Examples:

1. Minimum Fuel Optimal Control:

- System:
  \[
  x(t+1) = Ax(t) + bu(t) \quad t = 0, \ldots, N-1
  \]
  We want to go from the origin to some final state using minimum fuel consumption: \( x(0) = 0, \quad x(N) = \text{given} \)
  Fuel consumption:
  \[
  \sum_{t=0}^{N-1} |u(t)|
  \]

- LP:
  \[
  \min_{u(t), \ldots, u(t-1), x(t), \ldots, x(N)} \quad K_0 + K_1 + \ldots + K_{N-1}
  \]
  \[
  x(t+1) = Ax(t) + bu(t) \quad t = 0, \ldots, N-1
  \]
  \[
  x(0) = 0
  \]
  \[
  x(N) = \text{given}
  \]
  \[
  u(t) \leq K_t \quad t = 0, \ldots, N-1
  \]
  \[
  -u(t) \leq -K_t \quad t = 0, \ldots, N-1
  \]

2. Network Flow Problem:

- We have a network with a number of sources and sinks.
- Each sink requests some amount of flow.
- Each source can provide some flow over a range.
- Find the minimum-cost flow supply.

- Node: \( b_i \rightarrow \) node \( i \) \( \Rightarrow b_i \leq b_i \leq b_i \) max
  Special case: \( b_i^{\text{max}} = b_i^{\text{min}} = \) sink
Link: \( x_{ij} \) \( x_{ij}^{\text{min}} \leq x_{ij} \leq x_{ij}^{\text{max}} \)  

\[ \text{Cost} = C_{ij} x_{ij} \]

(Assume that the network is a directed graph denoted as \( \overrightarrow{G} \))

\[
\text{LP: } \min \sum_{(i,j) \in \overrightarrow{E}} C_{ij} x_{ij}
\]

\[
\begin{align*}
&\text{s.t. } x_{ij} \leq x_{ij}^{\text{max}}, \, (i,j) \in \overrightarrow{E} \\
&\quad \text{for } i \in \{1, \ldots, n\} \\
&\quad b_i^{\text{min}} \leq b_i \leq b_i^{\text{max}}, \, i \in \{1, \ldots, m\} \\
\end{align*}
\]

\[
\sum_{j \in W(i)} x_{ij} = 0 \quad \forall \, i \in W(j) 
\]

- Conservation of flow at node \( i \)

3. Hybrid vehicle:

- Engine:

Input: \( P_{e}(t) \)

Output: \( P_{e}(t) \)

- Brake:

Input: \( P_{b}(t) \)

Output: \( P_{b}(t) \)

- Motor/Generator:

Input: \( P_{g}(t) \)

Output: \( P_{g}(t) \)

- Battery:

Input: \( E(t) \)

Output: \( E(t) \)

- Required wheel power:

\( P_{req}(t) \), \( P_{req}(t-1), \ldots, P_{req}(T) \)

- Engine provide power:

\[ 0 \leq P_{e}(t) \leq P_{e}^{\text{max}} \]

\[ t = 1, \ldots, T \]

- Brake dissipate power:

\[ P_{b}(t) \geq 0 \]

- Motor/generator power:

\[ P_{g}^{\text{min}} \leq P_{g}(t) \leq P_{g}^{\text{max}} \]

\[ t = 1, \ldots, T \]

(Charge/discharge battery)

\[ F_{\text{cost}} = \sum_{t=1}^{T} F(P_{e}(t)) \]

- Fuel cost:

- Modeling of battery:

\[ E(t+1) = E(t) - P_{g}(t) \]

\[ 0 \leq E(t) \leq E^{\text{max}} \]

\[ t = 1, \ldots, T \]

- Modeling of \( P_{req} \):

\[ P_{req}(t) = P_{e}(t) + P_{g}(t) - P_{b}(t) \]

\[ \Rightarrow \text{convex optimization} \]
4. Channel Capacity (Gaussian channel)

\[ Y = X + Z, \quad Z \sim \text{normal} (0, N) \]

- Assume we want to submit a bit \( (0, 1) \).
- We send \( X \) with power \( P: \quad 0 \rightarrow -\sqrt{P}, \quad 1 \rightarrow \sqrt{P} \)
- Probability distribution for \( Y \):

\[
\begin{array}{c}
\text{if } 1 \text{ submitted} \\
\text{if } 0 \text{ submitted}
\end{array}
\]

So, if \( P \) is large, we can recover the correct signal with a high probability.

- Assume \( X = (x_1, \ldots, x_n) \) and \( \frac{1}{n} \sum_{i=1}^{n} x_i^2 \leq P \)
  
  Average power for transmission

- We want to use a channel for transmitting the symbols \( 1, 2, \ldots, M \):

  \[
  \begin{array}{c}
  1 \\
  2 \\
  \vdots \\
  M
  \end{array}
  \]

  \[
  \begin{array}{c}
  0 0 1 0 \\
  1 0 0 1 \\
  \vdots \\
  \end{array}
  \]

  Binary of size \( K \)

  \[
  \begin{array}{c}
  X_1 = (x_1, \ldots, x_n) \\
  X_2 = (x_2, \ldots, x_n) \\
  \vdots \\
  X_k = (x_{2k-1}, \ldots, x_n)
  \end{array}
  \]

  \[
  X_k = (x_{2k-1}, \ldots, x_n)
  \]

  \[
  \text{Coded}
  \]

  \[
  \text{Decoder}
  \]

  Recover the signal with a high probability

Technique:

\[
X^X_i
\]

Code design: sphere packing

\[
\text{Rate} = \frac{K}{n} \quad \text{bits per transmission}
\]

Capacity = \( \text{max} \) rate as \( n \) goes to infinity
Capacity of Gaussian channel:
\[ C = \frac{1}{2} \log(1 + \frac{P}{N}) \text{ bits per transmission} \]

Bandlimited channel with white noise:
Assume that the channel cuts off all frequencies greater than \( W \):
\[ \Rightarrow \text{Capacity} = W \log(1 + \frac{P}{N \cdot W}) \text{ bits per second} \]
\((N \cdot W = \text{power spectral density})\)

Problem: Optimal power and bandwidth allocation

- Total power is fixed:
\[ P_{\text{tot}} = P_1 + \ldots + P_L \quad (P_i \geq 0) \]

- Total bandwidth is fixed:
\[ W_{\text{tot}} = W_1 + \ldots + W_L \quad (W_i \geq 0) \]

- maximize the total utility:
\[ \sum_{i=1}^{L} U_i(R_i) \]

Note: \[-W_i \log \left(1 + \frac{P_i}{N \cdot W_i}\right) = \text{concave} \quad f(x) \text{ concave in } x \]
\[ tf\left(\frac{x}{t}\right) \text{ concave in } (x,t) \]

\[ \Rightarrow \min_{P_1, \ldots, P_L, W_1, \ldots, W_L, R_1, \ldots, R_L} \sum_{i=1}^{L} U_i(R_i) \]
\[ \begin{align*}
&\quad \quad \quad \quad \quad \quad P_i, W_i \geq 0 \quad i=1, \ldots, L \\
&\quad \quad \quad \quad \quad \quad \sum P_i = P_{\text{tot}} \\
&\quad \quad \quad \quad \quad \quad \sum W_i = W_{\text{tot}} \\
&\quad \quad \quad \quad \quad \quad R_i - W_i \log \left(1 + \frac{P_i}{N \cdot W_i}\right) \leq 0 \quad i=1, \ldots, L \quad (*) \\
\end{align*} \]

Note: \( R_i \uparrow \Rightarrow \text{obj} \downarrow \Rightarrow \quad (*) \) is binding at optimality
Robust optimization:

\[
\begin{align*}
\min_{x} \ & f_{0}(x) \\
\text{s.t.} \ & f_{i}(x) \leq 0 \quad i = 1, \ldots, m
\end{align*}
\]

Assume that there are some uncertainties in modeling \( f_{j} \)'s.

\[ f_{j}(x) \rightarrow f_{j}(x|u) \quad \text{and} \quad u \text{ is a random variable} \]

Stochastic optimization:

\[
\begin{align*}
\min_{x} \ & E_{u} \left( f_{0} \right) \\
\text{s.t.} \ & E_{u} \left( f_{i} \right) \leq 0 \quad i = 1, \ldots, m
\end{align*}
\]

\[
E_{u} \left( f_{j}(x|u) \right) = \int f_{j}(x|u) \pi(u) \, du
\]

(weighed sum of convex functions)

Worst case optimization:

\[
\begin{align*}
\min_{x} \ & \sup_{u} f_{i}(x|u) \\
\text{s.t.} \ & \sup_{u} f_{i}(x|u) \leq 0 \quad i = 1, \ldots, m
\end{align*}
\]

\[
\rightarrow \text{convex}
\]

So far, we have introduced LP, AP, ACAP, and SOCP.

Geometric programming (QP):

- Consider a function \( f : \mathbb{R}^{n}_{+} \rightarrow \mathbb{R} \) : \( f(x) = \sum_{i=1}^{n} a_{i} x_{i}^{\alpha_{i}} \)
- \( f(x) \) is called a monomial.
- Posynomial = a sum of monomials : \( \sum_{i=1}^{L} c_{i \lambda} x_{1}^{a_{1 \lambda}} x_{2}^{a_{2 \lambda}} \cdots x_{n}^{a_{n \lambda}} \)
- Posynomial: closed under addition and multiplication.
- Monomial: closed under division and multiplication.
\(-\text{GP:}\ \ \ \min f_{\text{GP}}(x)
\)
\(\text{st. } f_i(x) \leq 1 \quad i = 1, \ldots, m, \quad f_i's = \text{polynomials}
\)
\(h_j(x) = 1 \quad j = 1, \ldots, p \quad h_j's = \text{monomials}
\)

Some interesting circuit problems can be cast as GP.

Example:

\[
\begin{align*}
\min & \ \ \frac{x}{y} \\
\text{s.t. } & \ 2y \leq x \leq 3 \\
& \ x^2 + 3\frac{y}{z} \leq \sqrt{y} \\
& \ \frac{x}{y} = z^2 \\
& \ \frac{1}{3} x \leq 1 \\
& \ 2x^{-\frac{1}{2}}y \leq 1 \\
& \ x^y y^{-\frac{1}{2}} z^{-\frac{1}{2}} \leq 1 \\
& \ x^y y^{-\frac{1}{2}} z^{-\frac{1}{2}} = 1
\end{align*}
\]

Thm: GP can be cast as a convex optimization.

- \(\log(e^{x_1} + e^{x_2})\) is convex 
  \(\Rightarrow \nabla^2 = \frac{e^{x_1} e^{x_2}}{(e^{x_1} + e^{x_2})^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \geq 0\)

\(\Rightarrow \log(\leq e^{x_i})\) convex 
\(\Rightarrow \log(\leq e^{a_i y_i})\) convex

- Trick: \(x_i \rightarrow e^{y_i}\)

\(\Rightarrow \ c x_1 a_1 x_2 a_2 \ldots x_m a_m = 1 \rightarrow c e^{a_i y_i} = 1 \rightarrow e^{a_i y_i} = \text{constant}

\(\Rightarrow f_i(x) \leq 1 \rightarrow \log(\frac{1}{k_i)}e^{a_i y_i}) \leq 0\)

Convex

Example: Given \(M \in \mathbb{R}^{m \times n}\), design a diagonal matrix \(D > 0\) such that \(\|DMD^{-1}\|_F\) is minimized (Frobenius norm diagonal scaling).

\[
\min_{D > 0} \|DMD^{-1}\|_F^2 = \min_{D > 0} \sum_{i,j} m_i^2 d_i d_j^{-2} \rightarrow \text{GP}
\]
Semidefinite Program (SDP):

\[ M = \{ x \mid x_1 F_1 + \ldots + x_n F_n + G \leq 0 \} : \text{Convex} \]

\[ \text{SDP:} \quad \min_x c^T x \]

\[ \text{s.t.} \quad x_1 F_1 + \ldots + x_n F_n + G \leq 0 \quad \rightarrow \quad \text{Convex} \]

\[ A x = b \]

Example:

\[ \min_{x \in \mathbb{R}^2} x_1 - 2x_2 \]

\[ \left[ \begin{array}{cc}
    x_1 + x_2 & x_1 - 3x_2 \\
    x_1 - 3x_2 & x_1 + 2x_2
  \end{array} \right] \geq 0 \]

\[ \Rightarrow x_1 \left[ \begin{array}{cc}
    -1 & -1 \\
    -1 & -1
  \end{array} \right] + x_2 \left[ \begin{array}{cc}
    3 & 0 \\
    0 & -2
  \end{array} \right] \leq 0 \]

- \[ \leq x_i F_i \] is called a linear matrix inequality (LMI).

Standard Form of LP:

\[ \min_{x \in \mathbb{R}^n} \quad c^T x \]

\[ \text{s.t.} \quad a_i^T x = b_i, \quad i = 1, \ldots, k \]

\[ d_i^T x \leq e_i, \quad i = 1, \ldots, m \]

- Trick:

\[ x \in \mathbb{R}^n \quad \rightarrow \quad x = x^+ - x^-, \quad x^+, x^- \geq 0 \]

- Trick:

\[ d_i^T x \leq e_i \quad \rightarrow \quad d_i^T x + s_i = e_i \]

\[ \text{and} \quad s_i \geq 0 \]

\[ \text{slacks} \]

"Standard LP"
- Standard Form of SDP:

\[
\begin{align*}
\min_{X \in \mathbb{S}^n} & \quad \langle c, X \rangle \\
\text{s.t.} & \quad \langle A_i, X \rangle = b_i \quad \Rightarrow \quad \text{s.t.} \quad \text{tr}(AX) = b_i, \quad i = 1, \ldots, p \\
& \quad X \succeq 0
\end{align*}
\]

\[\Rightarrow \text{SDP = linear objective and linear scalar/matrix inequalities/equalities}\]

- Example:

\[
\begin{align*}
\min_{x_1, x_2} & \quad x_1 + 3x_2 \\
\text{s.t.} & \quad x_1, x_2 \geq 0 \\
& \quad x_1x_2 \geq 1 \quad \rightarrow \text{non-convex} \\
& \quad 2x_1 - 4x_2 \leq 5
\end{align*}
\]

\[\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \succeq 0\]

- Schur complement:

\[
\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \iff C > 0, \quad A - BC^{-1}B^T > 0
\]

- LP ≤ QCQP ≤ SOCQP ≤ SDP

proof: \[
\begin{align*}
\text{SOCQP} = \min_X & \quad f^T X \\
\text{s.t.} & \quad V AiX + bi \leq Ci^TX + d_i, \quad i = 1, \ldots, m \quad (**) \\
& \quad F_X = g
\end{align*}
\]

\[(**) \iff (AiX + bi)^T(AiX + bi) \leq (Ci^TX + d_i)^2
\]

\[\iff \begin{bmatrix} CiX + d_i & (AiX + bi)^T \\ (AiX + bi)^T & (CiX + d_i)I \end{bmatrix} \succeq 0\]

- How to find \(\min \lambda_{ij}\) of a symmetric matrix \(M\)?

\[
\begin{align*}
\max_{\lambda} & \quad \lambda \\
\text{s.t.} & \quad M \geq \lambda I
\end{align*}
\]
Example: matrix norm minimization

\[ A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad A_i \in \mathbb{R}^{p \times q} \]
\[ \min_{x \in \mathbb{R}^n} \| A(x) \|_2 \quad \rightarrow \quad \min_{\lambda, x} \lambda \]
\[ \text{s.t. } A(x)^T A(x) \leq \lambda I \]
\[ \rightarrow \min_{t, x} t^2 \quad \rightarrow \min_{t, x} t \]
\[ \text{s.t. } A(x)^T A(x) \leq t^2 I \]

Example: Eigenvalue optimization via SDP:

\[ A(x) = A_0 + x_1 A_1 + \cdots + x_n A_n, \quad A_i \in \mathbb{S}^m \]
\[ \text{eigs of } A(x): \lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x) \]
\[ (a): \text{maximize } \lambda_m(x): \max_{\lambda, x} \lambda \quad \text{s.t. } A(x) \geq \lambda I \]
\[ (b): \text{minimize } \lambda_1(x): \min_{\lambda, x} \lambda \quad \text{s.t. } A(x) \leq \lambda I \]
\[ (c): \text{minimize the spread of eigs, } \lambda_1(x) - \lambda_m(x): \]
\[ \min_{\lambda_1, \lambda_m, x} \lambda_1 - \lambda_m \quad \text{s.t. } A(x) \geq \lambda_m I \]
\[ A(x) \leq \lambda_1 I \]
\[ (d): \text{minimize the condition number } \frac{\lambda_1(x)}{\lambda_m(x)}: \]
\[ \min_{\lambda_1, \lambda_m, x} \frac{\lambda_1}{\lambda_m} \quad \text{s.t. } A_0 + \sum \lambda_i A_i \leq \lambda_m I \]
\[ A_0 + \sum \lambda_i A_i \leq \lambda_1 I \]
\[ \frac{1}{\lambda_m} A_0 + \sum \frac{\lambda_i}{\lambda_m} A_i \leq \frac{1}{\lambda_1} I \]
\[ \frac{1}{\lambda_m} A_0 + \sum \frac{\lambda_i}{\lambda_m} A_i \leq t I \]
Example: optimization with logistic model:

- $X \in \mathbb{S}_{0,1}$ random variable, $\text{Prob}(X = 1) = p = \frac{e^{a^T x + b}}{1 + e^{a^T x + b}}$
- $x \in \mathbb{R}^n$: parameter affecting probability
  (advertising effort, retail price, discounted price, packaging expense, ...)
- Constraint: $Fx \leq g$ (linear)

(a): maximize buying probability: $\max_x \frac{e^{a^T x + b}}{1 + e^{a^T x + b}}$

$\rightarrow \min_x \frac{1 + e^{a^T x + b}}{e^{a^T x + b}} \rightarrow \min_x e^{-a^T x - b} + 1 \rightarrow \text{convex}$

- $c^T x + d \geq 0$: profit from selling the product.

(b): maximize the expected profit: $\max_x \left( \frac{e^{a^T x + b}}{1 + e^{a^T x + b}} \right) (c^T x + d)$

$s.t. c^T x + d \geq 0$

$\rightarrow \min_x (e^{-a^T x - b} + 1) \frac{1}{c^T x + d} \rightarrow \min_{x,t} t$

$s.t. c^T x + d \geq 0$

$s.t. \frac{e^{-a^T x - b} + 1}{c^T x + d} \leq t$

$t = e^y$

$\rightarrow \min_{x,y} y$

$s.t. (e^{-a^T x - b} + 1)(e^{-y}) \leq c^T x + d$

$\rightarrow e^{-a^T x - b - y} + e^{-y} \leq c^T x + d$

$\rightarrow \text{convex}$

How to write QCQP as SDP:

\[
\begin{align*}
\min_{x, y} & \quad x^T p_0 x + q_0^T x + r_0 \\
\text{s.t.} & \quad x^T p_i x + q_i^T x + r_i \leq 0, \quad i = 1, \ldots, n \\
& \quad (x^T p_i y_i)(p_i^T x) \leq -y_i - q_i^T x
\end{align*}
\]

$\rightarrow \left[ \begin{array}{cc}
-y_i - q_i^T x & x^T p_i y_i \\
 p_i^T x & I
\end{array} \right] \succeq 0$