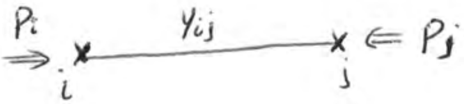
Example: $\min \| \underbrace{A_0 + A_1 x_1 + \cdots + A_n x_n}_{A(x)} \|_2$

$$= \min \sqrt{\lambda_{max}(A(x)^T A(x))} \iff \begin{array}{ll} \min & t \\ \text{s.t.} & \begin{bmatrix} tI & A(x) \\ A(x)^T & tI \end{bmatrix} \succeq 0 \end{array}$$

Back to optimization for distribution networks:



OPF: $\min \; f_i(P_i) + f_j(P_j)$

s.t. $- \; P_k \le P_k^{max} \quad k = i, j$

$- \; V_k^{min} \le |V_k| \le V_k^{max}$

$-$ Laws of physics

- This is the generalization of the previous case $|v_i| = 1$ studied geometrically.

$$\begin{cases} P_i = Re(V_i (V_i - V_j)^* y_{ij}^*) \\ P_j = Re(V_j (V_j - V_i)^* y_{ij}^*) \end{cases}$$, P_i and P_j are non-convex functions of V_i and V_j.

Define: $W_{(ij)} = \begin{bmatrix} V_i \\ V_j \end{bmatrix} \begin{bmatrix} V_i^* & V_j^* \end{bmatrix} \Rightarrow \begin{cases} P_i = Re([W_{i,i} - W_{i,j}] y_{ij}^*) \\ P_j = Re([W_{j,j} - W_{j,i}] y_{ij}^*) \end{cases}$

- good news: P_i and P_j Linear in W.
- bad news: $W_{(ij)}$ non-convex in V_i and V_j

$$W_{(ij)} = \begin{bmatrix} V_i \\ V_j \end{bmatrix} \begin{bmatrix} V_i^* & V_j^* \end{bmatrix} \iff \underbrace{W_{(ij)} \succeq 0}_{SDP}, \; \underbrace{\text{Rank}(W_{(ij)}) = 1}_{\text{non-convex}}$$

- Convexification: Drop $\text{Rank}(W_{(ij)}) = 1$

$\Rightarrow$ Convexified OPF: $\min \; f_i(P_i) + f_j(P_j)$

s.t. $- \; P_k \le P_k^{max} \quad k = i, j$

$- \; (V_k^{min})^2 \le W_{kk} \le (V_k^{max})^2$

$- \begin{cases} P_i = Re((W_{i,i} - W_{i,j}) y_{ij}^*) \\ P_j = Re((W_{j,j} - W_{j,i}) y_{ij}^*) \end{cases}, \; - \; W_{(ij)} \succeq 0$

- claim: OPF and convexified (relaxed) OPF are equivalent.

- proof: Let $W_{(ij)}^{opt}$ be one solution of relaxed OPF

- Define: $W_{(ij)}^{\varepsilon} = W_{(ij)}^{opt} + \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}$, $\varepsilon \geq 0$

- since $y_{ij} = \frac{1}{r+jX}$, $Re(y_{ij}) \geq 0$

$$\Rightarrow P_i^{\varepsilon} = P_i^{opt} - \varepsilon \, Re(y_{ij}) \leq P_i^{opt} , \quad P_j^{\varepsilon} \leq P_j^{opt}$$

$$\Rightarrow \begin{cases} P_k^{\varepsilon} \leq P_k^{opt} \leq P_k^{max} & k = i, j \\ f_i(P_i^{\varepsilon}) + f_j(P_j^{\varepsilon}) \leq f_i(P_i^{opt}) + f_j(P_j^{opt}) & (f \text{ is increasing}) \end{cases}$$

$\Rightarrow P^{\varepsilon}$ is better than P^{opt}.

- How about $W_{(ij)}^{\varepsilon} \succeq 0$? There exists an $\varepsilon \geq 0$ such that $W_{(ij)}^{\varepsilon} \succeq 0$ and $\mathrm{Rank}(W_{(ij)}^{\varepsilon}) = 1$.

- proof: $W_{(ij)}^{opt} \succeq 0 \Longrightarrow \| W_{i,j}^{opt} \|^2 \leq W_{i,i}^{opt} \times W_{j,j}^{opt}$

$$\Rightarrow \exists \, \varepsilon > 0 : \| W_{i,j}^{opt} + \varepsilon \|^2 = W_{i,i}^{opt} \times W_{j,j}^{opt}$$

(consider $\varepsilon = 0$ and $\varepsilon = \infty$, then use continuity).

$\Rightarrow$ Convexified OPF has a rank-1 solution.

- Note that convexified OPF = SDP.
- It can even be SOCP:

$$W_{(ij)} \succeq 0 \iff W_{i,i} \times W_{j,j} \geq \| W_{i,j} \|^2 , \; W_{j,j} \geq 0$$

$$\iff (W_{i,i} + W_{j,j})^2 \geq \| W_{i,j} \|^2 \times 2 + W_{i,i}^2 + W_{j,j}^2 , \; W_{j,j} \geq 0$$

$$\iff W_{i,i} + W_{j,j} \geq \left\| \left[\sqrt{2}\, W_{i,j} \quad W_{i,i} \quad W_{j,j} \right] \right\| , \; W_{i,i} \geq 0, \; W_{j,j} \geq 0$$

General Distribution Network:

OPF:
$$\begin{cases} \min \sum_{i=1}^{n} f_i(p_i) \\ - \ p_i \le p_i^{max} \quad i \in \{1, \dots, n\} \\ - \ p_{ij} \le p_{ij}^{max} \quad (i,j) \in L \\ - \ V_i^{min} \le |V_i| \le V_i^{max} \quad i \in \{1, \dots, n\} \\ - \ \text{Laws of physics} \end{cases}$$

SDP (SOCP) Relaxation:
$$\begin{cases} \min \sum_{i=1}^{n} f_i(p_i) \\ p_i \le p_i^{max}, \quad i \in \{1, \dots, n\} \\ p_{ij} = Re((W_{i,i} - W_{i,j}) y_{ij}^*), \quad i \in \{1, \dots, n\} \\ p_{ij} \le p_{ij}^{max}, \quad (i,j) \in L \\ p_i = \sum_{j \in N(i)} p_{ij}, \quad i \in \{1, \dots, n\} \\ (V_i^{min})^2 \le W_{i,i} \le (V_i^{max})^2, \quad i \in \{1, \dots, n\} \\ W_{(ij)} = \begin{bmatrix} W_{i,i} & W_{i,j} \\ W_{i,j}^* & W_{j,j} \end{bmatrix} \succeq 0, \quad (i,j) \in L \end{cases}$$

- How to recover the solution of opf:

1- Find one solution $W_{(ij)}^{opt}$, $(i,j) \in L$

2- Given $(i,j) \in L$, find ε such that $W_{(ij)} \succeq 0$ and $W_{(ij)} = \text{rank } \underline{1}$

3- Write $W_{(ij)}$ as $\begin{bmatrix} V_i^{opt} \\ V_j^{opt} \end{bmatrix} \begin{bmatrix} V_i^{opt} & V_j^{opt} \end{bmatrix}^*$ to get optimal voltages.

- what if we have an angle constraint $|\theta_{ij}| \le \theta_{ij}^{max}$ $(< 90^\circ)$

$$\Longleftrightarrow \left| \tan \theta_{ij} = \frac{Im(W_{i,j})}{Re(W_{i,j})} \right| \le \tan \theta_{ij}^{max} \Longleftrightarrow |Im(W_{i,j})| \le \alpha Re(W_{i,j}$$

(ε-modification doesn't violate thi

Lagrangian Duality:

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t.}\ & f_i(x) \le 0 \quad i=1,\dots,m \quad \leftarrow \lambda_i \\ & h_i(x) = 0 \quad i=1,\dots,p \quad \leftarrow \nu_i \end{aligned}$$

$$L(x,\lambda,\nu) = f_0(x) + \sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \quad \leftarrow \text{Lagrangian}$$

- Lagrangian Dual: $g: \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}$

$$g(\lambda,\nu) = \inf_{x\in D} L(x,\lambda,\nu) \Rightarrow \underline{g: \text{concave}}$$

(inf on a linear function)

- property $\underline{1}$: if $\lambda \geq 0 \Rightarrow g(\lambda,\nu) \leq p^*$

proof: Consider $\lambda \geq 0$ and feasible $\tilde{x}$:

$$f_0(\tilde{x}) \geq L(\tilde{x},\lambda,\nu) \geq \inf_{x\in D} L(x,\lambda,\nu) = g(\lambda,\nu)$$

Example: $\min\ x^T x$ s.t. $Ax=b$

$$L(x,\nu) = x^T x - \nu^T(Ax-b) \Rightarrow \nabla_x L(x,\nu)=0:\ x = -\tfrac{1}{2}A^T\nu$$

$$\Rightarrow g(\nu) = -\frac{1}{4}\nu^T A^T A \nu - b^T \nu \Rightarrow \boxed{p^* \geq \left(-\frac{1}{4}\right)\nu^T A^T A \nu - b^T\nu \quad \forall \nu}$$

This immediately gives a lower bound.

Example: (LP) $\min\ c^T x$ s.t. $Ax=b$, $x \geq 0$

$$\Rightarrow L(x,\lambda,\nu) = c^T x - \lambda^T x + \nu^T(Ax-b)$$
$$= -\nu^T b + (c^T - \lambda^T + \nu^T A)x$$

$$\Rightarrow g(\lambda,\nu) = \begin{cases} -b\nu^T & \text{if } A^T\nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\Rightarrow \boxed{p^* \geq -b\nu^T \text{ if } A^T\nu + c \geq 0}$$

Example: $\min\ x^T W x$ s.t. $x_i^2 = 1$

- 2^n feasible points, hard to solve.

$$g(\nu) = \inf_x \left(x^T W x + \sum_i \nu_i (x_i^2 - 1)\right)$$

$$= \inf_x x^T (W + \mathrm{diag}(\nu)) x - \mathbf{1}^T \nu = \begin{cases} -\mathbf{1}^T \nu & W + \mathrm{diag}(\nu) \succeq 0 \\ -\infty & \text{otherwise} \end{cases}$$

- Consider $\nu = -\lambda_{min}(W)\mathbf{1} \implies p^* \geq n\lambda_{min}(W)$

Dual Problem: maximize $g(\lambda, \nu)$ s.t. $\lambda \geq 0$

- Denote the solution as d^*.
- Dual feasible is a lower bound on primal feasible

$\implies \boxed{d^* \leq p^* \quad \text{weak duality}}$

Note that: primal might be non-convex, but dual is always convex.

Example:

primal) $\min c^T x$ s.t. $Ax = b$, $x \geq 0$, Dual) $\max -b^T \nu$ s.t. $A^T \nu + c \geq 0$

- Application: $\begin{cases} \min x^T W x \\ \text{s.t. } x_i^2 = 1 \end{cases}$ hard to solve.

But $\begin{cases} \max -\mathbf{1}^T \nu \\ \text{s.t. } W + \mathrm{diag}(\nu) \succeq 0 \end{cases}$ is easy (SDP) and provides a lower bound (no solution,

- Strong Duality: $\min f_0(x)$ s.t. $f_i(x) \leq 0 \quad i = 1, \dots, m$, $Ax = b$ $\Leftarrow$ Convex

Assume it is strictly feasible: $\exists x \in \mathrm{int}\, D: f_i(x) < 0, Ax = b$

$$\implies p^* = d^* > -\infty$$

- strict feasibility is <u>only</u> a sufficient condition.

<u>Example:</u> - strong duality always for LP if primal feasible.

- strong duality always for QP

<u>Complementary slackness:</u>

- Assume that strong duality holds for a non-convex/convex optimization
- x^*: primal optimal, (λ^*, v^*): dual optimal

$$f_0(x^*) = g(\lambda^*, v^*) = \inf_x \left(f_0(x) + \sum_{i=1}^{m} \lambda_i^* f_i(x) + \sum_{i=1}^{p} v_i^* h_i(x) \right)$$
$$\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^* f_i(x^*) + \sum_{i=1}^{p} v_i^* h_i(x^*))$$
$$\leq f_0(x^*)$$

$$\Rightarrow \begin{cases} 1- x^* \text{ minimizes } L(x, \lambda^*, v^*). \\ 2- \lambda_i^* f_i(x^*) = 0 \quad , i=1, \dots, n \end{cases}$$

<u>Karush-Kuhn-Tucker (KKT) conditions:</u> (for differentiable f_i, h_j)

$$\begin{cases} 1- f_i(x) \leq 0 \,, \; i=1,\dots,m, \; h_j(x)=0 \,, \; j=1,\dots,p & \leftarrow \text{primal feasibility} \\ 2- \lambda \geq 0 & \leftarrow \text{dual feasibility} \\ 3- \lambda_i f_i(x) = 0 \,, \; i=1,\dots,m & \leftarrow \text{complementary slackness} \\ 4- \nabla f_0(x) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) + \sum_{j=1}^{p} v_j \nabla h_j(x) = 0 & \leftarrow \text{zero gradient} \end{cases}$$

we have:

- If strong duality holds and (x, λ, v) is optimal, then it satisfies KKT.
- If (x, λ, v) satisfies KKT for a convex optimization, they are optimal.
- If slater's condition holds, then optimality $\overset{\text{iff}}{\Longleftrightarrow}$ KKT

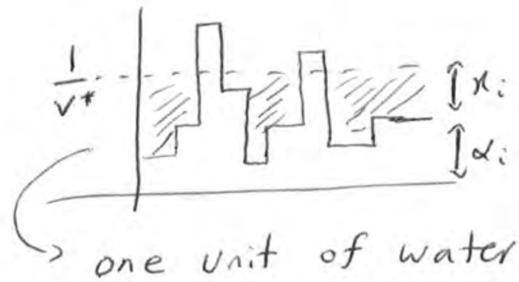
Example: water-filling $(\alpha_i > 0)$

$$\begin{rcases} \min & -\sum_{i=1}^{n} \log(x_i+\alpha_i) \\ \text{s.t.} & x \geq 0 \quad 1^T x = 1 \end{rcases} \overset{\text{KKT}}{\Longrightarrow} \begin{array}{l} x \geq 0, \quad 1^T x = 1, \\ \lambda \geq 0, \quad \lambda_i x_i = 0, \quad \dfrac{1}{x_i+\alpha_i} + \lambda_i = \nu \end{array}$$

- If $\nu < 1/\alpha_i \Rightarrow \lambda_i = 0$ and $x_i = \dfrac{1}{\nu} - \alpha_i$
- If $\nu \geq \dfrac{1}{\alpha_i} \Rightarrow \lambda_i = \nu - \dfrac{1}{\alpha_i}$ and $x_i = 0$

$$\Rightarrow \quad 1^T x = \sum_{i=1}^{n} \max\{0, \tfrac{1}{\nu} - \alpha_i\} = 1$$

$\Rightarrow$ find ν^* and then x_i^*:



perturbation and sensitivity analysis:

- $$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0 \\ & h_i(x) = 0 \end{array} \Rightarrow \begin{array}{ll} \max & g(\lambda, \nu) \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

- $$\begin{array}{ll} \min & f_0(x) \\ \text{s.t.} & f_i(x) \leq a_i \\ & h_i(x) = b_i \end{array} \Rightarrow \begin{array}{ll} \max & g(\lambda, \nu) - a^T\lambda - b^T\nu \\ \text{s.t.} & \lambda \geq 0 \end{array}$$

  Define: $p^*(a,b)$ optimal value as a function of $\underline{a}$ & $\underline{b}$

- Assume that strong duality holds.
- λ^* and ν^*: dual optimal for unperturbed Problem.

$$\begin{aligned} p^*(a,b) &\geq g(\lambda^*, \nu^*) - a^T\lambda^* - b^T\nu^* \\ &= p^*(0,0) - a^T\lambda^* - b^T\nu^* \end{aligned}$$

- Interpretation:
  - λ_i^* Large $\Rightarrow p^*$ increases greatly by tightening constraint i
  - λ_i^* small $\Rightarrow p^*$ decreases a Litle by Loosening cons $\underline{i}$.
  - ν_i^* Large and positive $\Rightarrow p^*$ increases greatly if we take $b_i < 0$.

  ⋮

- Convex problem $\Rightarrow$ $p^*(\underline{a},\underline{b})$ is convex in $\underline{\underline{a}}$ & $\underline{\underline{b}}$.

- If $p^*(a,b)$ is differentiable at $(0,0)$, then

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial a_i}, \quad \nu_i^* = -\frac{\partial p^*(0,0)}{\partial b_i}$$

shadow price

Semidefinite program:

primal SDP:

$\min \ c^T x$

s.t. $x_1 F_1 + \cdots + x_n F_n \preceq G \leftarrow$ $Z \in S_+^k$ dual vari[able]

$$\Rightarrow g(Z) = \inf_x L(x,Z) = \inf_x \left(c^T x + \operatorname{trace}\left(Z\left(\sum_i x_i F_i - G\right)\right)\right)$$

$$= \begin{cases} -\operatorname{trace}(ZG) & \text{if } \operatorname{tr}(F_i Z) + c_i = 0 \quad i=1,\dots,n \\ -\infty & \text{otherwise} \end{cases}$$

$\Rightarrow$ dual SDP:

$\max \ -\operatorname{trace}(GZ)$

s.t. $Z \succeq 0$, $\operatorname{tr}(F_i Z) + c_i = 0$ $\quad i=1,\dots,n$

and $p^* = d^*$ if primal SDP is strongly feasible.

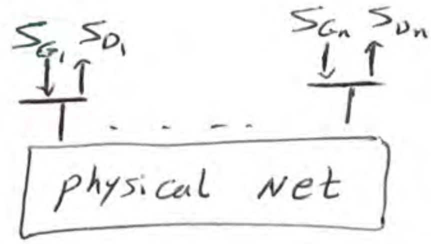
Numerical Algorithm:

- Consider unconstraind case: $\min f(x) \to$ convex

$x^{(1)}, x^{(2)}, \dots \to x^*$, $\quad x^{(k+1)} = x^{(k)} + t^{(k)} \Delta x^{(k)}$ such that $f(x^{(k+1)}) < f(x^{(k)})$

- Gradient descent: $\Delta x = -\nabla f(x)$
- steepest descent: $\Delta x = \operatorname{argmin} \{ \nabla f(x)^T v \mid \|v\| = 1 \}$ $\hookrightarrow$ given norm, Like $x^T P$
- Newton step: $\Delta x = -\nabla^2 f(x)^{-1} \nabla f(x)$

Optimal Power Flow for Transmission Networks:



OPF: $\min \sum_{k=1}^{n} f_k(P_{G_k})$

s.t. ✓ - $P_k^{min} \le P_k \le P_k^{max}$

- $Q_k^{min} \le Q_k \le Q_k^{max}$

✓ - $V_k^{min} \le |V_k| \le V_k^{max}$

- $|S_{kl}| \le S_{kl}^{max}$

- $|P_{kl}| \le P_{kl}^{max}$

- $|V_k - V_l| \le \Delta V_{kl}^{max}$

✓ - power balance equations

Ignore some of the constraints to simplify the derivation

Define: $X = \begin{bmatrix} Re(V) \\ Im(V) \end{bmatrix}$, $W = XX^*$

$$\Rightarrow P_k = P_{G_k} - P_{D_k} = Re(V_k I_k^*) = Re(V^* e_k e_k^* I) = Re(V^* \underbrace{e_k e_k^* Y}_{Y_k} V)$$

$$= X^T \begin{bmatrix} Re(Y_k) & -Im(Y_k) \\ Im(Y_k) & Re(Y_k) \end{bmatrix} X = \frac{1}{2} X^T \begin{bmatrix} Re(Y_k + Y_k^T) & Im(Y_k^T - Y_k) \\ Im(Y_k - Y_k^T) & Re(Y_k + Y_k^T) \end{bmatrix} X$$

$$= X^T \bar{Y}_k X = \text{Trace}(\bar{Y}_k W)$$

$$\Rightarrow \text{OPF}: \begin{cases} \min \sum_{k=1}^{n} f_k(P_{G_k}) \\ P_k^{min} \le P_{G_k} \le P_k^{max} \\ P_{G_k} = \text{Trace}(\bar{Y}_k W) + P_{D_k} \\ (V_k^{min})^2 \le \text{Trace}\left(\begin{bmatrix} e_k e_k^T & 0 \\ 0 & e_k e_k^T \end{bmatrix} W\right) \le (V_k^{max})^2 \\ \text{Rank}(W) = 1, \quad W \succeq 0 \end{cases}$$

- Relaxation: Remove $\text{Rank}(W) = 1 \Rightarrow$ SDP
- If there is rank-1 solution, relaxation is tight.

- Dual OPF: $P_k^{min} \leq P_G \leq P_k^{max} \longrightarrow (\underline{\lambda}_k, \bar{\lambda}_k)$

$$P_{G_k} = \text{trace}(\bar{Y}_k X X^*) + P_{D_k} \longrightarrow \nu_k$$

$$(v_k^{min})^2 \leq tr(H_k X X^*) \leq (V_k^{max})^2 \longrightarrow (\underline{\gamma}_k, \bar{\gamma}_k)$$

$$\Rightarrow L(p, X, \underline{\lambda}, \bar{\lambda}, \nu, \underline{\gamma}, \bar{\gamma}) = \underbrace{\sum_{k=1}^{n} \left(f_k(P_{G_k}) + (-\underline{\lambda}_k + \bar{\lambda}_k + \nu_k) P_{G_k} \right)}_{A}$$

$$+ tr\left(\left[\sum_{k=1}^{n} -\nu_k \bar{Y}_k - \underline{\gamma}_k H_k + \bar{\gamma}_k H_k \right] X X^* \right) \longrightarrow B$$

$$+ \sum_{k=1}^{n} \left(\underline{\lambda}_k P_k^{min} - \bar{\lambda}_k P_k^{max} + (v_k^{min})^2 \underline{\gamma}_k - (v_k^{max})^2 \bar{\gamma}_k - \nu_k P_{D_k} \right) \rightarrow C$$

$\min_{X,p} A \longrightarrow \sum_{k=1}^{n} -\bar{f}_k(\underline{\lambda}_k - \bar{\lambda}_k - \nu_k) \hookrightarrow$ conjugate function

$\min_{X,p} B = 0$ if $\sum_{k=1}^{n} (-\nu_k \bar{Y}_k - \underline{\gamma}_k H_k + \bar{\gamma}_k H_k) \succeq 0$

$\min_{X,p} C = C$

$\Rightarrow$ Dual OPF: $\max_{\theta} f(\theta) \rightarrow$ scalar

s.t. $A(\theta) \succeq 0$ $\leftarrow$ matrix

- Interesting point: Dual of Dual opf = Relaxed OPF

(dual of dual = itself if optimization is convex)

$$\begin{tikzcd} \boxed{\text{OPF}} \arrow[r, "\text{Dual}"] \arrow[rr, bend right, dashed, "\text{Convexification}"'] & \boxed{\text{Dual opf}} \arrow[r, leftrightarrow, "\text{Dual}"] & \boxed{\text{Relaxation}} \end{tikzcd}$$

- stron duality between Dual opf & SDP relaxation.
- Dual variable for $A(\theta) \dashrightarrow W \Rightarrow Tr(A(\theta) W) = 0$
- Relaxation is tight $\Longleftrightarrow d^* = p^*$ (strong duality)

- Relaxation is tight if $\mathrm{Rank}(W^{opt}) = 1$
- Ok even if $\mathrm{Rank}(W^{opt}) = 2$ (reference for angle is not considered).

- To get $\mathrm{Rank}(W^{opt}) = 2$, it's enough that

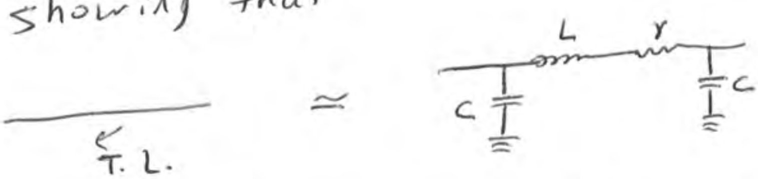
  $\mathrm{Rank}(A(\theta^{opt})) \geq 2n-2$:

  $\mathrm{Tr}(A(\theta^{opt})\, W^{opt}) = \mathrm{Tr}\left(\sum_{k=1}^{m} \mu_k U_k U_k^* W^{opt}\right) = \sum_{k=1}^{m} \mu_k U_k^* W^{opt} U_k$

  $\Rightarrow W^{opt} [U_1 U_2 \cdots U_m] = 0$, $m \geq 2n-2$ $\Longrightarrow$ $\mathrm{Rank}(W^{opt}) \leq 2$

- How to check strong duality: solve dual OPF, check $\mathrm{Rank}(A(\theta^{opt}))$

- A detailed discussion is provided in recent papers showing that this condition likely holds because



The physics of T.L.

- Special case: OPF in DC case: (all parameters are real and lines are resistor).

- It can be shown that $\underbrace{A(\theta)}_{n \times n,\ \text{not}\ 2n \times 2n} = \begin{bmatrix} ? & - & - \\ - & ? & - \\ - & - & ? \end{bmatrix} \succeq 0$

- Perron-Frobenius: max eig of positive matrix is simple and real.

- $\underbrace{-A(\theta)}_{\text{eig}\ \alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq 0} + \varepsilon I = \begin{bmatrix} + & + & + \\ + & + & + \\ + & + & + \end{bmatrix}$ for large enough ε.

$\Rightarrow$ $\max(-\alpha_n + \varepsilon, \ldots, -\alpha_1 + \varepsilon) =$ simple

$\Rightarrow \alpha_1 \neq \alpha_2$ $\Rightarrow$ $\mathrm{rank}(A(\theta^{opt})) \geq n-1$, Done!