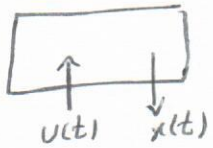
Examples:

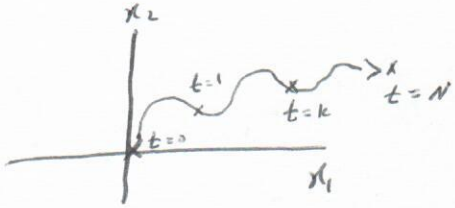
1- Minimum Fuel optimal control:

system:



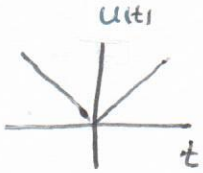
$$x(t+1) = \underset{n\times n}{A}\, x(t) + \underset{n\times 1}{b}\, u(t) \qquad t = 0, \dots, N-1$$

we want to go from the origin to some final state using minimum fuel consumption: $x(0) = 0, \quad x(N) = \text{given}$



Fuel consumption:

$$\sum_{t=0}^{N-1} |u(t)|$$



LP:

$$\min_{\substack{u(0),\dots,u(N-1),\\ x(0),\dots,x(N)\\ k_0,\dots,k_{N-1}}} \quad k_0 + k_1 + \dots + k_{N-1}$$

$$x(t+1) = Ax(t) + bu(t) \qquad t = 0, \dots, N-1$$

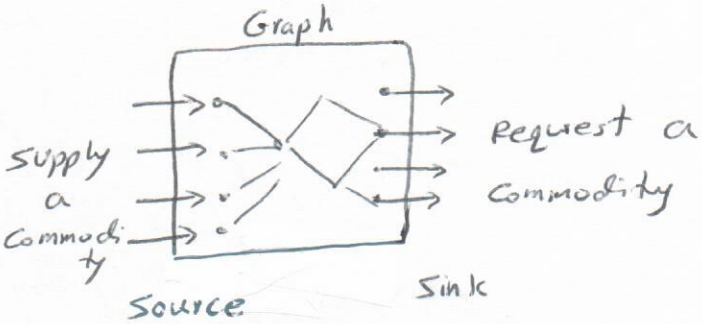
$$x(0) = 0$$

$$x(N) = \text{given}$$

$$u(t) \le k_t \qquad t = 0, \dots, N-1$$

$$-u(t) \le -k_t \qquad t = 0, \dots, N-1$$

2- Network Flow problem:



- we have a network with a number of sources and sinks.
- Each sink requests some amount of flow.
- Each source can provide some flow over a range
- Find the minimum-cost flow supply.

Node: $\xrightarrow{b_i}$ node i $\quad \Rightarrow \quad b_i^{min} \le b_i \le b_i^{max}$

special case: $b_i^{max} = b_i^{min} = -$ $\Rightarrow$ sink

Link: $\underset{i}{\times} \xrightarrow{x_{ij}} \underset{j}{\times}$ $\qquad x_{ij}^{min} \le x_{ij} \le x_{ij}^{max}$

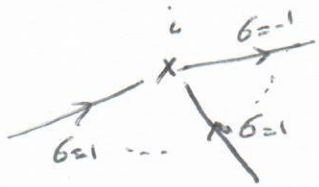
$$\text{cost} = c_{ij}\, x_{ij}$$

(Assume that the network is a directed graph denoted as $\vec{G}$)

LP: $\displaystyle\min_{\substack{x_{ij} : (i,j)\in\vec{G} \\ b_i : 1\le i\le n}} \quad x_{ij}\, c_{ij}$

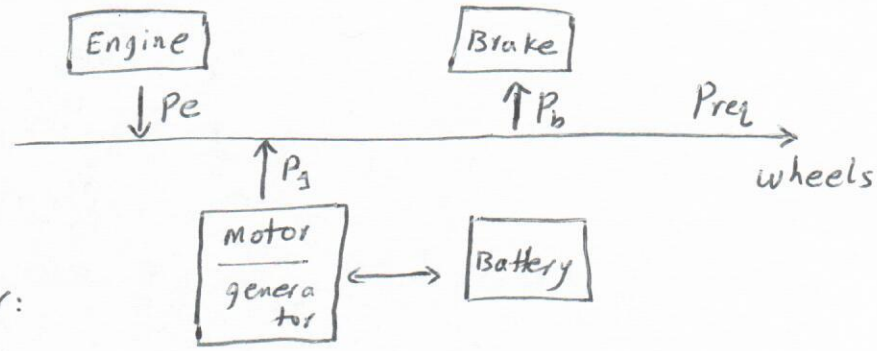
s.t. $-x_{ij}^{min} \le x_{ij} \le x_{ij}^{max} \quad (i,j)\in\vec{G}$

$-b_i^{min} \le b_i \le b_i^{max} \quad i\in\{1,\dots,n\}$



$$\sum_{j\in N(i)} \sigma_{ij}\, x_{ij} = 0 \quad \Leftarrow$$ - Conservation of flow at node $\underline{\underline{i}}$

3- Hybrid vehicle:



- required wheel power:

$P_{req}(1), P_{req}(2), \dots, P_{req}(T)$

- Engine provide power:

$$0 \le P_e(t) \le P_e^{max}$$
$$t = 1,\dots,T$$

- Brake dissipate power:

$$P_b(t) \ge 0$$

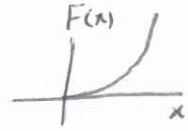
- motor / generator power:

$$P_g^{min} \le P_g(t) \le P_g^{max}$$
$$t = 1,\dots,T$$

(charge / discharge battery)

- Fuel cost:

$$\sum_{t=1}^{T} F(P_e(t))$$



- Modeling of battery:

$$E(t+1) = E(t) - P_g(t)$$

$$0 \le E(t) \le E^{max}$$
$$t = 1,\dots,T$$

- modeling of P_{req}:

$$P_{req}(t) = P_e(t) + P_g(t) - P_b(t)$$

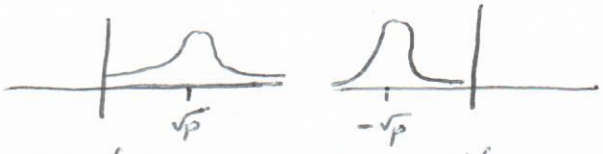
$\Rightarrow$ Convex optimization

4. Channel capacity: (Gaussian channel)

$$X \longrightarrow \boxed{\text{noisy channel}} \longrightarrow Y$$

$$Y = X + \underset{\text{noise}}{Z}, \quad Z \sim \text{normal}(0, N)$$

- Assume we want to submit a bit (0, 1).
- we send X with power $\underline{\underline{P}}$: $0 \longrightarrow -\sqrt{P}$, $1 \longrightarrow \sqrt{P}$
- probability distribution for Y:



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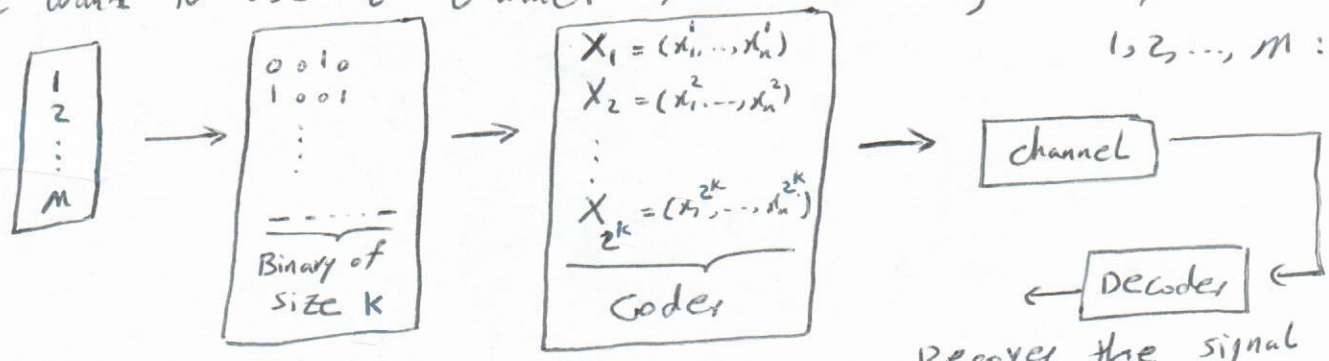
- So, if P is large, we can recover the correct signal with a high probability.

- Assume $X = (x_1, \dots, x_n)$ and $\underbrace{\frac{1}{n}\sum_{i=1}^{n} x_i^2}_{\text{average power for transmission}} \leq P$

- we want to use a channel for transmitting the symbols $1, 2, \dots, m$:



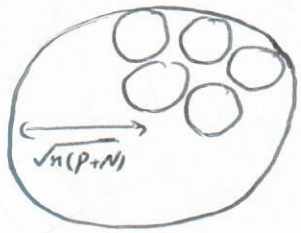
Recover the signal with a high probability

Technique:

radius: $\sqrt{n(N+\varepsilon)}$

Related to probability of error

Code design: sphere packing



Rate $= \frac{k}{n}$ bits per transmission

Capacity = max Rate as $\underline{\underline{n}}$ goes to infinity

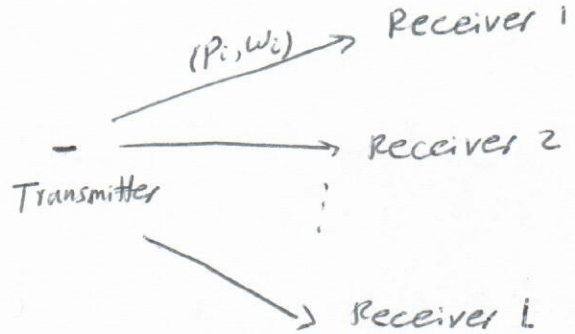
- Capacity of Gaussian channel:

$$C = \frac{1}{2}\log\left(1+\frac{P}{N}\right) \text{ bits per transmission}$$

- Bandlimited channel with white noise:
- Assume that the channel cuts out all frequencies greater than W: $\Rightarrow$ Capacity $= W\log\left(1+\frac{P}{N_0 W}\right)$ bits per second

($N_0/2$ = power spectral density)

problem: optimal power and bandwidth allocation



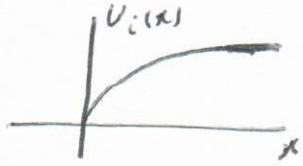
- Total power is fixed:

$$P_{tot} = P_1 + \cdots + P_L \qquad (P_i \geq 0)$$

- Total bandwidth is fixed:

$$W_{tot} = W_1 + \cdots + W_L \qquad (W_i \geq 0)$$

- maximize the total utility: $\sum_{i=1}^{L} U_i(R_i)$



Note: $-W_i\log\left(1+\frac{P_i}{N_0 W_i}\right)$ = convex ($f(x)$ convex in x $\Downarrow$ $t f\left(\frac{x}{t}\right)$ convex in (x,t))

$$\Rightarrow \min_{\substack{P_1,\dots,P_L \\ W_1,\dots,W_L \\ R_1,\dots,R_L}} -\sum_{i=1}^{L} U_i(R_i)$$

$$P_i, W_i \geq 0 \quad i=1,\dots,L$$

$$\sum P_i = P_{tot}$$

$$\sum W_i = W_{tot}$$

$$R_i - W_i\log\left(1+\frac{P_i}{N_0 W_i}\right) \leq 0 \quad i=1,\dots,L \qquad (*)$$

Note: $R_i \uparrow \Rightarrow$ obj $\downarrow$ $\Rightarrow$ $(*)$ is binding at optimality

- Robust optimization:

$$\min_x \; f_0(x)$$
$$\text{s.t.} \quad f_i(x) \le 0 \quad i=1,\dots,m$$

Assume that there are some uncertainties in modeling f_j's.

$f_j(x) \longrightarrow f_j(x,u)$ and u is a random variable

- Stochastic optimization:

$$\min_x \; E_u(f_0(x,u))$$
$$\text{s.t.} \quad E_u(f_i(x,u)) \le 0 \quad i=1,\dots,m$$

$\longrightarrow$ Convex :

$$E_u(f_j(x,u)) = \int f_i(x,u) \underbrace{p(u)}_{\ge 0} du$$

(weighted sum of convex functions)

- Worst-case optimization:

$$\min_x \; \sup_u f_0(x,u)$$
$$\text{s.t.} \quad \sup_u f_i(x,u) \le 0 \quad i=1,\dots,m$$

$\longrightarrow$ Convex

- So far, we have introduced LP, QP, QCQP, and SOCP.

- Geometric Programming (GP):

- Consider a function $f: \mathbb{R}^n_{++} \longrightarrow \mathbb{R}$: $f(x) = c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where $c > 0$, $a_i \in \mathbb{R}$
- $f(x)$ is called a monomial.
- posynomial = a sum of monomials : $\sum_{k=1}^{L} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$
- posynomial : closed under addition and multiplication.
- Monomial : closed under division and multiplication.

- GP: $\min f_0(x)$

  s.t. $f_i(x) \le 1 \quad i = 1, \dots, m$, f_i's = posynomials

  $h_j(x) = 1 \quad j = 1, \dots, p$ $\quad h_j$'s = monomials

- Some interesing circuit problems can be cast as QP.

- Example:

$$\min_{x,y,z \ge 0} \frac{x}{y} \qquad\qquad \min_{x,y,z \ge 0} \; x y^{-1}$$

$$2yz \le x \le 3 \qquad\qquad \frac{1}{3} x \le 1$$

$$x^2 + 3\frac{y}{z} \le \sqrt{y} \quad \longrightarrow \quad 2x^{-1}yz \le 1$$

$$x^2 y^{-1/2} + 3 y^{1/2} z^{-1} \le 1$$

$$\frac{x}{y} = z^2 \qquad\qquad x y^{-1} z^{-2} = 1$$

Thm: GP can be cast as a convex optimization.

- $\log(e^{x_1} + e^{x_2})$ is convex $\longrightarrow \nabla^2 = \dfrac{e^{x_1} e^{x_2}}{(e^{x_1} + e^{x_2})^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \succeq 0$

$\Rightarrow \log(\sum e^{x_i})$ convex $\rightarrow \log(\sum e^{\sum a_i x_i})$ convex

- Trick: $\underset{\ge 0}{x_i} \longrightarrow e^{y_i}$

$$\Rightarrow \quad c\, x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = 1 \longrightarrow c\, e^{\sum a_i y_i} = 1 \rightarrow \underbrace{\sum a_i y_i = \text{constant}}_{\text{Linear}}$$

$$\Rightarrow \quad f_i(x) \le 1 \longrightarrow \underbrace{\log\Big(\sum_{k=1}^{l} e^{\sum a_{ik} y_{ik}} c\Big)}_{\text{Convex}} \le 0$$

- Example: Given $M \in \mathbb{R}^{n \times n}$, design a diagonal matrix $D > 0$ such that $\|DMD^{-1}\|_F$ is minimized (Frobenius norm diagonal scaling).

$$\min_{D>0} \|DMD^{-1}\|_F^2 \longrightarrow \min_{D>0} \sum_{i,j} m_{ij}^2 d_i^2 d_j^{-2} \longrightarrow GP$$

- Semidefinite program (SDP):

$$\mathcal{M} = \{ x \mid x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \} \quad : \text{convex}$$

$$\text{SDP:} \quad \min_x \; c^T x$$
$$\text{s.t.} \quad x_1 F_1 + \cdots + x_n F_n + G \preceq 0 \quad \rightarrow \text{convex}$$
$$Ax = b$$

Example: $\min_{x_1, x_2} \; x_1 - 2x_2$

$$\begin{bmatrix} x_1 + x_2 + 1 & x_1 - 3x_2 \\ x_1 - 3x_2 & x_1 + 2x_2 \end{bmatrix} \succeq 0 \longrightarrow x_1 \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 & 3 \\ 3 & -2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \preceq 0$$

Example: $\min_x \; c^T x$

$$\left.\begin{aligned} \text{s.t.} \quad & \sum x_i F_i + G \preceq 0 \\ & \sum x_i \bar{F}_i + \bar{G} \preceq 0 \end{aligned}\right\} \Rightarrow \sum x_i \begin{bmatrix} F_i & 0 \\ 0 & \bar{F}_i \end{bmatrix} + \begin{bmatrix} G & 0 \\ 0 & \bar{G} \end{bmatrix} \preceq 0$$

- $\sum x_i F_i$ is called a linear matrix inequality (LMI).

- Standard Form of LP:

$$\min_{x \in \mathbb{R}^n} \; c^T x$$
$$a_i^T x = b_i \quad i = 1, \dots, k$$
$$d_i^T x \leq e_i \quad i = 1, \dots, m$$

$\longrightarrow$

$$\min_{x \in \mathbb{R}^n} \; c^T x$$
$$a_i^T x = b_i \quad i = 1, \dots, p$$
$$x \geq 0$$

$\downarrow$

$$\min_{x \in \mathbb{R}^n} \; \langle c, x \rangle$$
$$\langle a_i, x \rangle = b_i \quad i = 1, \dots, p$$
$$x \geq 0$$

"Standard LP"

- Trick:

$x \in \mathbb{R} \longrightarrow x = x^+ - x^-, \; x^+, x^- \geq 0$

- Trick:

$d_i^T x \leq e_i \longrightarrow d_i^T x + s_i = e_i$ and $s_i \geq 0$

slack param

- Standard Form of SDP:

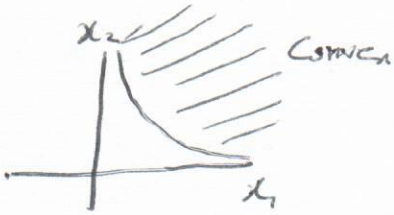
$$\min_{X \in S^n} \langle C, X \rangle \quad \text{s.t.} \quad \langle A_i, X \rangle = b_i, \quad X \succeq 0 \implies \min_X \operatorname{tr}(CX) \quad \text{s.t.} \quad \operatorname{tr}(A_i X) = b_i \quad i=1,\dots,p, \quad X \succeq 0$$

$\Rightarrow$ SDP = linear objective and linear scalar / matrix inequalities / equalities

- Example:

$$\min_{x_1, x_2} \; x_1 + 3x_2$$

$$\left.\begin{array}{l} x_1, x_2 \geq 0 \\ x_1 x_2 \geq 1 \longrightarrow \text{non-convex} \\ 2x_1 - 4x_2 \leq 5 \end{array}\right\} \xrightarrow{\text{Feasible set}}$$



$$\longrightarrow \begin{bmatrix} x_1 & 1 \\ 1 & x_2 \end{bmatrix} \succeq 0$$

- Schur complement: $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \iff C \succ 0, \; A - BC^{-1}B^T \succ 0$

- $LP \subseteq QP \subseteq QCQP \subseteq \underbrace{SOCP \subseteq SDP}_{?}$

proof: $SOCP = \min_x f^T x$

$$\|A_i x + b_i\|_2 \leq c_i^T x + d_i \qquad i=1,\dots,m \qquad (*)$$

$$Fx = g$$

$$(*) \iff (A_i x + b_i)^T (A_i x + b_i) \leq (c_i^T x + d_i)^2$$

$$\iff \begin{bmatrix} c_i x + d_i & (A_i x + b_i)^T \\ A_i x + b_i & (c_i x + d_i) I \end{bmatrix} \succeq 0$$

- How to find min eig of a symmetric matrix M? $\max_\lambda \lambda$ s.t. $M \succeq \lambda I$

- Example: matrix norm minimization

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad A_i \in \mathbb{R}^{p \times q}$$

$$\min_{x \in \mathbb{R}^n} \|A(x)\|_2 \longrightarrow \begin{array}{ll} \min\limits_{\lambda, x} & \lambda \\ \text{s.t.} & A(x)^T A(x) \leq \lambda I \end{array}$$

$$\longrightarrow \begin{array}{ll} \min\limits_{t, x} & t^2 \\ \text{s.t.} & A(x)^T A(x) \leq t^2 I \end{array} \longrightarrow \begin{array}{ll} \min\limits_{t, x} & t \\ \text{s.t.} & \begin{bmatrix} tI & A(x)^T \\ A(x) & tI \end{bmatrix} \succeq 0 \end{array}$$

- Example: Eigenvalue optimization via SDP:

$$A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad A_i \in S^m$$

eigs of $A(x)$: $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x)$

(a): maximize $\lambda_m(x)$: $\quad \max\limits_{\lambda, x} \lambda \quad \text{s.t.} \quad A(x) \succeq \lambda I$

(b): minimize $\lambda_1(x)$: $\quad \min\limits_{\lambda, x} \lambda \quad \text{s.t.} \quad A(x) \preceq \lambda I$

(c): minimize the spread of eigs, $\lambda_1(x) - \lambda_m(x)$:

$$\min_{\lambda_1, \lambda_m, x} \lambda_1 - \lambda_m \quad \text{s.t.} \quad \begin{array}{l} A(x) \succeq \lambda_m I \\ A(x) \preceq \lambda_1 I \end{array}$$

(d): Minimize the condition number $\dfrac{\lambda_1(x)}{\lambda_m(x)}$:

$$\min_{\lambda_1, \lambda_m, x} \frac{\lambda_1}{\lambda_m} \quad \text{s.t.} \quad \begin{array}{l} A_0 + \sum x_i A_i \succeq \lambda_m I \\ A_0 + \sum x_i A_i \preceq \lambda_1 I \end{array}$$

$$\downarrow$$

$$\min_{\lambda_1, \lambda_m, x} \frac{\lambda_1}{\lambda_m} \quad \text{s.t.} \quad I \preceq \underbrace{\frac{1}{\lambda_m}}_{\bar{\lambda}_m} A_0 + \sum \underbrace{\frac{x_i}{\lambda_m}}_{\bar{x}_i} A_i \preceq \underbrace{\frac{\lambda_1}{\lambda_m}}_{t} I$$

$$\downarrow$$

$$\min_{t, \bar{\lambda}_m, \bar{x}} t \quad \text{s.t.} \quad I \preceq \bar{\lambda}_m A_0 + \sum \bar{x}_i A_i \preceq t I$$

- Example: optimization with logistic model:

  - $X \in \{0,1\}$ random variable, $\text{Prob}(X=1) = p = \dfrac{e^{a^T x + b}}{1 + e^{a^T x + b}}$

  - $x \in \mathbb{R}^n$: parameter affecting probability (advertising effort, retail price, discounted price, packaging expense, ...)

  - Constraint: $Fx \leq g$ (linear)

(a): maximize buying probability: $\max_x \dfrac{e^{a^T x + b}}{1 + e^{a^T x + b}}$

$$\rightarrow \min_x \frac{1 + e^{a^T x + b}}{e^{a^T x + b}} \rightarrow \min_x e^{-a^T x - b} + 1 \rightarrow \text{Convex}$$

  - $c^T x + d \ (\geq 0)$: profit from selling the product.

(b): maximize the expected profit: $\max_x \left(\dfrac{e^{a^T x + b}}{1 + e^{a^T x + b}} \right) (c^T x + d)$

s.t. $c^T x + d \geq 0$

$$\rightarrow \min_x \left(e^{-a^T x - b} + 1 \right) \frac{1}{c^T x + d} \rightarrow \min_{x,t} t$$

s.t. $c^T x + d \geq 0$ $\qquad$ s.t. $\dfrac{e^{-a^T x - b} + 1}{c^T x + d} \leq t$

$c^T x + d \geq 0$

$$t = e^y \rightarrow \min_{x,y} y \quad \text{s.t. } \left(e^{-a^T x - b} + 1 \right)\left(e^{-y} \right) \leq c^T x + d$$

$$\rightarrow e^{-a^T x - b - y} + e^{-y} \leq c^T x + d$$

$\rightarrow$ Convex

- How to write QCQP as SDP:

$$\min_{x \in \mathbb{R}^n} x^T P_0 x + q_0^T x + r_0$$

s.t. $x^T P_i x + q_i^T x + r_i \leq 0$

$i = 1, \dots, m$

$\rightarrow$ Use epigraph technique to move objective into constraint

$$\rightarrow (x^T P_i^{1/2})(P_i^{1/2} x) \leq -r_i - q_i^T x$$

$$\rightarrow \begin{bmatrix} -r_i - q_i^T x & x^T P_i^{1/2} \\ P_i^{1/2} x & I \end{bmatrix} \succeq 0$$