- For indices $p \neq q$:

$$
\begin{cases}
J^{11}_{pq} = \dfrac{\partial P_p(x)}{\partial \theta_q} = |V_p||V_q|\left(G_{pq}\sin\theta_{pq} - B_{pq}\cos\theta_{pq}\right) \\[2ex]
J^{21}_{pq} = \dfrac{\partial Q_p(x)}{\partial \theta_q} = -|V_p||V_q|\left(G_{pq}\cos\theta_{pq} + B_{pq}\sin\theta_{pq}\right) \\[2ex]
J^{12}_{pq} = \dfrac{\partial P_p(x)}{\partial |V_q|} = |V_p|\left(G_{pq}\cos\theta_{pq} + B_{pq}\sin\theta_{pq}\right) \\[2ex]
J^{22}_{pq} = \dfrac{\partial Q_p(x)}{\partial |V_q|} = |V_p|\left(G_{pq}\sin\theta_{pq} - B_{pq}\cos\theta_{pq}\right)
\end{cases}
$$

- For indices $p = q$:

$$
\begin{cases}
J^{11}_{pp} = \dfrac{\partial P_p(x)}{\partial \theta_p} = -Q_p - B_{pp}|V_p|^2 \\[2ex]
J^{21}_{pp} = \dfrac{\partial Q_p(x)}{\partial \theta_p} = P_p - G_{pp}|V_p|^2 \\[2ex]
J^{21}_{pp} = \dfrac{\partial P_p(x)}{\partial |V_p|} = \dfrac{P_p}{|V_p|} + G_{pp}|V_p| \\[2ex]
J^{22}_{pp} = \dfrac{\partial Q_p(x)}{\partial |V_p|} = \dfrac{Q_p}{|V_p|} - B_{pp}|V_p|
\end{cases}
$$

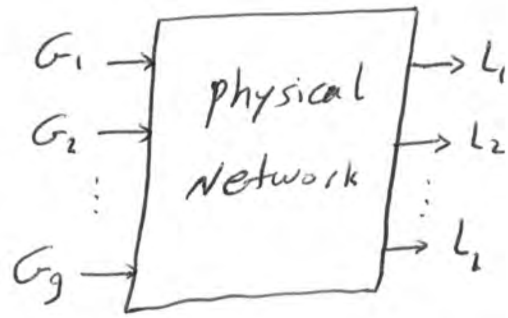
- Decoupled power flow:

Reasoning: - G_{pq} is small
- θ_{ij} is small

$$
A(x^v) = \begin{bmatrix} J_{11}^v & J_{12}^v \\ J_{21}^v & J_{22}^v \end{bmatrix} \longrightarrow A(x^v) = \begin{bmatrix} J_{11}^v & 0 \\ 0 & J_{22}^v \end{bmatrix}
$$

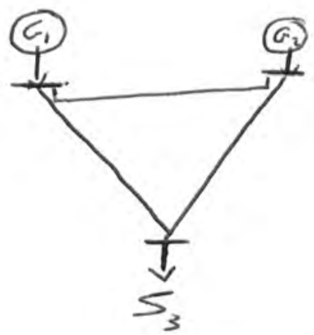
$$
\Rightarrow \begin{cases} J_{11}^v \Delta\theta^v = \Delta P(x^v) \\[1ex] J_{22}^v \Delta|V|^v = \Delta Q(x^v) \end{cases}
$$

- power flow (PF) is a special case of optimal power flow (OPF):



- Loads are given.
- Each generator has a cost
- minimize the total generation cost in such a way that network and physical constraints are satisfied.

Example:



minimize $f_1(P_1) + f_2(P_2)$

subject to:

physical constraint
$$
\begin{cases}
0.9 \le |V_i| \le 1.1 & i = 1,2,3 \\
|P_{ij}| \le P_{ij}^{max} & (i,j) \in L \\
|S_{ij}| \le S_{ij}^{max} & (i,j) \in L \\
|I_{ij}| \le I_{ij}^{max} & (i,j) \in L \\
P_i^{min} \le P_i \le P_i^{max} & i = 1,2 \\
Q_i^{min} \le Q_i \le Q_i^{max} & i = 1,2
\end{cases}
$$

Network constraint $\{$ power balance equations

involving $\sin\theta_{ij}$ and $\cos\theta_{ij}$

Remarks:

- Generation cost is often in terms of P_i and not S_i (fuel cost) $\longrightarrow$ Quadratic, piece-wise linear, ...
- OPF is solved every 5-10 mins in practice

Power Grid:

Layered Network

- Generation
- Transmission → Meshed graph (a single fault doesn't disconnect it)
- Distribution → Acyclic graph (mostly) (easier operation and protection)

To get some insight into OPF, consider acyclic graphs.

Optimization at distribution level (for smart grid):

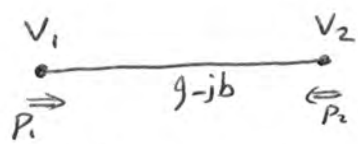
- Two-bus system:

- $P_i = P_{G_i} - P_{D_i}$, $i = 1, 2$

  (P_{G_i}: unknown; P_{D_i}: given)

- Goal: minimize $f_1(P_1) + f_2(P_2)$

- we have:



$\left(\frac{1}{r + jx} = \underbrace{g}_{+} - j\underbrace{b}_{+} \right)$

$$\begin{cases} P_1 = |V_1|^2 g + |V_1||V_2| b \sin(\theta) - |V_1||V_2| g \cos\theta \\ P_2 = |V_2|^2 g - |V_1||V_2| b \sin(\theta) - |V_1||V_2| g \cos\theta \end{cases} \quad (*)$$

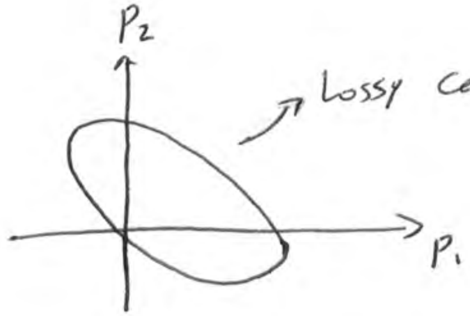
- Simplifying Assumptions (reasonable?):
  - $|V_1| = |V_2| = 1$
  - Ignore Q_1 & Q_2 (arbitrary compensation)

⇒ The design parameter is θ.

- How to optimize $f_1(P_1) + f_2(P_2)$ s.t. $(*)$?
- Initialization of Newton method is problematic.

- Geometric Approach: Find the feasible set of OPF

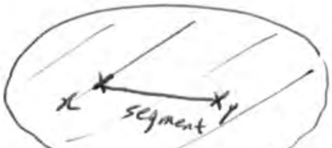
- $(\sin\theta, \cos\theta)$ is a circle $\Rightarrow$ Linear transformation leads to an ellipse

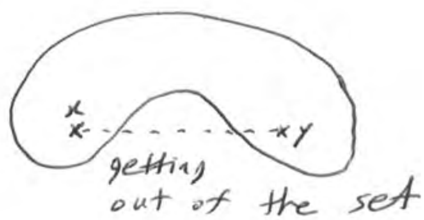


Question: How to do optimization over the boundary of an ellise ?

- Review: convex optimization

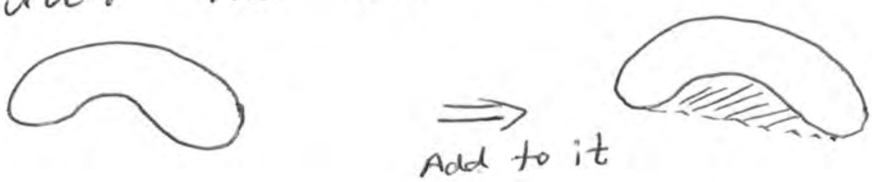
A set is convex if $\forall\, x, y \in S \Rightarrow \theta x + (1-\theta) y \in S$ for all $\theta \in [0,1]$





- Take it for granted that optimization over a convex set is easy (will be elaborated in the next two lectures).

- Convex hull: The smallest convex set containing $\underline{\underline{S}}$.



- Given $\underline{\underline{S}}$, characterize c.h. (S) ?

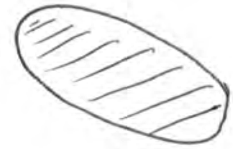
$\forall\, x_1, x_2, \dots, x_n$, $\forall\, \alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}^+$ s.t. $\alpha_1 + \alpha_2 + \dots + \alpha_n = 1$ $\Rightarrow$ $\underbrace{\sum_{i=1}^{n} \alpha_i x_i}_{\text{Convex combination}} \in$ c.h.(S)

- Back to the two-bus system:

feasible set of opf $\qquad\qquad$ convex hull

S

$\Longrightarrow$



S_c

Two optimization:

① $\min\ f_1(P_1)+f_2(P_2)$
s.t. $(P_1,P_2)\in S$

② $\min\ f_1(P_1)+f_2(P_2)$
s.t. $(P_1,P_2)\in S_c$

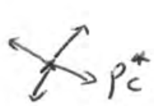
- Assumption: f_1 and f_2 are monotonic.

$\Longrightarrow$ Opt ① $\overset{\text{equivalent}}{=}$ Opt ②
(i.e. they will share the same solution)

<u>proof:</u>

- Denote the solutions as P^* and P_c^*
- Note that $f(P_c^*)\leq f(P^*)$ (always true)
- To show $f(P^*)\leq f(P_c^*)$, assume that $P^*\in \text{int}(S_c)$

S_c



going in one of these directions improves the solution
(monotonic f)

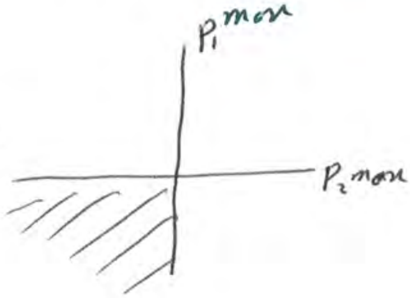
$\Rightarrow P_c^*\in$ boundary of S_c

$\Rightarrow P_c^*\in S$

$\Rightarrow f(P_c^*)=f(P^*)$

- The idea of solving Opt (2) instead of Opt (1) is called "convexification".

- Let's make it more practical: $P_i \le P_i^{max}, i=1,2$

 intersect with 

$\Rightarrow \begin{cases} 1\text{- empty} \\ 2\text{-} \end{cases}$ 

- Notice that S:  and S_c: 

- Opt (1) $\neq$ Opt (2) because if f_1 and f_2 are decreasing, then 

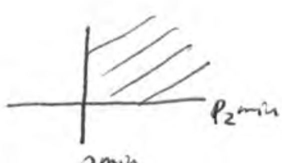
and $P_c^* \notin S$

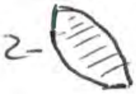
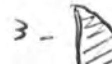
- Practical Assumption: f_1 and f_2 are increasing cost functions.

$\Rightarrow$ 

and $P_c^* \in S \implies$ Opt (1) = Opt (2)

- How about $P_i \ge P_i^{max}, i=1,2$? Very interesting case, will get back to it later.

 intersect with 

$\Rightarrow \begin{cases} 1\text{- empty} \\ 2\text{-} \\ 3\text{-} \end{cases}$   $\overset{\text{c.h.}}{\Longrightarrow} \begin{cases} 1\text{-empty} \\ 2\text{-} \\ 3\text{-} \end{cases}$  

- How about line flow limits $|P_{ij}| \le P_{ij}^{max}$?

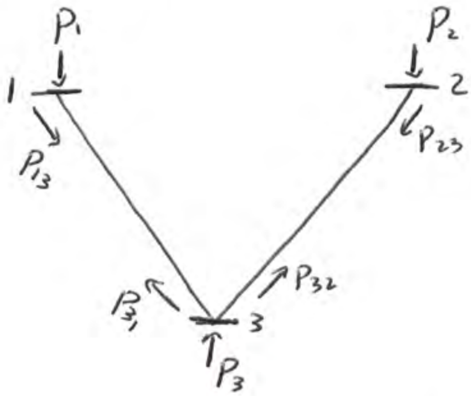
It doesn't matter for a two-bus system: $P_1 = P_{12}$, $P_2 = P_{21}$

General Network:

minimize $f(P_1, P_2, \ldots, P_n)$

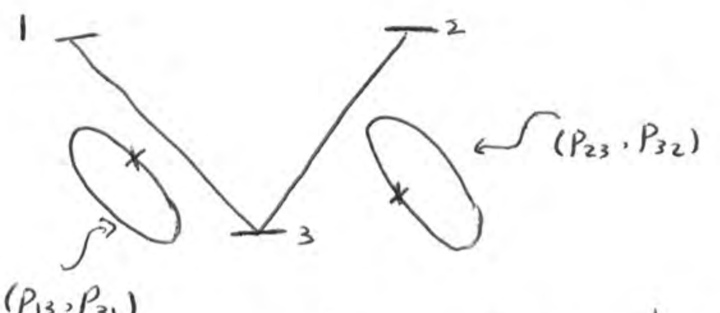
s.t.
- $|V_i| = 1$, $i = 1, \ldots, n$
- $P_i \le P_i^{max}$ $i = 1, \ldots, n$
- $|P_{ij}| \le P_{ij}^{max}$ $(i,j) \in l$
- Laws of physics relating P_i to V_j's.

Example:



- Feasible injection region: (P_1, P_2, P_3)
- Feasible flow region: $(P_{13}, P_{31}, P_{23}, P_{32})$

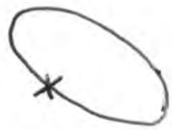
- Problem can be formulated in terms of θ_{13}, θ_{23}.
- This makes OPF very nonlinear.
- Trick: Use lifting technique, i.e., use four variables $P_{13}, P_{31}, P_{23}, P_{32}$ rather than two variables θ_{13}, θ_{32}



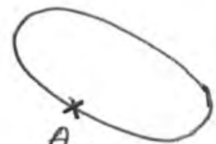
- A point (P_1, P_2, P_3) corresponds to two line flow points.

- Claim: Given two arbitrary line flow points, an injection point can always be constructed

(P_{13}, P_{31}) and (P_{23}, P_{32})



θ_{13}



θ_{23}

- Given θ_{13} and θ_{23}, find $\theta_1, \theta_2, \theta_3$.

$$\begin{cases} \theta_{13} = \theta_1 - \theta_3 \\ \theta_{23} = \theta_2 - \theta_3 \end{cases} \longrightarrow \begin{array}{l} \text{- Assume } \theta_1 = 0 \\ \text{- Find } \theta_2 \text{ and } \theta_3 \text{ uniquely} \end{array}$$

(This is the only place when the assumption of "acyclic graph" helps.)

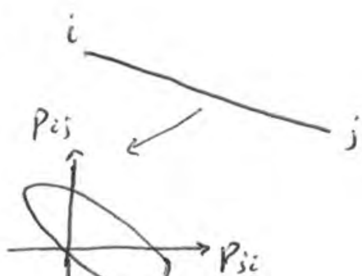
$\Rightarrow$ { - Each line has a feasible set for its two parameters
- points can be chosen independently from disparate line feasible sets

- Constraint $|P_{ij}| \leq P_{ij}^{max}$ and $|P_{ji}| \leq P_{ji}^{max}$ can be simplified if $P_{ij}^{max} = P_{ji}^{max}$.

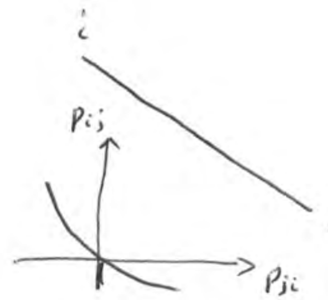
$$P_{ij} + P_{ji} = \text{Loss} \geq 0 \quad \Longrightarrow \quad P_{ij}, P_{ji} \leq P_{ij}^{max} = P_{ji}^{max}$$

implies $|P_{ij}|, |P_{ji}| \leq P_{ij}^{max} = P_{ji}^{max}$

-



$\xrightarrow{\text{Apply } P_{ij}, P_{ji} \leq P_{ij}^{max}}$



$\Rightarrow$ This constraint is local unlike $P_i = \sum_{j \in N(i)} P_{ij} \leq P_i^{max}$

- Define: Set $F_{ij} =$

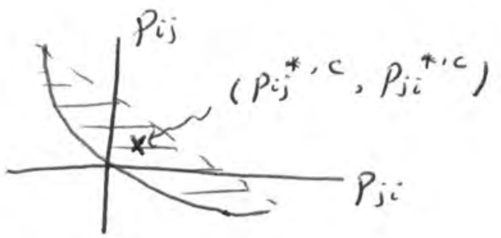
OPF:

$$\begin{cases} \text{Minimize} & f(p_1, p_2, \dots, p_n) \quad \text{increasing function} \\ \text{s.t.} & (p_{ij}, p_{ji}) \in F_{ij} \quad (i,j) \in L \\ & p_i = \sum_{j \in N(i)} p_{ij} \leq p_i^{max} \quad i = 1, 2, \dots, n \end{cases}$$

- Note that opf in θ_{ij}'s had $\underline{\underline{n-1}}$ variables (tree structure), while opf is p_{ij}'s has $2(n-1)$ variables (two for each line).
- Hard to solve because F_{ij} is non-convex
- Replace $(p_{ij}, p_{ji}) \in F_{ij}$ with $(p_{ij}, p_{ji}) \in c.h.(F_{ij})$
- Feasible set got bigger, so the optimal objective value of Opt (2) is a lower bound on that of Opt(1).

- Claim: optimal solution of Opt (2) is a feasible point of Opt (1).

- Proof by contradiction: Assume $\exists\ (i,j) \in L$ such that $(p_{ij}^{*,c}, p_{ji}^{*,c}) \in F_{ij}^c$ and $\notin F_{ij}$



$\rightarrow$ move downward $\swarrow \times\, p^{*,c}$

- This reduces p_{ij} and p_{ji}
- This reduces p_i and p_j
- we have got a better feasible point.

- Fixed voltage magnitudes $|V_1|, |V_2|$ helped a lot.

- How to incorporate $Q_i \leq Q_i^{max}, i=1,2,\dots,n$?

$$\Rightarrow \begin{bmatrix} P_1 - |V_1|^2 g \\ P_2 - |V_2|^2 g \end{bmatrix} = \begin{bmatrix} b & -g \\ -b & -g \end{bmatrix} \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \times |V_1||V_2|$$

$$\begin{bmatrix} q_1 + |V_1|^2 b \\ q_2 + |V_2|^2 b \end{bmatrix} = \begin{bmatrix} -g & -b \\ g & -b \end{bmatrix} \begin{bmatrix} \sin\theta \\ \cos\theta \end{bmatrix} \times |V_1||V_2|$$

$$\Rightarrow \begin{bmatrix} q_1 + |V_1|^2 b \\ q_2 + |V_2|^2 b \end{bmatrix} = \begin{bmatrix} -g & -b \\ g & -b \end{bmatrix} \begin{bmatrix} -g & g \\ b & b \end{bmatrix} \times \begin{bmatrix} P_1 - |V_1|^2 g \\ P_2 - |V_2|^2 g \end{bmatrix} \times \frac{-2}{bg}$$

- Observation: $\boxed{\text{$q_1$ and q_2 are linear functions in P_1 and P_2 (no effect on convexification)}}$

- Define $\alpha = g^2 - b^2$, $\beta = g^2 + b^2 \Rightarrow \beta^2 \geq \alpha^2$

$$(-1)\begin{bmatrix} -g & -b \\ g & -b \end{bmatrix} \begin{bmatrix} -g & g \\ b & b \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \beta & -\alpha \end{bmatrix}, \quad \begin{bmatrix} -\alpha & \beta \\ \beta & -\alpha \end{bmatrix}^{-1} = \begin{bmatrix} + & + \\ + & + \end{bmatrix}$$

$$\Rightarrow \begin{aligned} P_1 &= x_1 q_1 + x_2 q_2 + \text{Constant} \\ P_2 &= x_1' q_1 + x_2' q_2 + \text{Constant} \end{aligned} \quad \text{where } x_1, x_2, x_1', x_2' \geq 0$$

- Back to the proof: consider $(p_{ij}^{*,c}, p_{ji}^{*,c}) \in F_{ij}^c, \notin F_{ij}$

- Do the following: $q_{ij}^{*,c} \rightarrow q_{ij}^{*,c} - \varepsilon$, $q_{ji}^{*,c} \rightarrow q_{ji}^{*,c} - \varepsilon$

$\Rightarrow P_{ij}, P_i, Q_i$ all reduce and still feasible

$\Rightarrow$ There is a feasible point better than the optimal one. X