- Need to know "convex optimization" to generalize to variable $|V_i|$ or to include $P_i^{min} \le P_i$.

Convex optimization:

optimization: $\min f_0(x)$ s.t. $f_i(x) \le 0$, $x = (x_1, \dots, x_n)$ variables; $f_0 : \mathbb{R}^n \to \mathbb{R}$ objective; $f_i : \mathbb{R}^n \to \mathbb{R}$ constraint

optimal solution x^*: has the smallest value of f_0 among all vectors satisfying the constraints.

- very hard to solve in general (long computation time)
- certain classes can be solved efficiently.

Example: least-squares $\underset{x \in \mathbb{R}^n}{\text{minimize}} \|Ax - b\|_2^2$

- Analytical solution: $x^* = (A^TA)^{-1}A^Tb$
- Computation time: $O(n^2 K)$ $(A \in \mathbb{R}^{K \times n})$
- Almost all optimizations have no analytical solutions.
- So, we need an iterative algorithm.
- The easiest non-trivial case: $\min c^T x$ s.t. $a_i^T x \le b_i$ $i = 1, \dots, m$
- This is linear programming (LP).
- computation time: $O(n^2 m)$ if $m \ge n$

- Convex optimization: $\min f_0(x)$ s.t. $f_i(x) \le b_i$ $i = 1, \dots, m$

where $f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$ $i = 0, 1, \dots, m$

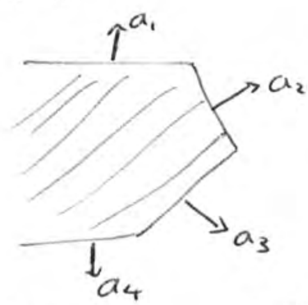
$\forall\ \alpha, \beta \in [0, 1]$ s.t. $\alpha + \beta = 1$

- No analytical solution in general
- Computation time proportional to $\max(n^3, n^2 m, F)$, where F is cost of evaluating f_i, f_i', f_i'' (partials)

More details on convex optimization:

Convex set: 

- polyhedra: $\{x \mid \underbrace{Ax \leq b}_{\text{half space}}, \underbrace{Cx = d}_{\text{halfplane}}\}$



- positive semidefinite matrices: $S^n = \{X \in \mathbb{R}^{n\times n} \mid X = X^T\}$

$S^n_+ = \{X \in S^n \mid X \succcurlyeq 0\}$

- $S \subseteq \mathbb{R}^n$ convex $\xrightarrow{f(x) = Ax + b \text{ (affine)}}$ $f(S) = \{f(x) \mid x \in S\}$ convex

Applications:
- scaling, translation, projection
- linear matrix inequality (LMI):

$\{x \mid x_1 A_1 + x_2 A_2 + \cdots + x_n A_n \preccurlyeq B\}$ $(A_i, B \in S^p)$

Example: $\{(x,y) \mid (y-1) \geq (x+2)^2\}$ 

$= \left\{(x,y) \;\middle|\; \begin{bmatrix} y-1 & x+2 \\ x+2 & 1 \end{bmatrix} \succcurlyeq 0 \right\}$ convex

- Convex function: $f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$ $\quad 0 \leq \theta \leq 1$

strictly convex if $f(\theta x + (1-\theta)y) < \theta f(x) + (1-\theta) f(y)$, $0 < \theta < 1$

- Concave: f is concave if $-f$ is convex.

- Convex:
  - affine $ax+b$
  - exponential e^{ax}
  - powers x^a on $\mathbb{R}_{++}$ for $a \geq 1$ or $a \leq 0$
  - negative entropy $x \log x$ on $\mathbb{R}_{++}$

- Concave:
  - affine $ax+b$
  - powers x^a on $\mathbb{R}_{++}$ for $0 \leq a \leq 1$
  - Logarithm $\log x$ on $\mathbb{R}_{++}$

- Other examples:
  - Norm: $\|x+y\| \geq \|x\| + \|y\|$ $\Rightarrow$ every norm is convex

    $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$

    $\|x\|_\infty = \max_k |x_k|$

    $\|X\|_2 = \sqrt{\lambda_{max}(X^T X)}$
  - Affine: $a^T x + b$ or $\text{trace}(A^T X)$

- Restriction to a line:

$f: \mathbb{R}^n \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

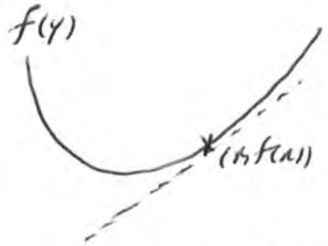
$g(t) = f(x+tv)$ $\Rightarrow$ f is convex in x iff g is convex in t for every x and v.

- Example: $f(X) = \log \det(X)$ is convex on S^n_{++}.

- f is differentiable if $\underline{\text{dom } f}$ is open and the gradient $\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$ exists.

- <u>First-order condition:</u> differentiable f is convex iff

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) \quad \forall x, y \in \text{dom } f$$



first-order approximation is global underestimator.

- f is twice differentiable if $\underline{\text{dom } f}$ is open and the Hessian $\nabla^2 f(x)$ exists: $\nabla^2 f(x)_{i,j} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j}$

- <u>second-order condition:</u> twice-differentiable f is convex iff

$$\nabla^2 f(x) \succeq 0 \qquad \forall x \in \text{dom } f$$

- <u>Examples:</u>

1- $f(x) = \frac{x^T P x}{2} + q^T x + r \implies \nabla f(x) = Px + q \implies \nabla^2 f(x) = P$

convex iff $P \succeq 0$

2- least-squares objective: $f(x) = \|Ax - b\|_2^2$

$\nabla f(x) = 2A^T(Ax - b) \implies \nabla^2 f(x) = 2A^T A \succeq 0$ convex

3- Quadratic over linear: $f(x,y) = \frac{x^2}{y}$

$\implies \nabla^2 f(x,y) = \frac{2}{y^3} \begin{bmatrix} y \\ -x \end{bmatrix} \begin{bmatrix} y & -x \end{bmatrix} \succeq 0$ if $y > 0$ (convex over $\mathbb{R} \times \mathbb{R}_{++}$)

- More complicated examples:
  - $f(x) = \log \sum_{k=1}^{n} e^{x_k}$ convex (log-sum-exp)
  - $f(x) = \left(\prod_{k=1}^{n} x_k\right)^{1/n}$ on $\mathbb{R}^n_{++}$ concave (geometric mean)

Operations preserving convexity:

- f convex $\Longrightarrow$ αf convex for $\alpha \geq 0$
- f_1, f_2 convex $\Rightarrow$ $f_1 + f_2$ convex
- f convex $\Rightarrow$ $f(Ax+b)$ convex

Example: Log barrier for linear inequality:

$$f(x) = -\sum_{i=1}^{m} \log(b_i - a_i^T x), \quad \text{dom } f = \{x \mid a_i^T x < b_i\}$$

- pointwise maximum:

  $f_1, f_2, \dots, f_n$ convex $\Longrightarrow$ $\max(f_1(x), \dots, f_n(x))$ convex

Example: $f(x) = \max_i (a_i^T x + b_i)$ convex

- pointwise supremum

  $f(x,y)$ convex in $\underline{x}$ $\Longrightarrow$ $g(x) = \sup_{y \in A} f(x,y)$ convex

Examples: (1) distance to farthest point in a set $\underline{S}$:

$$f(x) = \sup_{y \in C} \|x - y\|$$

(2) max eig of a symmetric matrix X:

$$\lambda_{max}(X) = \sup_{\|y\|_2 = 1} \underbrace{y^T X y}_{\text{Linear in } X}$$

- Minimization:

  $f(x,y)$ convex in x, y, C convex

  $\Longrightarrow$ $g(x) = \inf_{y \in C} f(x,y)$ convex

Example: - distance to a set:

$$\text{dist}(x, S) = \inf_{y \in S} \|x - y\| \quad \text{convex if } \underline{S} \text{ convex}$$

- Conjugate function:

$$f^*(y) = \sup_{x \in \mathrm{dom}\, f} \left(y^T x - f(x) \right) \qquad \text{convex}$$

Optimization in standard form:

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t. } & f_i(x) \le 0 \qquad i = 1, \dots, m \\ & h_i(x) = 0 \qquad i = 1, \dots, p \end{aligned}$$

inequality cons.
equality cons.

p^* = inf of $f_0(x)$ over the feesible set.

- $p^* = -\infty$: problem is unbounded from below.
- $p^* = +\infty$: problem is infeasible
- $x \in \mathrm{dom}\, f_0$ is a feasible point if it satisfies the cons.
- x^* is locally optimal if $\exists\ R > 0$ such that x^* is optimal for:

$$\min\ f_0(x) \quad \text{s.t. } f_i(x) \le 0,\ h_j(x) = 0,\ \| x^* - x \|_2 \le R$$

- $f_0(x) = \frac{1}{x}$ over $\mathbb{R}_{++}$ $\Rightarrow p^* = 0$ but no optimal point
- $f_0(x)$:

many solutions!

Convex optimization:

$$\begin{aligned} \min\ & f_0(x) \\ \text{s.t. } & f_i(x) \le 0 \qquad i = 1, \dots, m \\ & \underbrace{a_i^T x = b_i}_{AX = b} \qquad i = 1, \dots, p \end{aligned}$$

Assumption:

- f_i, $i = 0, \dots, m$, is convex.
- No non-affine equality.

$$\min \; x_1^2 + x_2^2$$
$$\text{s.t.} \; \frac{x_1}{1+x_1^2} \geq 0 \qquad \text{Non-Convex} \xrightarrow{\text{Transformation}} \qquad x_1 \geq 0$$
$$(x_1+x_2)^2 = 0 \qquad\qquad\qquad\qquad x_1 + x_2 = 0$$

- What is good about convex optimization? Local solution is a global solution.

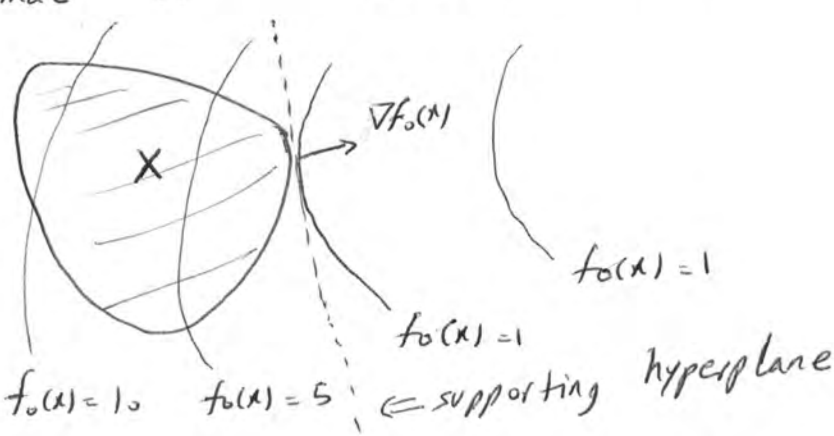
$\|x^l - z\| \leq R \implies z = \theta x^l + (1-\theta) x^g$

$f(z) \leq \theta f(x^l) + (1-\theta) f(x^g) < f(x^l)$

This can't be true.

- optimality criterion for differentiable f_0:

x^* is optimal iff $\nabla f_0(x)^T (y - x) \geq 0 \quad \forall$ feasible y



- Remark: No constraint $\implies \nabla f_0(x) = 0$

- Equivalence between optimizations:

1 $\quad Ax = b \iff x = Fz + x_0$ for some z

$$\begin{array}{ll} \min_x & f_0(x) \\ \text{s.t.} & f_i(x) \leq 0 \\ & Ax = b \end{array} \iff \begin{array}{ll} \min_z & f_0(Fz + x_0) \\ \text{s.t.} & f_i(Fz + x_0) \leq 0 \end{array}$$

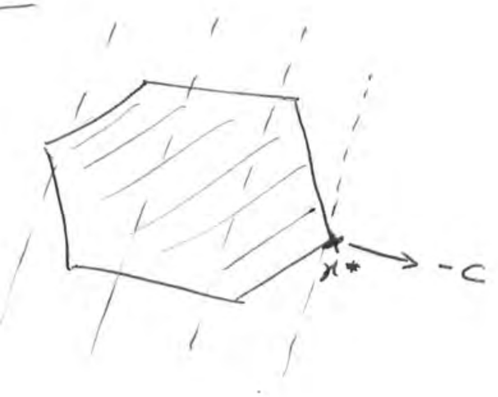
2 $\min f_0(x)$ s.t. $a_i^T x \le b_i$ $\iff$ $\min f_0(x)$, $a_i^T x + s_i = b_i$, $s_i \geq 0$

slack variables

- Different Types of optimizations:

- Linear program (LP): $\min\ c^T x + b$

$$G x \preceq h$$
$$A x = b$$



feasible set is a polyhedron.

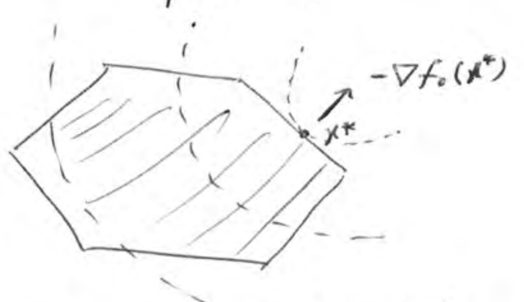
Example: $\min \max_i \left(a_i^T x + b_i\right)$

$$= \min t \quad \text{s.t.} \quad a_i^T x + b_i \le t$$

- Quadratic program (QP): $\min\ (1/2) x^T P x + q^T x + r$

$$G x \preceq h$$
$$A x = b$$

$P \in S_+^n$



$LP \subseteq QP$

Example: $\|Ax - b\|_2^2$, $l \le x \le u$ generalized Least squares.

- Quadratically constrained quadratic program (QCQP):

$$\min \left(\tfrac{1}{2}\right) x^T P_0 x + q_0^T x + r_0$$
$$\left(\tfrac{1}{2}\right) x^T P_i x + q_i^T x + r_i \le 0$$
$$A x = b$$

- $P_i \in S_+^n$
- $QP \subseteq QCQP$

- second-order cone program (SOCP):

$$\min \; e^T x$$
$$\text{s.t.} \; \|A_i x + b_i\|_2 \le c_i^T x + d_i \qquad (A_i, F = \text{matrices})$$
$$F x = g$$

- $(A_i x + b_i, \; c_i^T x + d_i)$ belongs to a cone $\mathbb{R}^{n+1}$ (considering the inequality cons.

$$\left(\tfrac{1}{2}\right) x^T P_i x + q_i^T x + r_i = \left\| \tfrac{1}{\sqrt{2}} P_i^{1/2} x + \sqrt{2}\, P_i^{1/2} q_i \right\|^2 + \text{Linear term}$$
$$\Rightarrow QCQP \subseteq SOCP$$

- Semidefinite program:

$$\min \; c^T x$$
$$\text{s.t.} \; x_1 F_1 + \cdots + x_n F_n + G \preceq 0$$
$$A x = b$$

$F_i, G \in S^n$

LMI Constraint

This is standard form, but includes many problems:

$$\begin{cases} \sum x_i \bar{F}_i \preceq 0 \\ \sum x_i \tilde{F}_i \preceq 0 \end{cases} \Rightarrow \left\{ \sum x_i \begin{bmatrix} \bar{F}_i & 0 \\ 0 & \tilde{F}_i \end{bmatrix} \preceq 0 \right.$$

- $\|A_i x + b_i\|_2 \le c_i^T x + d_i \iff \begin{bmatrix} (c_i^T x + d_i) I & A_i x + b_i \\ (A_i x + b_i)^T & c_i^T x + d_i \end{bmatrix} \succeq 0$

$$\left(\begin{bmatrix} A & B \\ B^T & D \end{bmatrix} \succ 0 \iff D \succ 0, \; A - B D B^T \succ 0 \right. \quad \text{schur complement}$$

$$\Rightarrow \boxed{LP \subseteq QP \subseteq QCQP \subseteq SOCP \subseteq SDP}$$

All convex

Example: $\min_x \lambda_{max}(A_0 + A_1 x_1 + \cdots + A_n x_n)$

$$\iff \min_{t \in \mathbb{R}} t \quad \text{s.t.} \quad A_0 + A_1 x_1 + \cdots + A_n x_n \preceq t I$$