# Per Unit Normalization:

- In reality, many transformers and voltage levels are involved.

- It's common to normalize the parameters.

- say $|V| = 1$ volt per unit or $\underline{1}$ p.u.

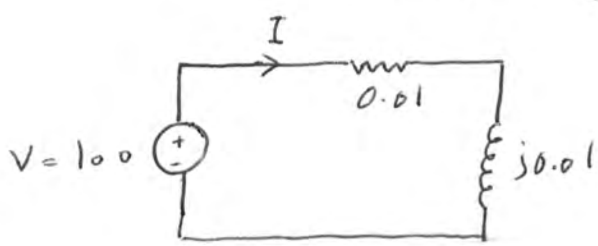
- pick base values in such a way that they satisfy the same kind of relationship as actual variables.

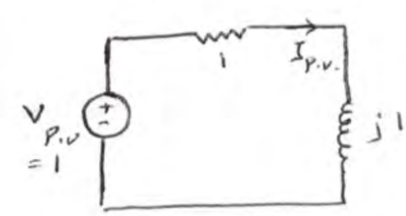
$$V = ZI \quad \Rightarrow \quad V_B = Z_B I_B$$

$$\frac{V}{V_B} = \frac{Z}{Z_B} \times \frac{I}{I_B} \quad \Rightarrow \quad V_{p.u.} = Z_{p.u.} \times I_{p.u.}$$

$$\text{quantity in p.u.} = \frac{\text{actual quantity}}{\text{base value of quantity}}$$

Example:
- $V_B = 100$, $Z_B = 0.01$
- Find $I_B$, $V_{p.u.}$, $Z_{p.u.}$, $I_{p.u.}$.



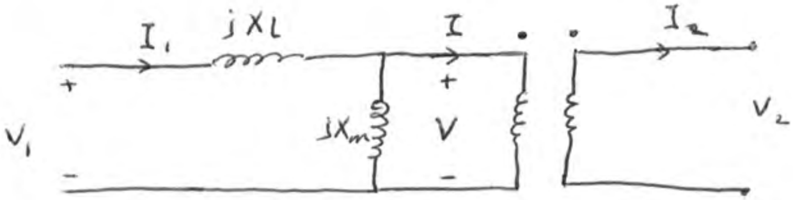




$$I_B = \frac{V_B}{Z_B} = 10^4, \quad V_{p.u.} = \frac{V}{V_B} = 1, \quad Z_{p.u.} = \frac{0.01 + j0.01}{Z_B} = 1 + j$$

$$I_{p.u.} = \frac{V_{p.u.}}{Z_{p.u.}} = 0.707 \angle -45^\circ$$

$$S = V I^{*}, \quad S_B = V_B I_B^{*} \Rightarrow S_{p.u.} = V_{p.u.} I_{p.u.}^{*}$$

- Advantage when a transformer is involved:




$V_{2B} = n V_{1B}$,

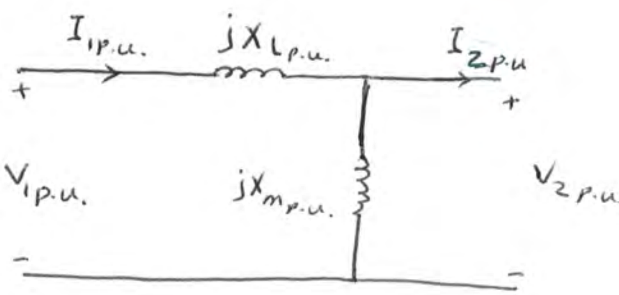
$S_{2B} = S_{1B} = S_B$

$I_{2B} = \dfrac{I_{1B}}{n}$,

$\Rightarrow Z_{2B} = n^2 Z_{1B}$

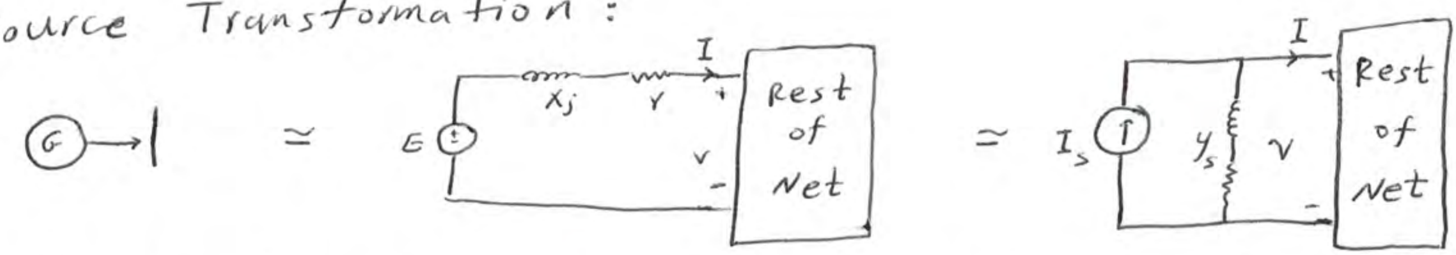
$$\Rightarrow V_{2p.u.} = \frac{V_2}{V_{2B}} = \frac{nV}{nV_{1B}} = V_{p.u.}$$

$$I_{2p.u.} = \frac{I_2}{I_{2B}} = \frac{\frac{1}{n} I}{\frac{1}{n} I_{1B}} = I_{p.u.}$$



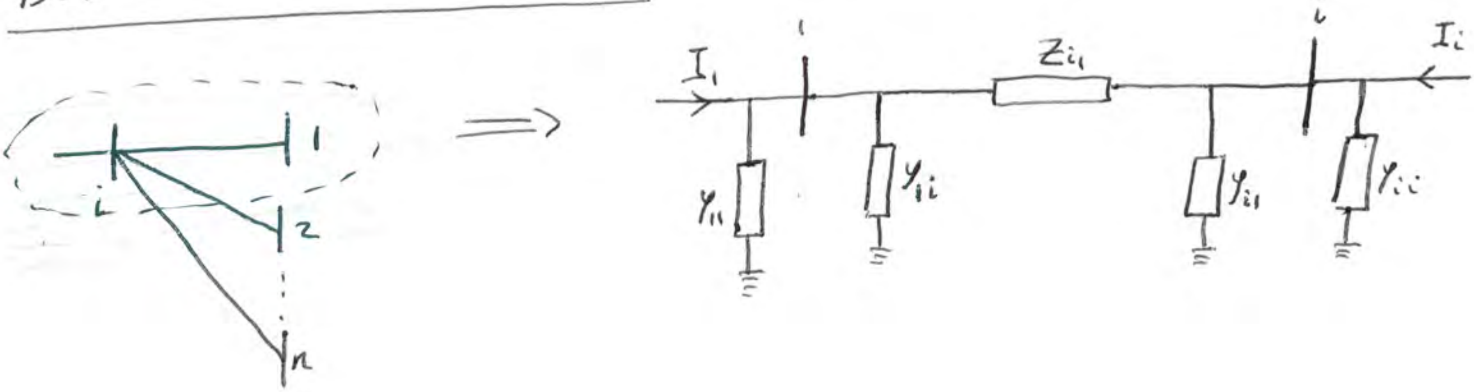

It seems as if there was no transformer.

- Source Transformation:




$$E = I(Xj + Y) + V \Rightarrow \frac{E}{Xj+Y} = I + \frac{V}{Xj+Y} \Rightarrow I_s = \frac{E}{Xj+Y}, \; Y_s = \frac{1}{Xj+Y}$$

Bus Admittance Matrix:



$$I_1 = y_{11}V_1 + \sum_{i=2}^{n} \left( y_{1i} V_1 + \frac{V_1 - V_i}{Z_{1i}} \right)$$

$$= \left( y_{11} + \sum_{i=2}^{n} y_{1i} + \sum_{i=2}^{n} \frac{1}{Z_{1i}} \right) V_1 - \sum_{i=2}^{n} \frac{1}{Z_{1i}} V_i$$

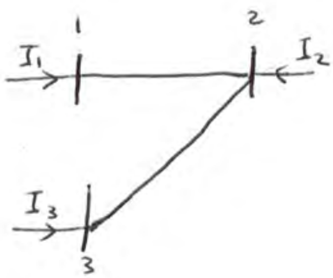
Define: $I = \begin{bmatrix} I_1 \\ \vdots \\ I_n \end{bmatrix}$ , $V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}$ , $Y = \left[ Y_{ij} \right]_{n \times n}$

where $Y_{ii} = y_{ii} + \sum_{j \neq i} y_{ij} + \frac{1}{Z_{ij}}$ , $Y_{ij} = -\frac{1}{Z_{ij}}$

$$\Rightarrow \boxed{I = YV}$$

This relates total injected currents to bus voltages

Example:




$$Y = \begin{bmatrix} \frac{B_{12}}{2} j + \frac{1}{Z_{12}} & -\frac{1}{Z_{12}} & 0 \\ -\frac{1}{Z_{12}} & \frac{B_{12}+B_{23}}{2} j + \frac{1}{Z_{12}} + \frac{1}{Z_{23}} & -\frac{1}{Z_{23}} \\ 0 & -\frac{1}{Z_{23}} & \frac{B_{31}}{2} j + \frac{1}{Z_{31}} \end{bmatrix}$$

Network solution: Given $I$, find $V$ ? ( $Y$ is fixed for a long period)

- Efficient method: Use L-U factorization: $Y = \triangle \times \nabla$

$$m = LU \Rightarrow \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

Find $U, L$ recursively $\Leftarrow$

$$= \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21} U_{11} & L_{21} U_{12} + U_{22} & L_{21} U_{13} + U_{23} \\ L_{31} U_{11} & L_{31} U_{12} + L_{32} U_{22} & L_{31} U_{13} + L_{32} U_{23} + U_{33} \end{bmatrix}$$

- Factorization is independent of $L$ and $U$

$$I = YV = L(UV) \implies \begin{array}{l} 1\text{- Find } \tilde{V} = UV \\ 2\text{- solve } I = L\tilde{V} \end{array}$$

Gaussian elimination

## Network Reduction:

Each network has some zero points. (buses)

zero current at bus $i$ : $I_i = 0$ (junction point)

$$\begin{bmatrix} \tilde{I}_1 \\ \tilde{I}_2 \end{bmatrix} = \begin{bmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{bmatrix} \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}, \quad \tilde{I}_2 = 0 \implies \tilde{I}_1 = (\tilde{Y}_{11} - \tilde{Y}_{12}\tilde{Y}_{22}^{-1}\tilde{Y}_{21})\tilde{V}_1$$

- why not think about impedance matrix:

$$V = Z_{bus} I \Rightarrow Z_{bus} = Y_{bus}^{-1}$$

Network is sparce $\implies$ Sparse $Y_{bus} \implies$ Dense $Z_{bus}$

(hidden structure)

## Power Flow Analysis:

- Consider a power network.
- Assume that loads withdraw power and generator inject power.
- we want to find steady-state powers and voltages.

- Generator bus $i$: $P_i$ and $|V_i|$ are given
(the reason will be discussed later)

- Load bus $i$: $P_i$ and $Q_i$ are given.

- Note that $S_i = P_i + Q_i j$ is the power injected to bus $\underline{i}$.

$$I = YV \ , \ Y = [Y_{ik}]_{n\times n} \Rightarrow S_i = V_i I_i^* = V_i \left(\sum_{k=1}^{n} Y_{ik} V_k\right)^* = \sum_{k=1}^{n} V_i Y_{ik}^* V_k^*$$

- Define:
$$V_i = |V_i| e^{j\measuredangle V_i} = |V_i| e^{j\theta_i}$$
$$\theta_{ik} = \theta_i - \theta_k \quad \text{(angle difference)}$$
$$Y_{ik} = G_{ik} + j B_{ik}$$

$$\Rightarrow S_i = \sum_{k=1}^{n} |V_i||V_k| e^{j\theta_{ik}} (G_{ik} - j B_{ik})$$
$$= \sum_{k=1}^{n} |V_i||V_k| (\cos B_{ik} + j \sin \theta_{ik})(G_{ik} - j B_{ik})$$

$$\Rightarrow \begin{cases} P_i = \sum_{k=1}^{n} |V_i||V_k| (G_{ik} \cos \theta_{ik} + B_{ik} \sin \theta_{ik}) \\ Q_i = \sum_{k=1}^{n} |V_i||V_k| (G_{ik} \sin \theta_{ik} - B_{ik} \cos \theta_{ik}) \end{cases}$$

- If all $\theta_i$'s are shifted by $\alpha$, powers won't chang.

$\Rightarrow$ we need a reference bus.

- slack (swing) bus: $|V_1|$ = known, $\measuredangle V_1 = 0$.

- Assume that:
  - Bus 1 is slack.
  - Buses $2 \ldots m$ are generator buses
  - Buses $(m+1) \ldots n$ are load buses

## Problem (power flow):

Given $V_1, (P_2, |V_2|), \ldots, (P_m, |V_m|), S_{m+1}, \ldots, S_n$

find $S_1, (Q_2, \measuredangle V_2), \ldots, (Q_m, \measuredangle V_m), V_{m+1}, \ldots, V_n$.

(note that $S_k = -P_{D_k} - Q_{D_k} j$ if $m+1 \leq k \leq n$.)

- In reality, there are more constraints (ignored for now).
- why isn't $P_1$ at the slack bus pre-specified?

Because that overconstrains the problem.

Example: In a lossless network, we have $P_1 = -\sum_{k=2}^{n} P_k$

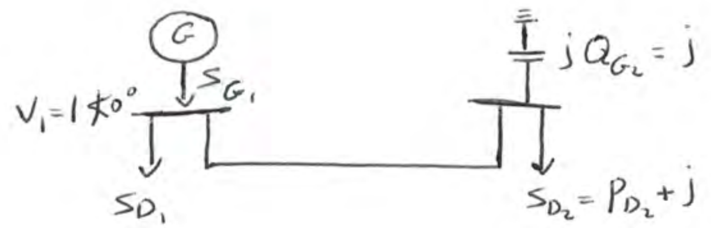
$\Rightarrow$ $P_1$ is not arbitrary

- Degree of freedom: $2n-1$ (we know everything if we know $|V_1|, \ldots, |V_n|, \measuredangle V_2, \ldots, \measuredangle V_n$).

$\Rightarrow$ only $2n-1$ parameters can be pre-specified.

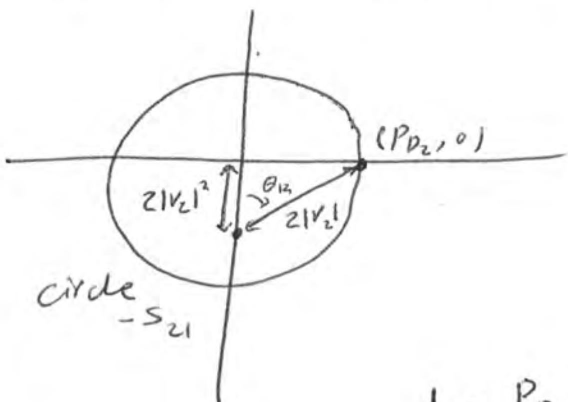
- power flow problem could be in a non-standard form where the $(2n-1)$ parameters known are not the same as $V_1, (P_2, |V_2|), \ldots, (P_m, |V_m|), S_{m+1}, \ldots, S_n$.

# Illustrative example:




Given $V_1$ and $S_2$,
find $S_1$ and $V_2$
as a function of $P_{D_2} \geq 0$.

$$S_2 = S_{G_2} - S_{D_2} = -P_{D_2}$$




$$\Rightarrow 4|V_2|^4 - 4|V_2|^2 + P_{D_2}^2 = 0$$

$$\Rightarrow |V_2|^2 = \frac{1}{2}\left(1 \pm \sqrt{1 - P_{D_2}^2}\right)$$

Three cases:
1- $P_{D_2} > 1 \Rightarrow$ no solution
2- $P_{D_2} = 1 \Rightarrow$ Unique solution
3- $P_{D_2} < 1 \Rightarrow$ Two real solutions

Assume: $P_{D_2} = 0.5 \Rightarrow V_2 = \begin{cases} 0.97 \angle -15^\circ \\ 0.26 \angle -75^\circ \end{cases}$

- voltage drop for $0.26 \angle -75^\circ$.
- we see $0.97 \angle -15^\circ$ in reality.
  (start from $1 \angle 0^\circ$ and move it slowly and continuously so we will get $0.97 \angle -15^\circ$ first.

## How to solve problem flow problem:

Example: Given $V_1, S_2, S_3, \ldots, S_n$
find $S_1, V_2, V_3, \ldots V_n$
$\Rightarrow$ Use Gauss iteration.

- $V_i = \frac{1}{Y_{ii}} \left( \frac{S_i^*}{V_i^*} - \sum_{k \neq i} Y_{ik} V_k \right)$

$$\Rightarrow \begin{cases} V_2 = \tilde{h}_2(V_2, \dots, V_n) \\ V_3 = \tilde{h}_3(V_2, \dots, V_n) \\ \vdots \\ V_n = \tilde{h}_n(V_2, \dots, V_n) \end{cases}$$

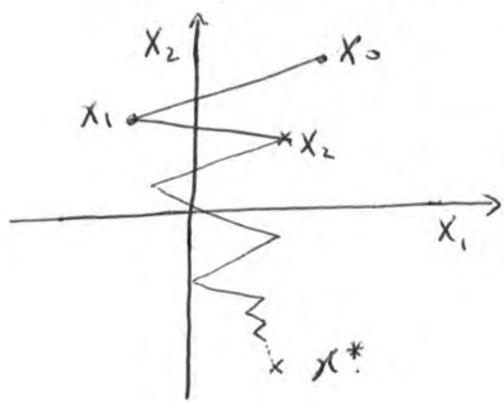
$\Rightarrow$ solve $X = h(X)$

strategy: - Guess $X^0$

- Iterate $X^{\nu+1} = h(X^\nu)$, $\nu = 0, 1, \dots$
- If lucky $X^0, X^1, X^2, \dots \longrightarrow X^*$

$\Rightarrow X^* = h(X^*)$

Illustration in $\mathbb{R}^2$:




What is Gauss-Seidel iteration?

- Find $x_1^{(\nu+1)}$ from $X^{(\nu)}$
- Use $x_1^{(\nu+1)}$ to get $x_2^{(\nu+1)}$

($x_1^{(\nu+1)}$ might be better than $x_1^{(\nu)}$)

$$x_1^{(\nu+1)} = h_1(x_1^\nu, x_2^\nu, \dots, x_n^\nu)$$
$$x_2^{(\nu+1)} = h_2(x_1^{\nu+1}, x_2^\nu, \dots, x_n^\nu)$$
$$x_3^{(\nu+1)} = h_3(x_1^{\nu+1}, x_2^{\nu+1}, x_3^\nu, \dots)$$
$$\vdots$$

- Contraction mapping: $d(h(X), h(Y)) \leq k\, d(X, Y)$ for some $k < 1$

$\Rightarrow d(h(X^{\nu+1}), h(X^\nu)) \leq k\, d(h(X^\nu), h(X^{\nu-1}))$

$k^\nu \to 0$ as $\nu \to \infty$ $\Rightarrow d(h(X^{\nu+1}), X^{\nu+1}) \to 0$

- How to verify existence of $d(\cdot,\cdot)$ ?

- Example: Assume $h: \mathbb{R} \to \mathbb{R}$, $|h'(x)| < 1$

$$\Rightarrow x_0, h(x_0), h^2(x_0), \ldots, h^n(x_0), \ldots \to x^*$$

Mean value theorem: Given $x, y$, there exist $z$ such that $h'(z) = \dfrac{h(y) - h(x)}{y - x}$

$\Rightarrow$ If $d(x,y) = \|x - y\|$, then

$$d(h(x), h(y)) = \|h(x) - h(y)\| = \|x - y\| \, |h'(z)|$$
$$< \|x - y\| = d(x,y)$$
$$\Rightarrow \underline{\text{contraction}}$$

- General case:

Jacobian matrix: $J(x) = \begin{bmatrix} \frac{\partial h_1(x)}{\partial x_1} & \cdots & \frac{\partial h_1(x)}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial h_n(x)}{\partial x_1} & \cdots & \frac{\partial h_n(x)}{\partial x_n} \end{bmatrix}$

property: $h_i(x + \Delta x) = h_i(x) + \dfrac{\partial h_i(x)}{\partial x_1} \Delta x_1 + \cdots + \dfrac{\partial h_i(x)}{\partial x_n} \Delta x_n$ + higher order terms.

$$\Rightarrow h(x + \Delta x) = h(x) + J(x) \Delta x + \text{h.o.t.}$$

Theorem: Given vectors $x$, $\Delta x$ and scaler $t$, we have: $h(x + \Delta x) - h(x) = \left( \int_0^1 J(x + t\Delta x) \, dt \right) \cdot \Delta x$

proof: $h_i(x+\Delta x) - h_i(x) = \int_0^1 \frac{h_i(x+t\Delta x)}{dt}\, dt$

$$= \int_0^1 \sum_{j=1}^{n} \frac{\partial h_i(x+t\Delta x)}{\partial x_j} \underbrace{\frac{\partial (x_i + t\Delta x_i)}{\partial t}}_{\Delta x_i} dt$$

- Assume that there is some norm $\|\cdot\|$ such that

$$\| J(x+t\Delta x)\| < 1 \quad \forall \quad t \in [0,1]$$

$$\Rightarrow \quad \| h(x+\Delta x) - h(x)\| \le \int_0^1 \| J(x+t\Delta x)\|\, dt \times \|\Delta x\|$$

$$< \|\Delta x\|$$

$$\Rightarrow \quad \| h(x) - h(y)\| < \| x - y\| \quad \Rightarrow \quad \underline{\text{Contraction}}$$

- Gauss iteration is basic and hard to converge
- We can go to Newton.
- Problem: Solve $f(x) = 0$

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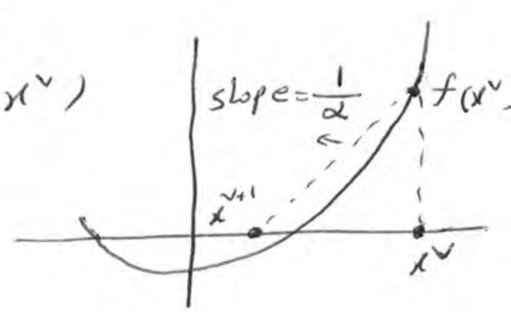
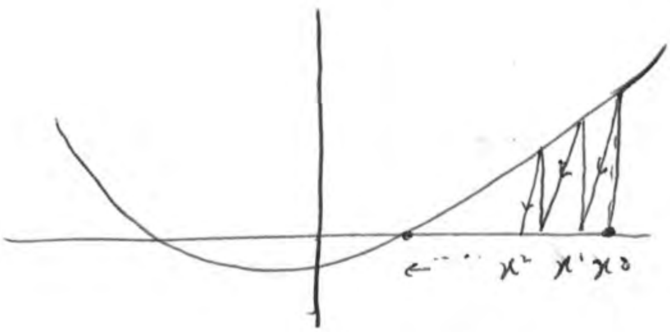
pick a non-singular matrix $A(x)$ and iterate

$$x^{v+1} = x^{v} - A(x^{v})^{-1} f(x^{v})$$

- special case: $A(x) =$ constant $\alpha$
- Scaler case: $x^{v+1} = x^{v} - \alpha f(x^{v})$

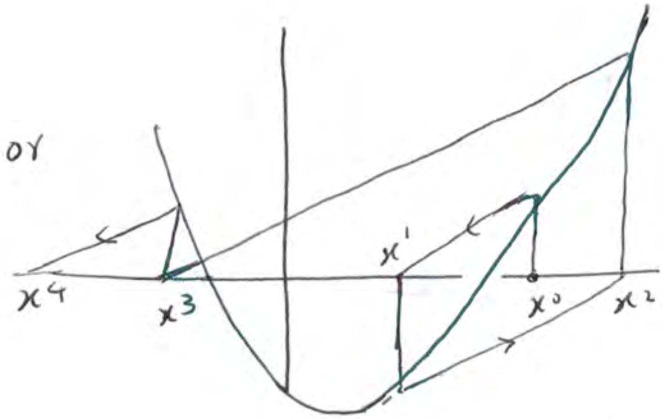


- what would happen in reality ?




$0 < \alpha \ll 1$

or




$\alpha \gg 1$ (unstable)

This suggests that: $x^{\nu+1} = x^{\nu} - f'(x^{\nu})^{-1} f(x^{\nu})$

- <u>Newton-Raphson:</u> $x^{\nu+1} = x^{\nu} - J(x^{\nu})^{-1} f(x^{\nu})$

When does Newton converge ?

- If the initial point $x^\circ$ is <u>close enough</u> to $x^*$.

- proof in scalar case:

$h(x) = x - f'(x)^{-1} f(x) \Rightarrow$ want to find $x^*$ such that $h(x^*) = x^*$

- Use contraction:
$$h'(x) = 1 - \frac{f'(x)^2 - f(x) f''(x)}{f'(x)^2} = \frac{f(x) f''(x)}{f'(x)^2}$$

$\Rightarrow h'(x^*) = 0 \Rightarrow \exists B \ni x^* : \forall x \in B$ we have $|h'(x)| < 1$.

- Let's revisit the iteration: (40)

$$x^{\nu+1} = x^{\nu} - J(x^{\nu})^{-1} f(x^{\nu})$$

- Taking the inverse of a huge matrix is cumber some

Define: $\Delta x^{\nu} = x^{\nu+1} - x^{\nu}$

$$\Rightarrow \quad J(x^{\nu})\,\Delta x^{\nu} = f(x^{\nu}) \xrightarrow{\text{Use L-U fact.}} \text{find } \Delta x^{\nu} \Downarrow \text{find } x^{\nu+1}$$

## Application in Power flow:

$$P_i = \sum_{k=1}^{n} |V_i||V_k| \left[ G_{ik} \cos(\theta_i - \theta_k) + B_{ik} \sin(\theta_i - \theta_k) \right]$$

$$Q_i = \sum_{k=1}^{n} |V_i||V_k| \left[ G_{ik} \sin(\theta_i - \theta_k) - B_{ik} \cos(\theta_i - \theta_k) \right]$$

$(\theta_i = \measuredangle V_i)$. - Given $V_1 = 1\measuredangle 0$, $P_i, Q_i$, find $V_i, \theta_i$ ?

- Varibles: $\theta = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$, $|V| = \begin{bmatrix} |V_2| \\ \vdots \\ |V_n| \end{bmatrix}$, $x = \begin{bmatrix} \theta \\ |V| \end{bmatrix}$

- Equations of interest: $\begin{cases} P_i = P_i(x) \\ Q_i = Q_i(x) \end{cases} \quad i = 2, \dots, n$

$$\Rightarrow \quad f(x) = \begin{bmatrix} P_2(x) - P_2 \\ \vdots \\ P_n(x) - P_n \\ \hline Q_2(x) - Q_2 \\ \vdots \\ Q_n(x) - Q_n \end{bmatrix} = 0$$

$$\Rightarrow \quad J = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial P_i(x)}{\partial \theta_k} & \frac{\partial P_i(x)}{\partial |V_k|} \\ \frac{\partial Q_i(x)}{\partial \theta_k} & \frac{\partial Q_i(x)}{\partial |V_k|} \end{bmatrix}$$