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MINGLING: MIXED-INTEGER ROUNDING WITH BOUNDS

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Abstract. Mixed-integer rounding (MIR) is a simple, yet powerful procedure for generating valid inequalities for mixed-integer programs. When used as cutting planes, MIR inequalities are very effective for mixed-integer programming problems with unbounded integer variables. For problems with bounded integer variables, however, cutting planes based on lifting techniques appear to be more effective. This is not surprising as lifting techniques make explicit use of the bounds on variables, whereas the MIR procedure does not.

In this paper we describe a simple procedure, which we call mingling, for incorporating variable bound information into mixed-integer rounding. By explicitly using the variable bounds, the mingling procedure leads to strong inequalities for mixed-integer sets with bounded variables. We show that facets of mixed-integer knapsack sets derived earlier by superadditive lifting techniques can be obtained by the mingling procedure. In particular, the mingling inequalities developed in this paper subsume the continuous cover and reverse continuous cover inequalities of Marchand and Wolsey [9] as well as the continuous integer knapsack cover and pack inequalities of Atamtürk [1, 3]. In addition, mingling inequalities give a generalization of the two-step MIR inequalities of Dash and Günlük [7] under some conditions.

Keywords: Cutting planes, mixed-integer rounding, superadditive lifting.

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1. Introduction

Mixed-integer rounding (MIR) is a general procedure for deriving valid inequalities for mixed-integer programming. The MIR cuts, introduced by Nemhauser and Wolsey [11, 12], are applied to a single constraint (possibly implied by other constraints) of a mixed-integer program (MIP), much like the Chvátal-Gomory integer rounding cuts [5] for pure integer programs. MIR cuts are equivalent to Gomory's mixed-integer cuts for MIPs [8] and split cuts of Cook, Kannan, and Schrijver [6] and are a special case of the disjunctive cuts of Balas [4].

Marchand and Wolsey [10] show that when applied carefully MIR cuts can give some of the well-known strong cuts for special mixed-integer sets. In their computational study, they make a convincing case that applying MIR cuts after aggregating constraints into a single constraint and complementing variables—that is, replacing a variable x satisfying $0 \leq x \leq u$ with $\bar{x} = u - x$ by appropriately updating the coefficients of the constraint—is very effective in solving MIPs. MIR cuts are implemented in major MIP solvers, more or less, by following this approach.

Unlike lifted inequalities for sets with special structures, the MIR cuts do not explicitly use the upper bounds of the variables; however, they use their lower bounds. Therefore, complementing variables allows mixed-integer rounding to make use of the upper bound information to some degree. In this paper we present a new way of incorporating upper bounds of the variables by a simple procedure, which we refer to as mingling the variables. By explicitly using the variable bounds, the mingling procedure leads to strong inequalities for mixed-integer sets with bounded variables. We show that facets of the mixed-integer knapsack sets derived earlier by superadditive lifting techniques are, indeed, mingling inequalities or two-step mingling inequalities.

Let us now recall the MIR inequalities. Consider the mixed-integer set given by

$$\sum_{i \in N} a_i x_i + s \geq b, \ x \in \mathbb{Z}_+^N, \ s \in \mathbb{R}_+, \tag{1}$$

where the *base inequality* $ax + s \geq b$ may be implied by constraints of an MIP. For any $\alpha > 0$, the α-MIR inequality for (1) is

$$\sum_{i \in N} \mu_{\alpha,b}(a_i) x_i + s \geq \mu_{\alpha,b}(b), \tag{2}$$

where

$$\begin{aligned} \mu_{\alpha,b}(a_i) &= r\lfloor a_i/\alpha \rfloor + \min\{r, r_i\}, \ i \in N, \\ \mu_{\alpha,b}(b) &= r\lfloor b/\alpha \rfloor + r, \end{aligned}$$

and

$$r = b - \alpha\lfloor b/\alpha \rfloor, \ r_i = a_i - \alpha\lfloor a_i/\alpha \rfloor, \ i \in N.$$

Observe that an α-MIR inequality is the 1-MIR inequality written after dividing the base inequality by $\alpha > 0$. In order to highlight the inequalities of this paper, it is important to remark that nonnegativity of $x_i, \ i \in N$, is necessary for the validity of the α-MIR inequality unless $a_i/\alpha \in \mathbb{Z}$ (see Marchand and Wolsey [10] for a simple proof of validity of the MIR inequalities).

Lemma 1. *[11] The MIR function $\mu_{\alpha,b}$ is nondecreasing and subadditive for $\alpha > 0$.*

If it is known that $a, b \geq 0$, then using the nonnegativity of the variables, we can first strengthen the base inequality as

$$\sum_{i \in N} \min\{a_i, b\} x_i + s \geq b \tag{3}$$

and then apply α-MIR to obtain

$$\sum_{i \in N} \mu_{\alpha,b}(\min\{a_i, b\}) x_i + s \geq \mu_{\alpha,b}(b), \tag{4}$$

which dominates α-MIR inequality (2) as $\mu_{\alpha,b}$ is nondecreasing.

In this paper, we present a similar strengthening idea when the coefficients of the base inequality are unrestricted in sign by using the lower bounds as well as the upper bounds of the variables. We illustrate this point with a simple set in the next section. In Sections 3 and 4 we present the mingling inequalities in general form and in Section 5 we show the connection of the mingling inequalities with other inequalities given in the literature before. Finally, in Section 6 we conclude with a few closing remarks.

2. A simple set

In this section, we describe a simple mixed-integer knapsack set to give an intuition for the inequalities that incorporate variable bound information into MIR. Consider the following three-variable set

$$S = \left\{ \ (x, s) \in \mathbb{Z}^2 \times \mathbb{R} : a_1 x_1 + x_2 + s \geq b, \ x_1 \geq 0, \ u_2 \geq x_2 \geq 0, \ s \geq 0 \ \right\},$$

where the coefficients satisfy $a_1 < 0 < b < 1 \leq u_2$ and $u_2 \in \mathbb{Z}$. Because $a_1 < 0$, coefficient improvement as in (3) is not applicable to the base inequality $a_1 x_1 + x_2 + s \geq b$. However, in this case, we can utilize the upper bound of x_2 in order to derive a valid inequality that generalizes the MIR inequalities.

2.1. Basic inequality.

Adding and subtracting $u_2 x_1$, we rewrite the base inequality of S as

$$(a_1 + u_2) x_1 + (x_2 - u_2 x_1) + s \geq b. \tag{5}$$

By considering the disjunction $x_1 = 0 \ \vee \ x_1 \geq 1$, we obtain from (5) the valid inequality

$$(a_1 + u_2) x_1 + b(x_2 - u_2 x_1) + s \geq b \tag{6}$$

for S. For $x_1 = 0$, inequality (6) is the 1-MIR inequality for (5) as $0 < b < 1$ and $s \geq 0$; and for $x_1 \geq 1$, inequality (6) is a relaxation of (5) as $b < 1$ and $x_2 - u_2 x_1 \leq 0$. This type of coefficient improvement using the upper bound of x_2 is of interest when $a_1 + u_2 < 0$ because otherwise, the 1-MIR inequality

$$\mu_{1,b}(a_1) x_1 + \mu_{1,b}(1) x_2 + s \geq \mu_{1,b}(b) \tag{7}$$

is at least as strong as (6). Writing (7) explicitly as

$$\min\{a_1 - \lfloor a_1 \rfloor, b\} x_1 + b(x_2 + \lfloor a_1 \rfloor x_1) + s \geq b \tag{8}$$

makes the comparison easier. Now, defining $k := \min\{u_2, -\lfloor a_1 \rfloor\}$, we can generalize (6) and (8) as

$$\min\{a_1 + k, b\} x_1 + b(x_2 - k x_1) + s \geq b. \tag{9}$$

Observe that if $u_2 < -\lfloor a_1 \rfloor$, then (9) is stronger than 1-MIR inequality (8). Otherwise, (9) is at least as strong as (6). Indeed, inequality (9) is facet-defining for conv(S), which is easily checked with the affinely independent points (x_1, x_2, s) of S listed below:

$$(0,0,b),\ (0,1,0),\ (1,k,(b-a_1-k)^+).$$

Hence, by using the upper bound of x_2, we have strengthened the basic MIR inequality (7). In Section 3, we generalize inequality (9) to obtain the *mingling inequality* (21).

Remark 1. We should point out that complementing x_2 in the base inequality and then applying 1-MIR does not lead to a new inequality as

$$\mu_{1,b}(a_1)x_1 + \mu_{1,b}(-1)(u_2 - x_2) + s \geq \mu_{1,b}(b - u_2),$$

which equals

$$\mu_{1,b}(a_1)x_1 - b(u_2 - x_2) + s \geq b - bu_2$$

for any $u_2 \in \mathbb{Z}$, is the 1-MIR inequality (7) obtained without complementing x_2.

2.2. **A two-step inequality.** Next we will derive a new inequality based on (9). First we consider the case $a_1 + k < b$. As inequality (9) and

$$(a_1 + k)x_1 + (x_2 - kx_1) + s \geq b$$

are both valid for S, their convex combination

$$(a_1 + k)x_1 + \beta(x_2 - kx_1) + s \geq b, \tag{10}$$

where $b \leq \beta \leq 1$, is valid as well. Choosing $\beta = \alpha\lceil b/\alpha\rceil \leq 1$ for some $\alpha > 0$ such that $b/\alpha \notin \mathbb{Z}$ and applying α-MIR to (10), we obtain

$$\mu_{\alpha,b}(a_1 + k)x_1 + \mu_{\alpha,b}(\alpha\lceil b/\alpha\rceil)(x_2 - kx_1) + s \geq \mu_{\alpha,b}(b), \tag{11}$$

or, equivalently,

$$\mu_{\alpha,b}(a_1 + k)x_1 + \mu_{\alpha,b}(b)(x_2 - kx_1) + s \geq \mu_{\alpha,b}(b). \tag{12}$$

Note that validity of the α-MIR inequality (11) crucially depends on the fact that β/α is integral since $x_2 - kx_1$ may not be nonnegative.

Next consider the case $a_1 + k \geq b$. We now write the base inequality as

$$(a_1 + (k-1))x_1 + (x_2 - (k-1)x_1) + s \geq b,$$

which, as $a_1 + (k-1) \leq 0$ in this case, can be relaxed to

$$1(x_2 - (k-1)x_1) + s \geq b.$$

Also inequality (9) can be written in a similar form as

$$b(x_2 - (k-1)x_1) + s \geq b.$$

Then,

$$\beta(x_2 - (k-1)x_1) + s \geq b, \tag{13}$$

where $b \leq \beta \leq 1$, is valid as well. Choosing $\beta = \alpha\lceil b/\alpha\rceil \leq 1$ for some $\alpha > 0$ such that $b/\alpha \notin \mathbb{Z}$ and applying α-MIR to (13), this time, we obtain

$$\mu_{\alpha,b}(b)(x_2 - (k-1)x_1) + s \geq \mu_{\alpha,b}(b). \tag{14}$$

Combining (12) and (14), we obtain the following valid inequality for S:

$$\mu_{\alpha,b}(\min\{a_1 + k, b\})x_1 + \mu_{\alpha,b}(b)(x_2 - kx_1) + s \geq \mu_{\alpha,b}(b). \tag{15}$$

We note that when $u_2 \geq -\lfloor a_1 \rfloor$, inequality (15) becomes the two-step MIR inequality [7]. In Section 3, we generalize inequality (15) to obtain *the two-step mingling inequality* (31).

Remark 2. It is of interest to know how inequality (15) compares with a direct application of MIR to inequality (9). In order to do so, we collect the terms for x_1 in (9) and rewrite it as

$$(\min\{a_1 + k, b\} - bk)x_1 + bx_2 + s \geq b.$$

Applying α-MIR to this inequality, we obtain

$$\mu_{\alpha,b}(\min\{a_1 + k, b\} - bk)x_1 + \mu_{\alpha,b}(b)x_2 + s \geq \mu_{\alpha,b}(b). \tag{16}$$

If α is chosen as above, then the difference between (15) and (16) is only the coefficient of x_1. However, because $\mu_{\alpha,b}$ is subadditive and $k \in \mathbb{Z}_+$, we have

$$\begin{aligned}\mu_{\alpha,b}(\min\{a_1 + k, b\}) &\leq \mu_{\alpha,b}(\min\{a_1 + k, b\} - bk) + \mu_{\alpha,b}(bk)\\ &\leq \mu_{\alpha,b}(\min\{a_1 + k, b\} - bk) + k\mu_{\alpha,b}(b)\end{aligned}$$

and, therefore, (15) is at least as strong as (16). The numerical example below illustrates that (15) dominates (16) strictly.

Example 1. Let set S be given as

$$-5x_1 + x_2 + s \geq 0.5,\ s \geq 0,\ x_1 \geq 0,\ 2 \geq x_2 \geq 0.$$

Then $k = \min\{2, 5\} = 2$ and inequality (9) is

$$-3x_1 + 0.5(x_2 - 2x_1) + s \geq 0.5.$$

For $\alpha = 0.3$, we have $r = 0.2$, and inequality (15)

$$-2x_1 + 0.4(x_2 - 2x_1) + s \geq 0.4$$

strictly dominates inequality (16)

$$-2.6x_1 + 0.4x_2 + s \geq 0.4.$$

We note that this inequality is not facet-defining for $\text{conv}(S)$.

3. A Mingling Procedure

In this section we generalize inequalities (9) and (15) to obtain valid inequalities for the mixed-integer knapsack set

$$K_{\geq} := \left\{(x, s) \in \mathbb{Z}^N \times \mathbb{R} : \sum_{i \in I} a_i x_i + \sum_{j \in J} a_j x_j + s \geq b,\ u \geq x \geq 0,\ s \geq 0\right\},$$

where (I, J) is the partitioning of N with $a_i > 0$ for $i \in I$ and $a_j < 0$ for $j \in J$. We allow the upper bound on each variable to be either a positive integer or infinite. Throughout this section we assume that $b \geq 0$ and derive valid inequalities for $K_{\geq}$ using upper bounds $u_i,\ i \in I$. In the next section, we derive valid inequalities when $b \leq 0$ using upper bounds $u_j,\ j \in J$.

3.1. **Mingling inequalities.** Let us first introduce some new notation. For $x \in \mathbb{Z}^N$, let $x(S) := \sum_{i \in S} x_i$ for $S \subseteq N$. Let $I^+ := \{1, \dots, n\}$ be a nonempty subset of $\{i \in I : a_i > b\}$ indexed in *nonincreasing* order of a_i's, $\kappa := \sum_{i \in I^+} a_i u_i$, and $\bar{J} := \{j \in J : a_j + \kappa < 0\}$.

For any $j \in J \setminus \bar{J}$, we next define a set I_j and numbers $0 \leq \bar{u}_{ij} \leq u_i$ for $i \in I_j$ such that $a_j + \sum_{i \in I_j} a_i \bar{u}_{ij} \geq 0$. More precisely, for $j \in J \setminus \bar{J}$, let

$$I_j := \{1, \dots, p(j)\}, \text{ where } p(j) := \min \left\{ p \in I^+ : a_j + \sum_{i=1}^{p} a_i u_i \geq 0 \right\} \tag{17}$$

and

$$k_j := \min \left\{ k \in \mathbb{Z}_+ : a_j + \sum_{i=1}^{p(j)-1} a_i u_i + a_{p(j)} k \geq 0 \right\}. \tag{18}$$

Furthermore, for $j \in J \setminus \bar{J}$, and $i \in I_j$, let

$$\bar{u}_{ij} = \begin{cases} u_i, & \text{if } i < p(j) \\ k_j, & \text{if } i = p(j) \end{cases} . \tag{19}$$

For $j \in \bar{J}$, we let $I_j := I^+$, $p(j) := n$, $k_j := u_n$ and $\bar{u}_{ij} = u_i$ for $i \in I_j$.

For $i \in I$, let $J_i := \{j \in J : i \in I_j\}$; hence, $J_i = \emptyset$ for $i \in I \setminus I^+$. Observe that the definitions of *mingling sets* I_j and J_i imply that they are nested. Precisely,

$$\text{for } i, k \in I^+, \quad a_i > a_k \quad \Rightarrow \quad J_k \subseteq J_i$$

and

$$\text{for } j, k \in J, \quad a_j < a_k \quad \Rightarrow \quad I_k \subseteq I_j.$$

The nestedness property of the mingling sets is crucial for the validity of the mingling inequalities introduced next.

Using the mingling sets defined above we can now write the base inequality in $K_{\geq}$ as follows:

$$\sum_{i \in I} a_i (x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{j \in J} (a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}) x_j + s \geq b. \tag{20}$$

The main result of this section is the derivation of the *mingling inequality*

$$\sum_{i \in I^+} b (x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} \min\{b, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\} x_j + s \geq b, \tag{21}$$

from (20). Observe that if $I^+ = \emptyset$, then inequality (21) reduces to the base inequality (20). The validity of the mingling inequality for $K_{\geq}$ is not obvious and does not follow from mixed-integer rounding of (20) because the terms $(x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j)$, $i \in I^+$, and $(a_j + \sum_{i \in I_j} a_i \bar{u}_{ij})$, $j \in J$, are not necessarily nonnegative. Furthermore, it does not seem to be possible to derive inequality (21) as a straightforward extension of (9). In the following, we first prove the validity and then the strength of the mingling inequality.

Proposition 1. The mingling inequality (21) is valid for $K_{\geq}$.

Proof. First we will write inequality (21) more explicitly with the aid of some new notation. For $j \in J$, let

$$\delta_j = a_j + \sum_{i=1}^{p(j)-1} a_i u_i + a_{p(j)} k_j = a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}. \tag{22}$$

We have $\delta_j < 0$ for $j \in \bar{J}$ and $\delta_j \geq 0$ for $j \in J \setminus \bar{J}$. For $j \in J \setminus \bar{J}$ and $i \in I_j$, let

$$\tilde{u}_{ij} = \begin{cases} u_i, & \text{if } i < p(j), \\ k_j, & \text{if } i = p(j) \text{ and } \delta_j < b, \\ k_j - 1, & \text{if } i = p(j) \text{ and } \delta_j \geq b. \end{cases} \tag{23}$$

Note that $\tilde{u}_{ij} = \bar{u}_{ij} - 1$ if $i = p(j)$ and $\delta_j \geq b$, and $\tilde{u}_{ij} = \bar{u}_{ij}$ otherwise. Also observe that for $j \in J \setminus \bar{J}$

$$a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij} = \begin{cases} \delta_j & \geq 0 \quad \text{if } \delta_j < b, \\ \delta_j - a_{p(j)} & \leq 0 \quad \text{if } \delta_j \geq b. \end{cases} \tag{24}$$

Now let $w = s + \sum_{i \in I \setminus I^+} a_i x_i$. We treat w as a nonnegative continuous variable. Instead of $\bar{u}$, using $\tilde{u}$ let us write the mingling inequality (21) explicitly as

$$w + \sum_{i \in I^+} b[x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in \bar{J}} (a_j + \kappa) x_j + \sum_{j \in J \setminus \bar{J}} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b. \tag{25}$$

Using the same notation, consider also the following relaxation of the base inequality

$$w + \sum_{i \in I^+} a_i [x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in \bar{J}} (a_j + \kappa) x_j + \sum_{j \in J \setminus \bar{J}} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b. \tag{26}$$

Let $(\bar{x}, \bar{s})$ be a feasible point of $K_{\geq}$. We will examine two cases and show that $(\bar{x}, \bar{s})$ satisfies inequality (25). First assume that $\bar{x}(\bar{J}) \geq 1$. In this case $[\bar{x}_i - u_i \bar{x}(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} \bar{x}_j] \leq 0$ for all $i \in I^+$ as $\bar{x}_i \leq u_i$. Therefore, inequality (25) is a relaxation of (26) for $(\bar{x}, \bar{s})$.

So, we can now assume that $\bar{x}(\bar{J}) = 0$. If $\bar{x}(J \setminus \bar{J}) \neq 0$, let $j' := \text{argmin}_{j \in J}\{a_j : \bar{x}_j \geq 1\}$ and note that $j' \in J \setminus \bar{J}$ as $\bar{x}(\bar{J}) = 0$. Now let $\ell := p(j')$; thus, $I_{j'} = \{1, \ldots, \ell\}$. In other words, $j' \in J_i$ for all $i = 1, \ldots, \ell$ and $j' \notin J_i$ for any $i = \ell + 1, \ldots, n$. Furthermore, observe that $\tilde{u}_{ij} = u_i$ for $i < \ell$ and $j \in J_i \setminus \bar{J}$, and $\bar{x}_j = 0$ for $j \in J_i \setminus \bar{J}$ for $i > \ell$. Then, for $i < \ell$, we have $\bar{x}_i - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} \bar{x}_j \leq \bar{x}_i - u_i \leq 0$; and for $i > \ell$, we have $\bar{x}_i - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} \bar{x}_j = \bar{x}_i \geq 0$. Thus,

$$\bar{x}_i - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} \bar{x}_j \begin{cases} \leq 0 & \text{if } i < \ell \\ \geq 0 & \text{if } i > \ell \end{cases} \quad \text{for all } i \in I^+. \tag{27}$$

If $\bar{x}(J \setminus \bar{J}) = 0$, then let $\ell = 1$ and notice that (27) still holds. As $(\bar{x}, \bar{s}) \in K_{\geq}$, it satisfies (26) and as $a_i \geq a_\ell$ for $i < \ell$ and $a_i \leq a_\ell$ for $i > \ell$ it also satisfies

$$w + \sum_{i \in I^+} a_\ell [x_i - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in J} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b. \tag{28}$$

Therefore, it has to satisfy the a_ℓ-MIR inequality for (28)

$$w + \sum_{i \in I^+} b[x_i - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in J} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b, \tag{29}$$

which is same as (25) when $x(\bar{J}) = 0$. □

We next show that the mingling inequality is facet-defining for $\text{conv}(K_{\geq})$ when I^+ is chosen to be $\{i \in I : a_i > b\}$. Furthermore, when $\bar{J} \neq \emptyset$, then it can also be facet-defining when I^+ is a proper subset of $\{i \in I : a_i > b\}$.

Proposition 2. The mingling inequality (21) is facet-defining for $\text{conv}(K_{\geq})$ if

$$b - \min\{\ a_j + \kappa : j \in \bar{J}\ \} \geq \max\{\ a_i : a_i > b,\ i \in I \setminus I^+\ \}.$$

Proof. We use the notation introduced in the proof of Proposition 1. In addition, let $J = \bar{J} \cup J' \cup J''$, where $J' = \{j \in J \setminus \bar{J} : \delta_j < b\}$ and $J'' = \{j \in J \setminus \bar{J} : \delta_j \geq b\}$, and let $I = I^+ \cup I' \cup I''$, where $I' = \{i \in I : a_i \leq b\}$ and $I'' = \{i \in I \setminus I^+ : a_i > b\}$. Furthermore, let $j^* = \text{argmin}\{a_j : j \in \bar{J}\}$, and note that if $\bar{J} = \emptyset$ then $I = I^+$ and $I'' = \emptyset$ by the assumption of the proposition.

It is easily seen that the following $|I| + |J| + 1$ affinely independent points are on the face defined by (21):

$$\begin{array}{lllll}
 & : s = b, & & & \\
i \in I^+ & : s = 0, & x_i = 1, & & \\
i \in I' & : s = b - a_i, & x_i = 1, & & \\
j \in \bar{J} & : s = b - a_j - \kappa, & x_i = u_i,\ i \in I^+, & x_j = 1, & \\
j \in J' & : s = b - \delta_j, & x_i = \bar{u}_{ij}, i \in I_j, & x_j = 1, & \\
j \in J'' & : s = 0, & x_i = \bar{u}_{ij}, i \in I_j, & x_j = 1, & \\
i \in I'' & : s = b - a_{j^*} - \kappa - a_i, & x_k = u_k,\ k \in I^+, & x_{j^*} = 1, & x_i = 1.
\end{array}$$

Each row above shows only the nonzero components of a point. □

3.2. **Two-step mingling inequalities.** Next we will derive a second class of inequalities based on the mingling inequalities

$$s + \sum_{i \in I^+} b(x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} \min\{b, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\} x_j \geq b. \quad (30)$$

For any $\alpha > 0$ such that $\alpha \lceil b/\alpha \rceil \leq \min_{i \in I^+} a_i$, let the *two-step mingling inequality* be defined as

$$s + \sum_{i \in I^+} \mu_{\alpha,b}(b)(x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in I \setminus I^+} \mu_{\alpha,b}(a_i) x_i + \sum_{j \in J} \mu_{\alpha,b}(\min\{b, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\}) x_j \geq \mu_{\alpha,b}(b), \quad (31)$$

where $\mu_{\alpha,b}$ is the MIR function described in Section 1.

Observe that if $I^+ = \emptyset$, then inequality (30) is same as the base inequality (20) and therefore inequality (31) simply becomes the α-MIR inequality obtained from inequality (20). If $I^+ \neq \emptyset$, however, this is not the case even though inequality (31) is obtained by applying the MIR function $\mu_{\alpha,b}$ to the coefficients of the variables in inequality (30). Notice that the terms $(x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j)$, $i \in I^+$, are not necessarily nonnegative and therefore validity of (31) does not follow from mixed-integer rounding. As shown in Remark 2, due to subadditivity of $\mu_{\alpha,b}$, mixed-integer rounding of inequality (30) produces a weaker inequality than inequality (31) for the same α. Finally, if $b/\alpha \in \mathbb{Z}$, then inequality (30) reduces to $s \geq 0$ as does the MIR inequality (2).

Proposition 3. The two-step mingling inequality (31) is valid for $K_{\geq}$.

Proof. We use the notation introduced in the proof of Proposition 1. Let $(\bar{x}, \bar{s})$ be a feasible point of $K_{\geq}$ and consider the relaxation (26) of the base inequality. If $\bar{x}(\bar{J}) = 0$, then using the same arguments as in the proof of Proposition 1, the inequality

$$s + \sum_{i \in I \setminus I^+} a_i x_i + a_\ell \sum_{i \in I^+} [x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in \bar{J}} (a_j + \kappa) x_j + \sum_{j \in J \setminus \bar{J}} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b \quad (32)$$

is valid for $(\bar{x}, \bar{s})$ for some $\ell \in I^+$. Consider, again, mingling inequality (21), written in its explicit form

$$s + \sum_{i \in I \setminus I^+} a_i x_i + b \sum_{i \in I^+} [x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in \bar{J}} (a_j + \kappa) x_j + \sum_{j \in J \setminus \bar{J}} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b. \quad (33)$$

As both inequalities (32) and (33) are valid for $(\bar{x}, \bar{s})$, inequality

$$s + \sum_{i \in I \setminus I^+} a_i x_i + \beta \sum_{i \in I^+} [x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j] + \sum_{j \in \bar{J}} (a_j + \kappa) x_j + \sum_{j \in J \setminus \bar{J}} (a_j + \sum_{i \in I_j} a_i \tilde{u}_{ij})^+ x_j \geq b \quad (34)$$

is valid for $(\bar{x}, \bar{s})$ for any β such that $b \leq \beta \leq a_\ell$. Then, choosing $b \leq \beta \leq \underline{a} := \min_{i \in I^+} a_i$ ensures validity of (34) for $(\bar{x}, \bar{s})$ provided that $\bar{x}(\bar{J}) = 0$.

On the other hand, if $\bar{x}(\bar{J}) \geq 1$, as $[\bar{x}_i - u_i \bar{x}(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} \bar{x}_j] \leq 0$ for all $i \in I^+$, inequality (34) is a relaxation of (26) for $(\bar{x}, \bar{s})$ provided that $\beta \leq \underline{a}$.

Hence, (34) is valid for all $(\bar{x}, \bar{s}) \in K_{\geq}$ provided that $b \leq \beta \leq \underline{a}$. By taking $\beta = \alpha \lceil b/\alpha \rceil$, (31) simply becomes the α-MIR inequality for (34). Note that, $\mu_{\alpha,b}(b) = \mu_{\alpha,b}(\alpha \lceil b/\alpha \rceil)$ by definition, and as β is an integer multiple of α, it is not necessary for the integer term $\sum_{i \in I^+} [x_i - u_i x(\bar{J}) - \sum_{j \in J_i \setminus \bar{J}} \tilde{u}_{ij} x_j]$ in (34) to be nonnegative for the corresponding α-MIR inequality to be valid for $K_{\geq}$ (see [7]). □

Remark 3. If $I^+ = \{i \in I : a_i > b\}$ and α is chosen such that $\min_{i \in I^+} a_i \geq \alpha \geq b$, then mingling inequality (21) dominates two-step mingling inequality (31). To see this, observe that for such α, we have $\mu_{\alpha,b}(a) = a$ for $0 \leq a \leq b$. On the other hand, $\mu_{\alpha,b}(a) \geq a$ for any $a \leq 0$ and $\alpha > 0$.

Proposition 4. The two-step mingling inequality (31) is facet-defining for $\text{conv}(K_{\geq})$ if $b > 0$, $\bar{J} = \emptyset$, $I^+ = \{i \in I : a_i \geq \alpha \lceil b/\alpha \rceil\}$, and $\alpha = a_i$ for some $i \in I$.

Proof. We show in Section 5.2 that if $I^+ = \{i \in I : a_i \geq \alpha \lceil b/\alpha \rceil\}$, then two-step mingling inequalities become the continuous integer cover inequalities [1] obtained by superadditive lifting with integer variables [2]. These inequalities are facet-defining for $\text{conv}(K_{\geq})$ when $b > 0$, $\bar{J} = \emptyset$, and $\alpha = a_i$ for some $i \in I$ as shown in Theorem 6 of Atamtürk [1]. □

4. Symmetric inequalities

In this section we present inequalities for the mixed-integer knapsack set $K_{\geq}$ when $b \leq 0$. The two classes of inequalities we present below are "symmetric" to the mingling inequality (21) and the two-step mingling inequality (31) developed for the case when $b \geq 0$.

To develop the symmetric inequalities, we use a basic observation that shows the correspondence between the facets of $\text{conv}(K_{\geq})$ and the facets of $\text{conv}(K_{\leq})$, where

$$K_{\leq} = \left\{(x,t) \in \mathbb{Z}^N \times \mathbb{R} : \ ax \leq b + t, \ u \geq x \geq 0, \ t \geq 0\right\}.$$

Based on this observation, we utilize the results in the previous section. Note that we do not restrict the sign of a or b in the following lemma.

Lemma 2. *Inequality $\pi x + s \geq \pi_o$ is valid for $K_{\geq}$ if and only if inequality $(a-\pi)x \leq b - \pi_o + t$ is valid for $K_{\leq}$. Moreover, $\pi x + s \geq \pi_o$ defines a facet of $\text{conv}(K_{\geq})$ if and only if $(a-\pi)x \leq b - \pi_o + t$ defines a facet of $\text{conv}(K_{\leq})$.*

Proof. To see the first part, by adding a slack variable s let us write $K_{\leq}$ as

$$K = \left\{(x,s,t) \in \mathbb{Z}^N \times \mathbb{R} \times \mathbb{R} : \ ax + s = b + t, \ u \geq x \geq 0, \ s,t \geq 0\right\}$$

and consider its "relaxation" $K_{\geq}$ obtained by dropping t. Because $\pi x + s \geq \pi_o$ is valid for $K_{\geq}$, it is also valid for K. Substituting $b + t - ax$ for s, we obtain $(a-\pi)x \leq b - \pi_o + t$ as a valid inequality for $K_{\leq}$. The other direction is the same. For the second part, observe that $\text{conv}(K_{\leq})$ is isomorphic to $\text{conv}(K)$, which is isomorphic to $\text{conv}(K_{\geq})$. Thus, there is a one-to-one correspondence between the facets of $\text{conv}(K_{\leq})$ and $\text{conv}(K_{\geq})$. □

Based on this observation, we next describe how to derive valid inequalities for $K_{\geq}$ when $b \leq 0$. First we multiply the base inequality $ax \geq b$ by -1 to obtain an equivalent representation of the set in $K_{\leq}$ form with a nonnegative right-hand-side. We then use Lemma 2, and utilize the facets of the corresponding $K_{\geq}$ set (again with a nonnegative right-hand-side) to obtain facets of the $K_{\leq}$ representation. The inequalities presented below generalize the reverse continuous cover inequality developed by Marchand and Wolsey [9] and the continuous integer knapsack pack inequality of Atamtürk [1, 3]. We compare them in detail later in Section 5.

4.1. Symmetric mingling inequalities.

We now consider the case $b \leq 0$ for $K_{\geq}$. Our approach this time is to update the coefficients of x_i, $i \in I$, in the base inequality of $K_{\geq}$ using the upper bounds of x_j, $j \in J$, to get a more convenient form (36). Toward this end, let $J^- := \{1, \ldots, m\}$ be a nonempty subset of $\{j \in J : a_j < b\}$, indexed in *nondecreasing* order of a_j's, $\nu := \sum_{j \in J^-} a_j u_j$, and $\bar{I} := \{i \in I : a_i + \nu > 0\}$. For $i \in I \setminus \bar{I}$, let

$$J_i := \{1, \ldots, p(i)\}, \ \text{where } p(i) := \min\left\{p \in J^- : a_i + \sum_{j=1}^{p} a_j u_j \leq 0\right\}$$

and

$$k_i := \min\left\{k \in \mathbb{Z}_+ : a_i + \sum_{j=1}^{p(i)-1} a_j u_j + a_{p(i)} k \leq 0\right\}.$$

For $i \in \bar{I}$, we let $J_i := J^-$, $p(i) := m$, and $k_i := u_m$. For and $i \in I$ and $j \in J_i$, let

$$\bar{u}_{ji} := \begin{cases} u_j, & \text{if } j < p(i), \\ k_i, & \text{if } j = p(i). \end{cases} \tag{35}$$

For $j \in J$, let $I_j := \{i \in I : j \in J_i\}$. Note $I_j = \emptyset$ for $j \in J \setminus J^-$. Using these mingling sets, the base inequality of $K_{\geq}$ can be written as

$$\sum_{j \in J} a_j(x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + \sum_{i \in I} (a_i + \sum_{j \in J_i} a_j \bar{u}_{ji}) x_i + s \geq b. \tag{36}$$

We define the *symmetric mingling inequality* corresponding to (36) as

$$\sum_{j \in J^-} (a_j - b)(x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + \sum_{i \in I} \min\{a_i + \sum_{j \in J_i} a_j \bar{u}_{ji} - b, 0\} x_i + s \geq 0. \tag{37}$$

Observe that if $J^- = \emptyset$, then inequality (37) reduces to $s \geq 0$.

Proposition 5. The symmetric mingling inequality (37) is valid for $K_{\geq}$. Furthermore, it is facet-defining for $\mathrm{conv}(K_{\geq})$ provided that

$$\min\{\ a_j : a_j < b,\ j \in J \setminus J^-\ \} \geq \max\{\ a_i + \nu : i \in \bar{I}\ \}.$$

Proof. After rewriting the base inequality of $K_{\geq}$ as

$$\sum_{j \in J} -a_j x_j + \sum_{i \in I} -a_i x_i \leq -b + s$$

in $K_{\leq}$ form, we use the corresponding $K_{\geq}$ set

$$\sum_{j \in J} -a_j x_j + \sum_{i \in I} -a_i x_i + s \geq -b \tag{38}$$

to generate mingling inequalities of Section 3.1 as $-b \geq 0$. Using the mingling sets defined in this section, the corresponding mingling inequality (21) for (38) is

$$\sum_{j \in J^-} -b(x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + \sum_{j \in J \setminus J^-} -a_j x_j + \sum_{i \in I} \min\{-b, -a_i - \sum_{j \in J_i} a_j \bar{u}_{ji}\} x_i + s \geq -b. \tag{39}$$

Translating this inequality for the original $K_{\leq}$ set using Lemma 2, we obtain

$$\sum_{j \in J^-} (-a_j + b) x_j - \sum_{j \in J^-} \sum_{i \in I_j} b \bar{u}_{ji} x_i + \sum_{i \in I} (-a_i - \min\{-b, -a_i - \sum_{j \in J_i} a_j \bar{u}_{ji}\}) x_i \leq s,$$

which is equivalent to (37). The facet condition follows from Proposition 2 and Lemma 2. □

4.2. **Symmetric two-step mingling inequalities.** In this section, we give the symmetric class of inequalities for two-step mingling inequalities for the case $b \leq 0$. For any $\alpha > 0$ such that $\max_{j \in J^-} a_j \leq \alpha \lfloor b/\alpha \rfloor$, let us define the *symmetric two-step mingling inequality* corresponding to (36) as

$$\begin{aligned} &\sum_{j \in J^-} (a_j + \mu_{\alpha,-b}(-b))(x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i) + \sum_{j \in J \setminus J^-} (a_j + \mu_{\alpha,-b}(-a_j)) x_j + \\ &\sum_{i \in I} (a_i + \sum_{j \in J_i} a_j \bar{u}_{ji} + \mu_{\alpha,-b}(\min\{-b, -a_i - \sum_{j \in J_i} a_j \bar{u}_{ji}\})) x_i + s \geq b + \mu_{\alpha,-b}(-b). \end{aligned} \tag{40}$$

Observe that if $b/\alpha \in \mathbb{Z}$, then the symmetric two-step mingling inequality (40) reduces to the base inequality (1). Therefore, we assume below that $b/\alpha \notin \mathbb{Z}$. If $J^- = \emptyset$, then inequality (40) reduces to

$$\sum_{i \in N} (a_i + \mu_{\alpha,-b}(-a_i))x_i + s \geq b + \mu_{\alpha,-b}(-b),$$

which equals the α-MIR inequality

$$\sum_{i \in N} \mu_{\alpha,b}(a_i)x_i + s \geq \mu_{\alpha,b}(b)$$

because $\mu_{\alpha,b}(a_i) = a_i + \mu_{\alpha,-b}(-a_i)$ for $a_i \in \mathbb{R}$ as checked below. Let $r = b - \alpha\lfloor b/\alpha \rfloor$ and $r_i = a_i - \alpha\lfloor a_i/\alpha \rfloor,\ i \in N$. If $a_i/\alpha \notin \mathbb{Z}$, then

$$\begin{aligned}
\mu_{\alpha,b}(a_i) - \mu_{\alpha,-b}(-a_i) &= r\lfloor a_i/\alpha \rfloor + \min\{r, r_i\} - (\alpha - r)\lfloor -a_i/\alpha \rfloor - \min\{\alpha - r, \alpha - r_i\} \\
&= r\lfloor a_i/\alpha \rfloor + \min\{r, r_i\} + (\alpha - r)\lceil a_i/\alpha \rceil - \alpha + \max\{r, r_i\} \\
&= \min\{r, r_i\} + \alpha\lceil a_i/\alpha \rceil - r - \alpha + \max\{r, r_i\} \\
&= \alpha\lfloor a_i/\alpha \rfloor - r + r + r_i \\
&= a_i
\end{aligned}$$

And if $a_i/\alpha \in \mathbb{Z}$, we have $\mu_{\alpha,b}(a_i) - \mu_{\alpha,-b}(-a_i) = r(a_i/\alpha) - (\alpha - r)(-a_i/\alpha) = a_i$.

Proposition 6. The symmetric two-step mingling inequality (40) is valid for $K_{\geq}$. Furthermore, it is facet-defining for $\operatorname{conv}(K_{\geq})$ if $b < 0$, $\bar{I} = \emptyset$, $J^- = \{j \in J : a_j \leq \alpha\lfloor b/\alpha \rfloor\}$, and $\alpha = a_j$ for some $j \in J$.

Proof. Applying Proposition 3 to inequality (39), for any $\alpha > 0$ such that $\alpha\lceil -b/\alpha \rceil \leq \min_{j \in J^-} -a_j$, we obtain

$$\begin{aligned}
&\sum_{j \in J^-} \mu_{\alpha,-b}(-b)(x_j - \sum_{i \in I_j} \bar{u}_{ji}x_i) + \sum_{j \in J \setminus J^-} \mu_{\alpha,-b}(-a_j)x_j \\
&\qquad + \sum_{i \in I} \mu_{\alpha,-b}(\min\{-b, -a_i - \sum_{j \in J_i} a_j\bar{u}_{ji}\})x_i + s \geq \mu_{\alpha,-b}(-b) \qquad (41)
\end{aligned}$$

Translating it to the original $K_{\leq}$ form using Lemma 2 gives

$$\begin{aligned}
&\sum_{j \in J^-} (-a_j - \mu_{\alpha,-b}(-b))x_j + \sum_{j \in J \setminus J^-} (-a_j - \mu_{\alpha,-b}(-a_j))x_j + \mu_{\alpha,-b}(-b) \sum_{j \in J^-} \sum_{i \in I_j} \bar{u}_{ji}x_i \\
&\qquad + \sum_{i \in I} (-a_i - \mu_{\alpha,-b}(\min\{-b, -a_i - \sum_{j \in J_i} a_j\bar{u}_{ji}\}))x_i \leq s - b - \mu_{\alpha,-b}(-b),
\end{aligned}$$

which is equivalent to (40). The facet condition follows from Proposition 4 and Lemma 2. □

5. Connections with other inequalities

In this section we present some well-known valid inequalities from the literature for knapsack sets and describe how to obtain them as (symmetric) mingling or (symmetric) two-step mingling inequalities. In particular, we consider the continuous cover and reverse continuous cover inequalities of Marchand and Wolsey [9] and the continuous integer knapsack cover and pack inequalities of Atamtürk [1, 3]. We would like to emphasize that all these inequalities can be obtained by mingling when the set I^+ (or, J^-, respectively) is taken to be $\{i \in I : a_i > b\}$ (or,

$\{j \in J : a_j < b\}$, respectively). When subsets of $\{i \in I : a_i > b\}$ ($\{j \in J : a_j < b\}$) are used for I^+ (J^-), the mingling procedure leads to new inequalities for these sets.

5.1. **Continuous 0-1 cover inequalities.** Consider the mixed 0-1 knapsack set

$$K^1_{\leq} := \Big\{(x, s) \in \{0,1\}^N \times \mathbb{R} \ : \ \sum_{i \in N} a_i x_i \leq b + s, \ s \geq 0\Big\},$$

where $a > 0$. A subset C of N is called a *cover* if $\lambda := \sum_{i \in C} a_i - b > 0$. Letting $\bar{x}_i = 1 - x_i$, $i \in C$, after rewriting the base inequality as

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda, \tag{42}$$

for $I^+ \subseteq \{i \in C : a_i > \lambda\}$ and $J = N \setminus C$, we obtain the mingling inequality

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in N \setminus C} \min\{\lambda, -a_j + \sum_{i \in I_j} a_i\} x_j + s \geq \lambda. \tag{43}$$

For $I^+ = \{i \in C : a_i > \lambda\}$, mingling inequality (43) reduces to

$$\sum_{i \in C} \min\{\lambda, a_i\} \bar{x}_i + \sum_{j \in N \setminus C} (-\lambda |I_j| + \min\{\lambda, -a_j + \sum_{i \in J_i} a_i\}) x_j + s \geq \lambda, \tag{44}$$

which is equivalent to the continuous cover inequality (Marchand and Wolsey [9]). We will illustrate inequality (44) in Example 2.

Now assume that there exists a $k \in C$ such that $\theta = a_k - \lambda > 0$. In this case writing the base inequality as

$$\sum_{i \in C \setminus k} a_i \bar{x}_i + \sum_{i \in N \setminus (C \setminus k)} -a_i x_i + s \geq -\theta, \tag{45}$$

for $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j > \theta\}$ and $I = C \setminus k$, we obtain the symmetric mingling inequality

$$\sum_{i \in C \setminus k} \min\{0, a_i - \sum_{j \in J_i} a_j + \theta\} \bar{x}_i + \sum_{j \in J^-} (\theta - a_j)(x_j - \sum_{i \in I_j} \bar{x}_i) + s \geq 0. \tag{46}$$

For $J^- = \{j \in N \setminus (C \setminus k) : a_j > \theta\}$, symmetric mingling inequality (46) reduces to

$$\sum_{i \in C \setminus k} (\min\{0, a_i - \sum_{j \in J_i} a_j + \theta\} + \sum_{j \in J_i} (a_j - \theta)) \bar{x}_i - \sum_{j \in N \setminus (C \setminus k)} (a_j - \theta)^+ x_j + s \geq 0, \tag{47}$$

which is the reverse continuous cover inequality [9].

5.2. **Continuous integer cover inequalities.** Consider now the mixed-integer knapsack set with finite upper bounds for all integer variables

$$K^u_{\leq} := \Big\{(x, s) \in \mathbb{Z}^N \times \mathbb{R} \ : \ \sum_{i \in N} a_i x_i \leq b + s, \ u \geq x \geq 0, \ s \geq 0\Big\},$$

where $a > 0$. A subset C of N is called a *cover* if $\lambda := \sum_{i \in C} a_i u_i - b > 0$. After letting $\bar{x}_i = u_i - x_i$, $i \in C$, by rewriting the base inequality as

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda, \tag{48}$$

for $I^+ \subseteq \{i \in C : a_i > \lambda\}$ and $J = N \setminus C$, we obtain the mingling inequality

$$\sum_{i\in I^+} \lambda(\bar{x}_i - \sum_{j\in J_i} \bar{u}_{ij} x_j) + \sum_{i\in C\setminus I^+} a_i \bar{x}_i + \sum_{j\in J} \min\{\lambda, -a_j + \sum_{i\in I_j} a_i \bar{u}_{ij}\} x_j + s \geq \lambda. \quad (49)$$

Now assume that there exists a $k \in C$ such that $\theta = a_k u_k - \lambda > 0$ and $\lambda/a_k \notin \mathbb{Z}$. Furthermore, let $\eta = \lceil \theta/a_k \rceil$ and $\rho = \theta - a_k \lfloor \theta/a_k \rfloor$. Then $\lceil \lambda/a_k \rceil = u_k - \eta + 1$ and $\lambda - a_k \lfloor \lambda/a_k \rfloor = a_k - \rho$. For $I^+ \subseteq \{i \in C : a_i \geq a_k \lceil \lambda/a_k \rceil\}$, the corresponding two-step inequality for (48) with $\alpha = a_k$ is then

$$\sum_{i\in I^+} (u_k - \eta + 1)(a_k - \rho)(\bar{x}_i - \sum_{j\in J_i} \bar{u}_{ij} x_j) + \sum_{i\in C\setminus I^+} \mu_{a_k,\lambda}(a_i)\bar{x}_i$$
$$+ \sum_{j\in J} \mu_{a_k,\lambda}(\min\{\lambda, -a_j + \sum_{i\in I_j} a_i \bar{u}_{ij}\}) x_j + s \geq (u_k - \eta + 1)(a_k - \rho). \quad (50)$$

Observe that if $u_k = 1$, then we have $a_k - \rho = \lambda$ and $\eta = 1$. In this case, inequalities (49) and (50) are the same if $C \setminus I^+ = \{i \in C : a_i \leq \lambda\}$ and $\bar{J} = \emptyset$ because $\bar{J} = \emptyset$ implies that $-a_j + \sum_{i\in I_j} a_i \bar{u}_{ij} \geq 0$ for all $j \in J$ and $\mu_{a_k,\lambda}(a) = a$ for $0 \leq a \leq \lambda < a_k$.

Alternatively, writing the base inequality as

$$\sum_{i\in C\setminus k} a_i \bar{x}_i + \sum_{i\in N\setminus(C\setminus k)} -a_i x_i + s \geq -\theta, \quad (51)$$

for $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j > \theta\}$ and $I = C \setminus k$, we obtain the symmetric mingling inequality

$$\sum_{i\in C\setminus k} \min\{0, a_i - \sum_{j\in J_i} a_j \bar{u}_{ji} + \theta\}\bar{x}_i + \sum_{j\in J^-} (\theta - a_j)(x_j - \sum_{i\in I_j} \bar{u}_{ji}\bar{x}_i) + s \geq 0. \quad (52)$$

For $J^- \subseteq \{j \in N \setminus (C \setminus k) : a_j \geq a_k \lceil \theta/a_k \rceil\}$, the corresponding symmetric two-step inequality for (51) with $\alpha = a_k$ is then

$$\sum_{j\in J^-} (-a_j + \eta\rho)(x_j - \sum_{i\in I_j} \bar{u}_{ji}\bar{x}_i) + \sum_{j\in J\setminus J^-} (-a_j + \mu_{a_k,\theta}(a_j)) x_j$$
$$+ \sum_{i\in C\setminus k} (a_i + \sum_{j\in J_i} \bar{u}_{ji} + \mu_{a_k,\theta}(\min\{\theta, -a_i + \sum_{j\in J_i} a_j \bar{u}_{ji}\}))\bar{x}_i + s \geq -\theta + \eta\rho. \quad (53)$$

Observe that if $u_k = 1$, then we have $\rho = \theta$ and $\eta = 1$. In this case, inequalities (52) and (53) are the same if $J \setminus J^- = \{j \in N \setminus (C \setminus k) : a_j \leq \theta\}$ and $\bar{I} = \emptyset$ because $\bar{I} = \emptyset$ implies that $-a_i + \sum_{j\in J_i} a_j \bar{u}_{ji} \geq 0$ for all $i \in I = C \setminus k$ and $\mu_{a_k,\theta}(a) = a$ for $0 \leq a \leq \theta < a_k$.

For a cover C, Atamtürk [1, 3] gives the following continuous integer knapsack cover and pack inequalities

$$\sum_{i\in C} -\Phi_k(-a_i)\bar{x}_i + \sum_{j\in J} -\gamma_k(a_j) x_j + s \geq (u_k - \eta + 1)(a_k - \rho) \quad (54)$$

and

$$\sum_{j\in J} -\Phi_k(a_j) x_j + \sum_{i\in C\setminus k} -\omega_k(-a_i)\bar{x}_i + s \geq -\theta + \eta\rho, \quad (55)$$

where

$$\Phi_k(a) = \begin{cases} (\eta - u_k - 1)(a_k - \rho) & \text{if } a < -\lambda, \\ a - \mu_{a_k,b}(a) & \text{if } -\lambda \leq a \leq \theta, \quad \text{for } k \in N \\ a - \eta\rho & \text{if } a > \theta, \end{cases}$$

and γ_k and ω_k are superadditive lifting functions [2] described explicitly in these references. By inspection, it can be verified that if $I^+ = \{i \in C : a_i \geq a_k \lceil \lambda / a_k \rceil\}$, then

$$\gamma_k(a_j) = \sum_{i \in I_j} \bar{u}_{ij}(u_k - \eta + 1)(a_k - \rho) - \mu_{\alpha,\lambda}(\min\{\lambda, (-a_j + \sum_{i \in I_j} a_i \bar{u}_{ij})\}) \text{ for } j \in J$$

and if $J^- = \{j \in N \setminus (C \setminus k) : a_j \geq a_k \lceil \theta / a_k \rceil\}$, then

$$\omega_k(a_i) = a_i + \sum_{j \in J_i} \bar{u}_{ji}(\eta\rho - 1) - \mu_{\alpha,\theta}(\min\{\theta, (a_i + \sum_{j \in J_i} a_j \bar{u}_{ji})\}) \text{ for } i \in C \setminus k.$$

Hence, inequalities (50) and (53) are equivalent to (54) and (55), respectively.

5.3. **C-MIR inequalities.** As mentioned in the Introduction, complemented MIR inequalities, given by Marchand and Wolsey [10], have been successfully implemented as cutting planes in commercial MIP solvers. These cuts involve obtaining a base inequality from the mixed-integer program via constraint aggregation and then complementing some of the variables that have finite upper bounds. An α-MIR inequality is then written for the complemented base inequality.

Here we apply the C-MIR inequalities to the mixed-integer knapsack set $K_{\leq}^{u}$ and compare them with the mingling inequalities. Let $C \subseteq N$ be a cover such that $\bar{a} := \max_{i \in C} a_i > \lambda$. Complementing $x_i,\ i \in C$, we can then write the complemented base inequality as

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{i \in N \setminus C} -a_i x_i + s \geq \lambda, \tag{56}$$

where $\bar{x}_i$ denotes $u - x_i$ as before. Consider the $\bar{a}$-MIR inequality for (56)

$$\sum_{i \in C} \min\{\lambda, a_i\} \bar{x}_i + \sum_{i \in N \setminus C} \mu_{\bar{a},\lambda}(-a_i) x_i + s \geq \lambda. \tag{57}$$

We now show that the mingling inequality (49)

$$\sum_{i \in I^+} \lambda(\bar{x}_i - \sum_{j \in J_i} \bar{u}_{ij} x_j) + \sum_{i \in C \setminus I^+} a_i \bar{x}_i + \sum_{j \in J} \min\{\lambda, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij}\} x_j + s \geq \lambda, \tag{58}$$

where $I^+ = \{i \in C : a_i > \lambda\}$ and $J = N \setminus C$, dominates (57). By definition of I^+ the coefficients for C are the same for both inequalities (57) and (58). For $j \in N \setminus C$, let $k_j = \lceil a_j / \bar{a} \rceil$. Then

$$\mu_{\bar{a},\lambda}(-a_j) = -\lambda k_j + \min\{\lambda, \bar{a} k_j - a_j\}. \tag{59}$$

On the other hand, the coefficient of $x_j,\ j \in J$, in (58) is

$$\sigma_j := -\lambda \sum_{i \in I_j} \bar{u}_{ij} + \min\{\lambda, \sum_{i \in I_j} a_i \bar{u}_{ij} - a_j\}. \tag{60}$$

We consider three cases: 1. if $\sum_{i \in I_j} \bar{u}_{ij} = k_j$, then $\sigma_j \leq \mu_{\bar{a},\lambda}(-a_j)$ as $\sum_{i \in I_j} a_i \bar{u}_{ij} \leq \bar{a} k_j$; 2. if $\sum_{i \in I_j} \bar{u}_{ij} \geq k_j + 1$, then $\sigma_j \leq \mu_{\bar{a},\lambda}(-a_j)$ as $\min\{\lambda, -\sum_{i \in I_j} a_i \bar{u}_{ij} - a_j\} - \min\{\lambda, \bar{a} k_j - a_j\} \leq \lambda$; and finally, 3. if $\sum_{i \in I_j} \bar{u}_{ij} \leq k_j - 1$, then

$$\sigma_j \leq \sum_{i \in I_j} (a_i - \lambda) \bar{u}_{ij} - a_j \leq \sum_{i \in I_j} (\bar{a} - \lambda) \bar{u}_{ij} - a_j \leq (\bar{a} - \lambda)(k_j - 1) - a_j \leq \mu_{\bar{a},\lambda}(-a_j).$$

If C is chosen to be a minimal 0-1 cover, (57) is the familiar 0-1 knapsack cover inequality

$$\sum_{i\in C}\lambda x_i + \sum_{i\in N\setminus C} -\mu_{\bar{a},\lambda}(-a_i)x_i \le \lambda(|C|-1)+s \tag{61}$$

lifted using the MIR function $\mu_{\bar{a},\lambda}$. The numerical example below illustrates the coefficients of $x_i,\ i\in N\setminus C$ for (61) and (43) for comparison.

Example 2. Consider a mixed 0-1 knapsack set given by

$$13x_1 + 10x_2 + 9x_3 + 8x_4 + 5x_5 + ax_6 \le 42 + s,\ x\in\{0,1\}^6,\ s\ge 0.$$

For cover $C=\{1,2,3,4,5\}$, we have $\lambda = 3$ and $\bar{a}=13$. Writing the knapsack inequality as

$$\sum_{i=1}^{5} a_i(1-x_i) - ax_6 + s \ge \lambda, \tag{62}$$

we see that the corresponding complemented $\bar{a}$-MIR inequality (61) is

$$\sum_{i=1}^{5}\lambda x_i - \mu_{13,\lambda}(-a)x_6 \le 4\lambda + s. \tag{63}$$

For the base inequality (62), we have $I^+=\{1,2,3,4,5\}$. The mingling set for x_6 is a function of its coefficient a. For instance, if $32<a\le 40$, then $I_6=\{1,2,3,4\}$, $J_i=\{6\}$ for $i=1,\dots,4$, and $J_5=\emptyset$. Rewriting the base inequality (56) as

$$\sum_{i=1}^{4} a_i[(1-x_i)-x_6] + a_5(1-x_5) + (-a+\sum_{i=1}^{4}a_i)x_6 + s \ge \lambda, \tag{64}$$

we obtain the corresponding mingling inequality

$$\sum_{i=1}^{4}\lambda[(1-x_i)-x_6] + \lambda(1-x_5) + \min\{\lambda, -a+\sum_{i=1}^{4}a_i\}x_6 + s \ge \lambda \tag{65}$$

or, equivalently,

$$\sum_{i=1}^{5}\lambda x_i + \overbrace{\left(4\lambda - \min\{\lambda, -a+\sum_{i=1}^{4}a_i\}\right)}^{\Gamma(a)} x_6 \le 4\lambda + s. \tag{66}$$

In Figure 2 we plot the coefficient of x_6 as a function of a for mingling and complemented MIR inequalities. In general, we have $\Gamma(a)\ge -\mu_{\bar{a},\lambda}(-a)$ for all $a\ge 0$.

5.4. **Two-step inequalities.** We next consider the case when variables do not have finite upper bounds. In this case, let $I^+\subseteq\{i\in I : a_i > b\}$, $I^+\ne\emptyset$, and let $\bar{a} := a_1 = \max\{a_i : i\in I^+\}$. As $u_i=\infty$ for all $i\in I^+$, we have $\bar{J}=\emptyset$, and therefore $p(j)=1$ and $I_j=\{1\}$ for all $j\in J$. Furthermore, $J_1=J$ and $J_i=\emptyset$ for $i>1$. Letting

$$k_j := \bar{u}_{1j} = -\lfloor a_j/\bar{a}\rfloor,\ \text{and}\ r_j := a_j - \bar{a}\lfloor a_j/\bar{a}\rfloor,\ j\in J,$$

inequality (21) becomes

$$b(x_1 + \sum_{j\in J}\lfloor a_j/\bar{a}\rfloor x_j) + \sum_{i\in I^+\setminus\{1\}} bx_i + \sum_{i\in I\setminus I^+} a_i x_i + \sum_{j\in J}\min\{b, r_j\}x_j + s \ge b,$$

which, if $I^+=\{i\in I : a_i > b\}$ reduces to

$$s + \sum_{j\in J}(\min\{b,r_j\} + b\lfloor a_j/\bar{a}\rfloor)x_j + \sum_{i\in I}\min\{a_i, b\}x_i \ge b,$$

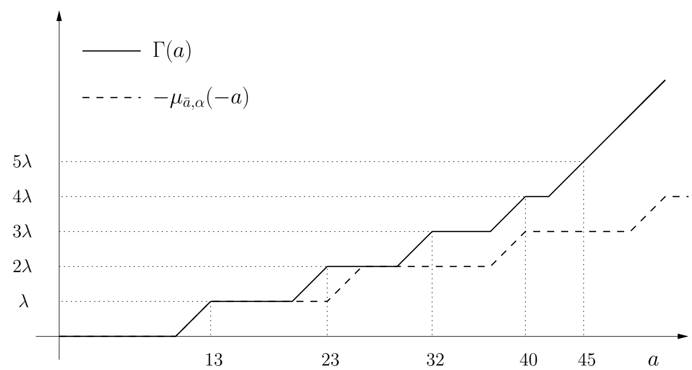

FIGURE 1. Coefficients of mingling and C-MIR inequalities compared.

or, equivalently, the $\bar{a}$-MIR inequality

$$s + \sum_{j \in J} \mu_{\bar{a},b}(a_j)x_j + \sum_{i \in I} \mu_{\bar{a},b}(a_i)x_i \geq \mu_{\bar{a},b}(b) = b$$

applied to the base inequality. In addition, for $\alpha > 0$ such that $\alpha \lceil b/\alpha \rceil \leq \min\{a_i : i \in I^+\}$ inequality (31) becomes

$$s + \sum_{i \in I^+} \mu_{\alpha,b}(b)x_i + \sum_{i \in I \setminus I^+} \mu_{\alpha,b}(a_i)x_i + \sum_{j \in J} [\mu_{\alpha,b}(\min\{b, r_j\}) - \mu_{\alpha,b}(b)k_j]x_j \geq \mu_{\alpha,b}(b),$$

which is the two-step MIR inequality developed by Dash and Günlük [7] when applied to a base inequality $ax \geq b$ that has $\max_{i \in N}\{a_i\} = 1 > b$.

6. Final remarks

Mingling is a simple procedure for incorporating upper bound information into mixed-integer rounding cuts. The fact that many strong inequalities for the fundamental knapsack sets can also be obtained via mingling suggests that mingling may be effective as a cut generation procedure for solving general mixed-integer programs. Furthermore, because mingling uses only MIR functions to describe the cuts, mingling inequalities can be easily implemented using existing MIR routines.

We also note that (symmetric) mingling and two-step mingling inequalities can (and should) be applied after aggregating constraints of the mixed-integer program to form base inequality that defines the knapsack set. An effective approach to achieve this has been described by Marchand and Wolsey [10] for MIR inequalities.

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