x_i$: # of employees $\boxed{\text{beginning}}$ work on day $i$, $i=1,2,\dots,7$

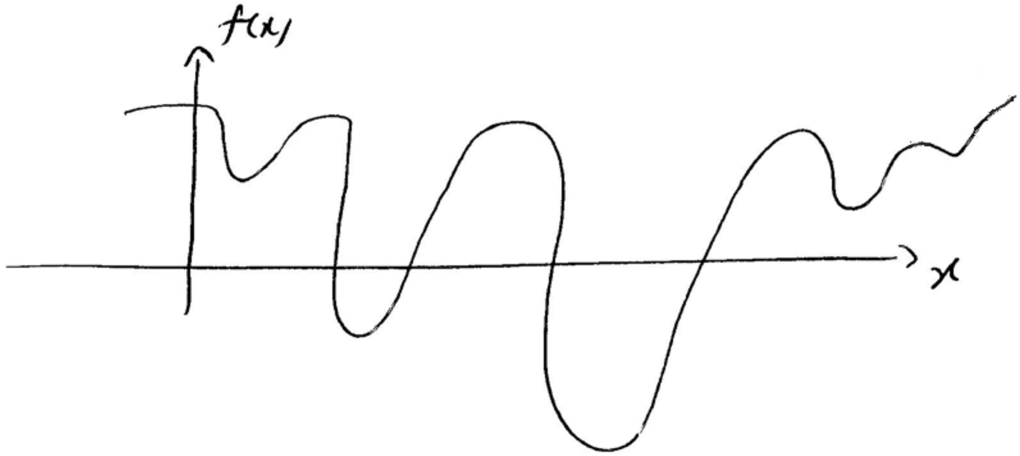
objective: minimize $x_1+x_2+\dots+x_7$

Constraints:

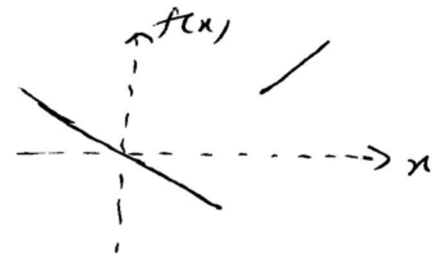
$$x_1+x_4+x_5+x_6+x_7 \geq 17 \longrightarrow \text{Monday}$$
$$x_1+x_2+x_5+x_6+x_7 \geq 13 \longrightarrow \text{Tuesday}$$
$$\vdots$$
$$x_i = 0,1,2,\dots \qquad i=1,2,\dots,7$$

- Unconstrained, Univariate optimization:

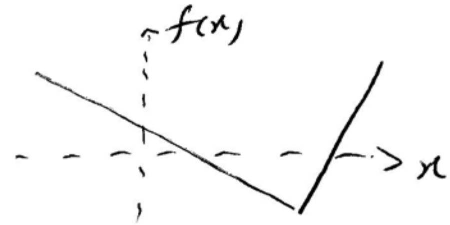
$$\min_{x \in \mathbb{R}} f(x)$$




- Functions: Discontinuous:




Continuous but non-differentiable:




Continuous and differentiable

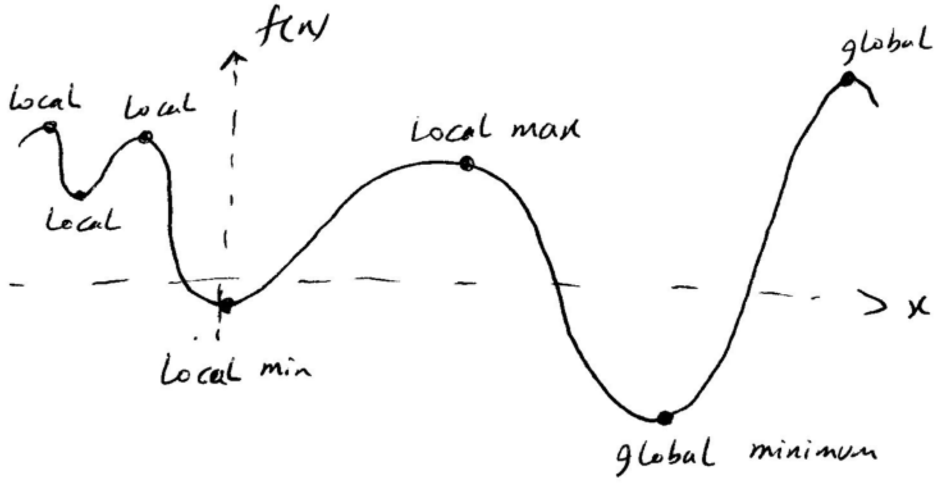

$$\boxed{f(x) \longrightarrow f'(x) \quad \text{derivative}}$$

- $x_*$ : Local minimum (maximum / solution)

$\exists$ $\varepsilon$ such that $\forall x$ satisfying $\|x - x_*\| < \varepsilon$

there exists — for every — we have $f(x) \geq f(x_*)$

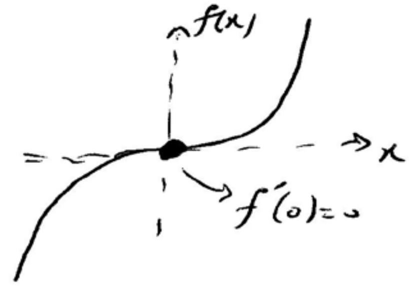
- Best local solution is called global solution.




- Continuous and differentiable:

$$x_* \text{ is Local } \xrightarrow[\xleftarrow{\text{not necessarily true}}]{} f'(x_*) = 0$$

not necessarily true




- If $f'(x_*) = 0 \implies x_*$ is called stationary point

1. local minimum
2. local maximum
3. saddle point

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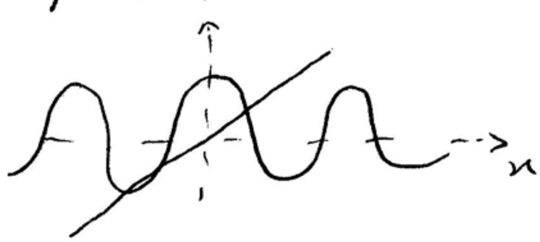
Example: Find intersection of $y = x$ and $y = \cos x$




$$x = \cos x \implies \min_x \, (x - \cos x)^2$$

- $\min\ e^x + (x-1)^2 \quad \rightarrow \quad f'(x_*) = 0$ :

$$e^x + 2x - 2 = 0$$

$\rightarrow$ Find all solutions and inspects which one gives a better (lower) objective

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- $\min\ \cos x \quad \rightarrow \quad \frac{\partial \cos x}{\partial x} = -\sin x = 0 \quad \Rightarrow \quad x = k\pi$, $k = \ldots -1\ 0\ 1 \ldots$

- $\max\ \cos x \quad \rightarrow \quad \min\ (-\cos x) \quad \rightarrow \quad \frac{\partial(-\cos x)}{\partial x} = 0$

$$\Rightarrow \quad x = k\pi$$

$k = \ldots -1\ 0\ 1 \ldots$

$\Rightarrow$ The proposed stationary condition doesn't distinguish between minima and maxima.

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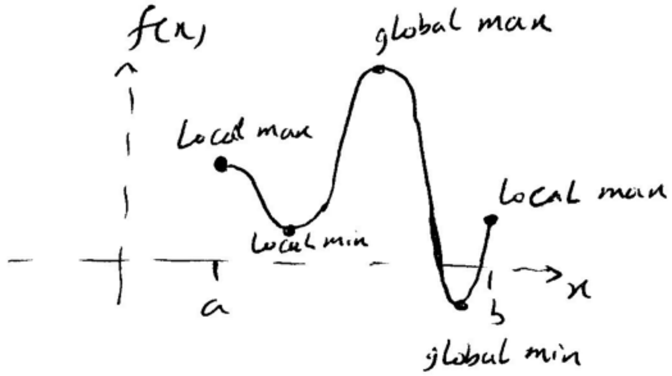
- Constrained, univariate optimization:

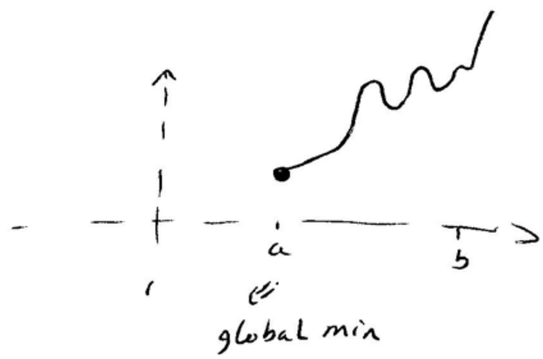
$$\min_{\substack{x \in \mathbb{R} \\ \underbrace{a \leq x \leq b}_{\text{interval constraint}}}} f(x) \quad \Rightarrow \quad \text{Compare all these values: } \underbrace{f(a),\ f(b)}_{\text{endpoints}},\ \underbrace{f(x_*) \text{ for every } x_* \text{ s.t. } f'(x_*) = 0}_{\text{stationary points}}$$

- $\min\ (x-2)^2$ s.t. $1 \leq x \leq 3$ $\quad \Rightarrow \quad (1-2)^2,\ (3-2)^2,\ (2-2)^2$

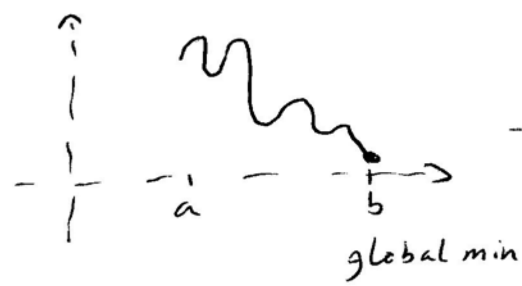
(because $2(x_* - 2) = 0$)



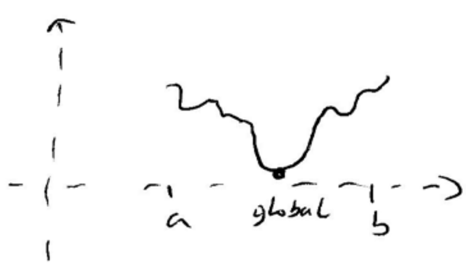

Various cases:




→ end point




→ end point




→ stationary point