1 Problem 5.3

(i) Imagine that when in state \( i \), dummy (or false) transitions which leave the system in state \( i \) occur at rate \( M - v_i \). Then all transitions (real plus dummy) occur at a Poisson rate \( M \) and so the number of such transitions by any time \( t \) is finite. Hence, the number of real transitions is also finite with probability 1.

(ii) let \( X_{n+1} \) denotes the time between the \( n^{th} \) and \( n + 1^{th} \) transition and let \( J_n \) denote the state at time \( n \). If we let

\[ N(t) = \sup \{ n : X_1 + \cdots + X_n \leq t \} \]

then \( N(t) \) denote the number of transitions by time \( t \). Let \( j \) be the first recurrent state that is reached and suppose it was reached at \( n_0 \) transition (which is finite by our assumption). Let \( n_1, n_2, \ldots \) be the successive integers \( n \) at which \( J_n = j \). Such integers exist as \( j \) is recurrent. Set \( T_0 = X_1 + \cdots + X_{n_0} \) and \( T_k = X_{n_k - 1} + \cdots + X_{n_k} \). \( T_k \) is the time between \( k - 1^{th} \) and \( k^{th} \) visit to \( j \). It follows that \( \{T_k, k \geq 1\} \) forms a renewal process and hence \( \sum_{k \geq 1} T_k = \infty \) with probability 1. But we have:

\[ \sum_{n=1}^{\infty} X_n = \sum_{k=1}^{\infty} = \infty \]

Hence if the number of transitions are infinite then the time for these transitions is also infinite with probability one.

2 Problem 5.4

Let \( T_i \) denote the time to go from \( i \) to \( i + 1 \), \( i \geq 0 \). Then \( \sum_{i=0}^{N-1} T_i \) is the time taken to go from 0 to \( N \). As \( T_i \) is exponential with rate \( \lambda_i \) and the time \( T_i \) are independent we have:

\[ E \left[ \exp \left( s \sum_{i=0}^{N-1} T_i \right) \right] = \Pi_{i=1}^{N-1} E[\exp(sT_i)] = \Pi_{i=1}^{N-1} \frac{\lambda_i}{\lambda_i - s} \]

Also

\[ E \sum_{i=0}^{N-1} T_i = \sum_{i=0}^{N-1} \frac{1}{\lambda_i} \]

and

\[ \text{Var} \sum_{i=0}^{N-1} T_i = \sum_{i=0}^{N-1} \frac{1}{\lambda_i^2} \]

3 Problem 5.5

For \( 0 \leq s_1 \leq s_2 \ldots \leq s_n \leq t \), we want to calculate

\[ P(S_i = s_1, S_{i+1} = s_2, \ldots, S_{i+k-1} = s_k | X(t) = i + k, X(0) = i) \]

which is same as

\[ \frac{P(T_i = s_1, T_{i+1} = s_2 - s_1, \ldots, T_{i+k-1} = s_k - s_{k-1}, T_{i+k} > t - s_k | X(0) = i)}{P(X(t) = i + k | X(0) = i)} \]
Using the independence of $T_m$'s and substituting their densities we get:

$$i\lambda e^{-i\lambda s_1} (i+1)\lambda e^{-(i+1)\lambda(s_2-s_1)} \ldots (i+k-1)\lambda e^{-(i+k-1)\lambda(s_{i+k}-s_{i+k-1})}e^{-(i+k)\lambda(t-s_{i+k-1})}$$

\[
\binom{i+k-1}{i-1} e^{-\lambda ti} (1 - e^{-\lambda t})^k
\]

Simplifying we get:

$$k!\prod_{m=1}^k \frac{\lambda e^{-\lambda(t-s_m)}}{1 - e^{-\lambda t}}$$

4 Problem 5.12

(i) Since $P_0 = \frac{1}{\lambda + 1/\mu} = \frac{\mu}{\lambda + \mu}$, it follows from renewal theory (renewal every time we come back to state 0),

$$\lim_{t \to \infty} \frac{N(t)}{t} = \frac{\alpha_0 \mu}{\lambda + \mu} + \frac{\alpha_1 \lambda}{\lambda + \mu}$$

(ii) The expected total time spent in state 0 by time $t$ is

$$E[T_0(t)] = \int_0^t P_{00}(s)ds = \frac{\mu}{\lambda + \mu} t + \frac{\lambda}{(\lambda + \mu)^2} \left(1 - e^{(\lambda + \mu)t}\right)$$

From this:

$$E[N(t)] = \alpha_0 E[T_0(t)] + \alpha_1 (t - E[T_0(t)])$$.