Flow pack facets of the single node fixed-charge flow polytope

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Abstract

We present a class of facet-defining inequalities for the single node fixed-charge flow polytope and provide a comparison of valid inequalities for this polytope. We also present computational results that show the effectiveness of these inequalities in solving fixed-charge network flow problems. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

The single node fixed-charge flow model is a basic structure that arises as an important relaxation of many mixed 0–1 integer programming (MBIP) problems with fixed charges. The model consists of a flow balance inequality for a single node with demand b and variable upper bounds on inflow and outflow arcs, which can be formulated as

$$S = \left\{ x \in \{0, 1\}^{|N|}, y \in \mathbb{R}_+^{|N|} : \sum_{i \in N^+} y_i - \sum_{i \in N^-} y_i \leq b, y_i \leq u_i x_i \quad \forall i \in N \right\},$$

where variable y_i is the flow on arc i with capacity u_i , x_i is a binary variable that indicates whether arc i is open or closed, and $N = N^+ \cup N^-$.

The single node fixed-charge flow model is interesting not only because it is a relaxation of the fixed-charge network flow problem, but also because it is possible to derive relaxations of the form S of a general MBIP problem. Therefore, valid inequalities for S can be used as cutting planes in branch-and-cut algorithms to solve MBIP problems. General purpose mixed-integer programming software packages, such as CPLEX,² MINTO [8] and bc-opt [4] use cutting planes derived from S among others. We refer the reader to [13] for a detailed discussion on using S and related structures as relaxations in mixed-integer programming. Valid inequalities for S have also been instrumental in developing strong cutting planes for a variety of problems, including lot-sizing problems [3,10] and facility location problems [1].

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² CPLEX is a trademark of ILOG, Inc.

The study of the polyhedral structure of the convex hull of S is initiated by Padberg et al. [9] for the case with $N^- = \emptyset$. They introduce the flow cover inequalities, which are generalized by Van Roy and Wolsey [12]. Gu et al. [5] strengthen these inequalities through sequence independent lifting for S . Stallaert [11] gives a complementary class of flow cover inequalities. Marchand and Wolsey [7] derive other inequalities for S from mixed 0–1 knapsack relaxations of S .

Here we introduce yet another class of strong inequalities for S and show the relationship between these inequalities and the ones introduced earlier. In Section 2, we present the flow pack inequalities for S and study their strength for a certain restriction of S . Then we strengthen and generalize them by lifting and compare them with other inequalities for S given in the literature. In Section 3, we conclude with a summary of computational experiments that show the effectiveness of the new inequalities in solving capacitated fixed-charge network flow problems with a branch-and-cut algorithm.

2. Flow pack facets

Without loss of generality, we assume that (A.1) $u_i > 0$ for all $i \in N$ (otherwise $y_i = 0$), (A.2) $b + \sum_{i \in N^-} u_i \geq u_k$ for all $k \in N^-$ (otherwise $x_k = 1$), and (A.3) $b + \sum_{i \in N^-} u_i > 0$ (otherwise $y_i = 0$ for all $i \in N^+$). Note that, if $N^- \neq \emptyset$, then (A.3) follows from (A.1) and (A.2). Under these assumptions, $\text{conv}(S)$ is full-dimensional. The linear relaxation of S is

$$P = \left\{ (x, y) \in \mathbb{R}_+^{2|N|} : \sum_{i \in N^+} y_i - \sum_{i \in N^-} y_i \leq b, \ y_i \leq u_i x_i, \ x_i \leq 1 \quad \forall i \in N \right\}.$$

Observe that an extreme point of P has at the most one fractional variable in x . For $C^+ \subseteq N^+$ and $C^- \subseteq N^-$, (C^+, C^-) is called a *flow cover* if $\sum_{i \in C^+} u_i - \sum_{i \in C^-} u_i = b + \lambda$ with $\lambda > 0$. A fractional extreme point $(\bar{x}, \bar{y})$ of P has $\bar{y}_k = u_k - \lambda$, $\bar{x}_k = (u_k - \lambda)/u_k$ for $k \in C^+$, or $\bar{y}_k = \lambda$, $\bar{x}_k = \lambda/u_k$ for $k \in L^- \subseteq N^- \setminus C^-$ with $\lambda < u_k$. The rest of the flow and variable upper bound variables in the cover are at their upper bounds, the flow variables not in the cover are at their lower bounds and the remaining variable upper bound variables are at either of their bounds. Letting $K = N^- \setminus (C^- \cup L^-)$, the well-known *flow cover inequality*

$$\sum_{i \in C^+} (y_i + (u_i - \lambda)^+(1 - x_i)) - \sum_{i \in L^-} \min\{u_i, \lambda\} x_i - \sum_{i \in K} y_i \leq b + \sum_{i \in C^-} u_i, \quad (1)$$

where $a^+ = \max\{a, 0\}$, cuts off the fractional extreme points characterized by flow cover (C^+, C^-) . If x_k , $k \in C^+$ is fractional, then (1) is violated by $(u_k - \lambda)\lambda/u_k$, else if x_k , $k \in L^-$ is fractional, then it is violated by $\lambda(1 - \lambda/u_k)$. Van Roy and Wolsey [12] show that flow cover inequality (1) is facet-defining for $\text{conv}(S)$ if $\max_{i \in C^+} u_i > \lambda$, $u_i > \lambda$ for all $i \in L^-$ and $C^- = \emptyset$.

An alternative characterization of the fractional extreme points of P gives rise to a different class of valid inequalities for S . For $C^+ \subseteq N^+$ and $C^- \subseteq N^-$, (C^+, C^-) is called a *flow pack* if $\sum_{i \in C^+} u_i - \sum_{i \in C^-} u_i + \mu = b$ with $\mu > 0$. A fractional extreme point of P has $\bar{y}_k = \mu$, $\bar{x}_k = \mu/u_k$ for $k \in L^+ \subseteq N^+ \setminus C^+$, or $\bar{y}_k = u_k - \mu$, $\bar{x}_k = (u_k - \mu)/u_k$ for $k \in C^-$ with $\mu < u_k$. The rest of the flow and variable upper bound variables in the pack are at their upper bounds, the flow variables not in the pack are at their lower bounds and the remaining variable upper bound variables are at either of their bounds. Letting $K = N^- \setminus C^-$, the *flow pack inequality*

$$\sum_{i \in C^+} y_i + \sum_{i \in L^+} (y_i - \min\{u_i, \mu\} x_i) + \sum_{i \in C^-} (u_i - \mu)^+(1 - x_i) - \sum_{i \in K} y_i \leq \sum_{i \in C^+} u_i \quad (2)$$

cuts off the fractional extreme points characterized by flow pack (C^+, C^-) . If x_k , $k \in L^+$ is fractional, then (2) is violated by $\mu(1 - \mu/u_k)$, else if x_k , $k \in C^-$ is fractional, then it is violated by $(u_k - \mu)\mu/u_k$. The flow pack inequality is a special case of the inequalities given in Stallaert [11] and may be viewed as a flow cover

inequality (1) for the relaxation of S , where the balance constraint is relaxed to $\sum_{i \in N^-} y_i - \sum_{i \in N^+} y_i - s \leq -b$ after introducing a slack variable s . Flow pack inequalities are facet-defining for the convex hull of the restriction $S_{C^+} = \{(x, y) \in S: x_i = 1 \text{ for } i \in C^+\}$ of S .

Proposition 1. *The following are necessary and sufficient conditions for flow pack inequality (2) to be facet-defining for $\text{conv}(S_{C^+})$:*

1. If $|C^+ \cup K| \geq 2$, then (a) $u_i > \mu$ for all $i \in L^+$, and (b) either $L^+ \neq \emptyset$ or $b + \sum_{i \in N^-} u_i > u_k > \mu$ for some $k \in C^-$.
2. If $|C^+ \cup K| = 1$, then (a) $u_i > \mu$ for all $i \in L^+$, and (b) either $L^+ \neq \emptyset$ or $u_k > \mu$ for some $k \in C^-$.
3. If $|C^+ \cup K| = 0$, then $|L^+| = 1$ and $u_i \geq \mu$ for $i \in L^+$.

Proof. We show only necessity here. For sufficiency, see [2]. Suppose $|C^+ \cup K| \geq 1$. Let $\zeta = \sum_{i \in C^+ \cup K} u_i$ and $\delta_i^j = (u_j - \mu)u_i/\zeta$. If $u_i \leq \mu$ for some $i \in L^+$, then the inequality obtained by adding $y_i \leq u_i x_i$ and the flow pack inequality with $L^+ \setminus \{i\}$ is at least as strong as the original inequality. Now suppose $L^+ = \emptyset$. If there is no $k \in C^-$ with $u_k > \mu$, then the flow pack inequality can be obtained by adding $y_i \leq u_i$, $i \in C^+$ and $-y_i \leq 0$, $i \in K$. Finally, suppose no $k \in C^-$ with $u_k > \mu$ satisfies $b + \sum_{i \in N^-} u_i > u_k$. Since by assumption (A.2), $b + \sum_{i \in N^-} u_i \geq u_k$ holds for all $k \in N^-$, $b + \sum_{i \in N^-} u_i = u_k$ for all $k \in C^-$ with $u_k > \mu$. Then $b + \sum_{i \in N^-} u_i = u_k \Leftrightarrow \zeta = u_k - \mu \Leftrightarrow u_i = \delta_i^k$ for all $i \in C^+ \cup K$. But whenever $x_k = 0$ for $k \in C^-$ with $u_k > \mu$, $y_i = 0$ for all $i \in C^+$ and $y_i = u_i$ for all $i \in K$ in any feasible solution. Hence if $|C^+ \cup K| \geq 2$, then there are not enough points on the face defined by (2). If $|C^+ \cup K| = 0$, then (2) reduces to a trivial inequality $y_i - \mu x_i \leq 0$, $i \in L^+$ and the conditions $|L^+| = 1$ and $u_i \geq \mu$ for $i \in L^+$ are clearly necessary and sufficient. $\square$

2.1. Lifting flow pack inequalities

In order to both strengthen and generalize flow pack inequality (2) for S , for a flow pack (C^+, C^-) , we let $L^- \subseteq N^- \setminus C^-$ and $K = N^- \setminus (C^- \cup L^-)$, and consider the projection $S^0 = \{(x, y) \in S: (x_j, y_j) = (1, u_j) \text{ for all } j \in C^+, (x_j, y_j) = (0, 0) \text{ for all } j \in L^-\}$. S^0 is full-dimensional if $\mu + \sum_{i \in K} u_i \geq u_k$ for all $k \in N^- \setminus L^-$. The inequality

$$\sum_{i \in L^+} (y_i - \min\{u_i, \mu\}x_i) + \sum_{i \in C^-} (u_i - \mu)^+(1 - x_i) - \sum_{i \in K} y_i \leq 0 \quad (3)$$

is facet-defining for $\text{conv}(S^0)$ under the conditions of Proposition 1. To lift (3) for S , we need to compute

$$\begin{aligned} f(z) = & -\max \sum_{i \in L^+} (y_i - \min\{u_i, \mu\}x_i) + \sum_{i \in C^-} (u_i - \mu)^+(1 - x_i) - \sum_{i \in K} y_i \\ \text{s.t. } & \sum_{i \in N^+ \setminus C^+} y_i - \sum_{i \in N^- \setminus L^-} y_i \leq b^0 - z \\ & 0 \leq y_i \leq u_i x_i, \quad x_i \in \{0, 1\}, \quad y_i \in \mathbb{R}, \quad i \in (N^+ \setminus C^+) \cup (N^- \setminus L^-), \end{aligned} \quad (4)$$

for $z \in \mathbb{R}_-$, where $b^0 = b - \sum_{i \in C^+} u_i$.

We show that optimization problem (4) can be solved easily. Let $(\bar{x}, \bar{y})$ be an optimal solution to (4), let $Y = \{i \in L^+: \bar{x}_i = 1\}$ and $Z = \{i \in C^-: \bar{x}_i = 0\}$. We may assume that $Y \subseteq \{i \in L^+: u_i > \mu\}$, $Z \subseteq \{i \in C^-: u_i > \mu\}$, and $\bar{y}_i = u_i$ for all $i \in C^- \setminus Z$, otherwise, we can find a solution with the same or better objective value satisfying these assumptions. For the moment, assume that $\bar{y}_i = 0$ for all $i \in K$ in an optimal solution. Then, $\sum_{i \in L^+} \bar{y}_i$ equals either $b^0 - z + \sum_{i \in C^- \setminus Z} u_i$ or $\sum_{i \in Y} u_i$, depending on whether the balance constraint is tight

or not. If $b^0 - z + \sum_{i \in C^- \setminus Z} u_i \leq \sum_{i \in Y} u_i$ (equivalently, $-\sum_{i \in Y \cup Z} u_i + \mu \leq z$), then

$$f(z)^1 = - \left(b - \sum_{i \in C^+} u_i + \sum_{i \in C^- \setminus Z} u_i + z - |Y|\mu + \sum_{i \in Z} (u_i - \mu) \right) = (|Y \cup Z| - 1)\mu + z.$$

If $b^0 - z + \sum_{i \in C^- \setminus Z} u_i > \sum_{i \in Y} u_i$ (equivalently, $-\sum_{i \in Y \cup Z} u_i + \mu > z$), then

$$f(z)^2 = - \left(\sum_{i \in Y} u_i - |Y|\mu + \sum_{i \in Z} (u_i - \mu) \right) = - \sum_{i \in Y \cup Z} (u_i - \mu).$$

Now, we argue that there exists an optimal solution such that $u_{\underline{l}} = \min\{u_i: i \in Y \cup Z\} \geq u_{\bar{r}} = \max\{u_i: i \in (L^+ \cup C^-) \setminus (Y \cup Z)\}$. For contradiction, suppose $u_{\underline{l}} < u_{\bar{r}}$. If the balance constraint is tight (Case 1), by exchanging $\underline{l}$ and $\bar{r}$ we obtain a solution with the same objective value, since $|Y \cup Z|$ remains unchanged. Otherwise (Case 2), we again exchange $\underline{l}$ and $\bar{r}$. If the new solution is still in Case 2, $f(z)^2$ decreases by $u_{\bar{r}} - u_{\underline{l}}$. If the new solution is in Case 1, the objective value decreases again since $f(z)^2 - f(z)^1 = -\sum_{i \in Y \cup Z} (u_i - \mu) - (|Y \cup Z| - 1)\mu - z = -\sum_{i \in Y \cup Z} u_i + \mu - z > 0$.

Therefore, we may assume that $Y \cup Z$ consists of the first $|Y \cup Z|$ elements of $\{i_1, i_2, \dots, i_r\} \equiv \{i \in C^- \cup L^+: u_i > \mu\}$ indexed in nonincreasing order of u_i . Then the lifting function $f(z)$ can be expressed in a closed form as

$$f(z) = \begin{cases} k\mu + z, & -w_{k+1} + \mu < z \leq -w_k, \quad k = 0, 1, \dots, r-1, \\ k\mu - w_k, & -w_k < z \leq -w_k + \mu, \quad k = 1, 2, \dots, r, \\ r\mu - w_r, & z \leq -w_r, \end{cases} \quad (5)$$

where $w_0 = 0$, $w_k = \sum_{i=1}^k u_{i_i}$ for $k = 1, 2, \dots, r$.

At this time, we remove the assumption that $\bar{y}_i = 0$ for all $i \in K$. Suppose $\bar{y}_i > 0$ for some $i \in K$. Then, the balance constraint must be tight, otherwise, a better solution is available by decreasing $\bar{y}_i$. We can assume that $\bar{y}_i = 0$ for all $i \in L^+$, since otherwise, we can obtain a solution with the same objective value or better by decreasing both $\sum_{i \in K} \bar{y}_i$ and $\sum_{i \in L^+} \bar{y}_i$ in the same amount and setting $x_i = 0$, $i \in L^+$ if applicable. Therefore, $\sum_{i \in K} \bar{y}_i = -b^0 + z - \sum_{i \in C^- \setminus Z} u_i > 0$, which is true only if $b^0 = b - \sum_{i \in C^+} u_i < 0$. Then for such a solution, $f(z)$ equals

$$f(z)^3 = - \left(\sum_{i \in Z} (u_i - \mu) + b - \sum_{i \in C^+} u_i - z + \sum_{i \in C^- \setminus Z} u_i \right) = (|Z| - 1)\mu + z.$$

Let $\bar{Z}$ give the minimum value $\bar{f}(z)^3$. $\bar{Z}$ consists of the elements of C^- ordered in nondecreasing u_i such that $b^0 - z + \sum_{i \in C^- \setminus \bar{Z}} u_i < 0$. Since $b^0 - z + \sum_{i \in C^- \setminus \bar{Z}} u_i < 0 \leq \sum_{i \in C^+} u_i$, it follows that $z > -\sum_{i \in \bar{Z}} u_i + \mu$, implying $f(z) \leq \bar{f}(z)^3$. It remains to check the feasibility of the solution described by $f(z)$. If $\max_{i \in L^+} u_i \leq \min_{i \in \bar{Z}} u_i$, then $f(z)$ is determined by only $\bar{Z}$. In this case, if $-\sum_{i \in \bar{Z}} u_i + \min_{i \in \bar{Z}} u_i \leq z$, then by definition of $\bar{Z}$, $b^0 - z + \sum_{i \in C^- \setminus \bar{Z}} u_i + \min_{i \in \bar{Z}} u_i \geq 0$ and hence the solution is feasible (when $y_i = 0$ for all $i \in K$). Else if $-\sum_{i \in \bar{Z}} u_i + \mu < z < -\sum_{i \in \bar{Z}} u_i + \min_{i \in \bar{Z}} u_i$, then $f(z) = (|\bar{Z}| - 1)\mu + z = \bar{f}(z)^3$. If $\max_{i \in L^+} u_i > \min_{i \in \bar{Z}} u_i$, then $f(z)$ is determined by a subset of L^+ and a strict subset Z of $\bar{Z}$ and since $b^0 - z + \sum_{i \in C^- \setminus Z} u_i \geq 0$ for $Z \subset \bar{Z}$, the solution is feasible. This completes the characterization of the lifting function $f(z)$.

It can be shown that f is superadditive on $\mathbb{R}_-$, which implies that the lifting is sequence independent [6,14], that is the lifting function $f(z)$ remains unchanged as the projected variable pairs (x_j, y_j) , $j \in C^+ \cup L^-$ are introduced to inequality (3) sequentially.

In order to write the lifted pack inequalities explicitly and show their relation to the lifted flow cover inequalities, we define the following function. For $u \in \mathbb{R}_+$, let

$$\phi_{L^+ \cup C^-}(u, \mu) = \begin{cases} k\mu, & w_k \leq u < w_{k+1} - \mu, \quad k = 0, 1, \dots, r-1, \\ k\mu - w_k + u, & w_k - \mu \leq u < w_k, \quad k = 1, 2, \dots, r, \\ r\mu - w_r + u, & w_r \leq u. \end{cases}$$

From Theorem 10 of Gu et al. [5], it can be seen that $\phi_{C^+ \cup L^-}$ is an approximate superadditive lifting function for flow cover inequalities. Then, after adding a slack variable s to the balance constraint of S and relaxing it to $\sum_{i \in N^-} y_i - \sum_{i \in N^+} y_i - s \leq -b$, by writing a lifted flow cover inequality for this relaxation, one obtains the following lifted flow pack inequality.

For $i \in L^-$, let $l = \operatorname{argmax}_{0 \leq h \leq r} \{u_i \geq w_h - \mu\}$. If $l = 0$, then let $G_i = \{(-1, 0)\}$. If $l > 0$, then $G_i = \{(-1, 0)\} \cup G_i^1 \cup G_i^2$, where

$$G_i^1 = \left\{ \left(\frac{\mu}{u_k} - 1, \mu \left(k - 1 - \frac{w_k - \mu}{u_k} \right) \right) : k = 2, 3, \dots, l \right\}$$

and

$$G_i^2 = \begin{cases} \emptyset & \text{if } u_i = w_l - \mu, \\ (0, l\mu - w_l) & \text{if } w_l - \mu < u_i \leq w_l \text{ or } u_i > w_r, \\ \left(\frac{\mu}{u_i - w_l + \mu} - 1, \mu \left(l - \frac{u_i}{u_i - w_l + \mu} \right) \right) & \text{if } u_i \leq w_r \text{ and } w_l < u_i < w_{l+1} - \mu. \end{cases}$$

Theorem 2. Let (C^+, C^-) be a flow pack. If S^0 is full-dimensional, $(\alpha_i, \beta_i) \in G_i$ for $i \in L^-$, and the conditions of Proposition 1 are satisfied, then the lifted flow pack inequality

$$\begin{aligned} & \sum_{i \in C^+} (y_i + \phi_{L^+ \cup C^-}(u_i, \mu)) (1 - x_i) + \sum_{i \in L^+} (y_i - \mu x_i) \\ & + \sum_{i \in C^-} (u_i - \min\{u_i, \mu\})^+ (1 - x_i) + \sum_{i \in L^-} (\alpha_i y_i + \beta_i x_i) - \sum_{i \in K} y_i \leq \sum_{i \in C^+} u_i, \end{aligned} \quad (6)$$

is facet-defining for $\operatorname{conv}(S)$.

Inequality (6) is facet-defining for $\operatorname{conv}(S)$, because the exact lifting function (5) for the flow pack inequalities can be written as $f(-u) = \phi_{L^+ \cup C^-}(u) - u$.

2.2. Comparison of valid inequalities

In contrast to the lifting function of the flow pack inequality (2), the lifting function of the flow cover inequality (1) is not superadditive and its lifting is not sequence independent [5]. An intuitive explanation for the absence of superadditivity in the cover lifting function is that the nonincreasing order of u_i may need to be violated to achieve feasibility when lifting a flow cover inequality; the lifting problem of flow cover inequalities is a restriction, whereas the lifting problem of flow pack inequalities is a relaxation.

In order to highlight the relation among valid inequalities for S , we write a form of the lifted flow cover inequalities, which uses $\phi_{C^+ \cup L^-}$ as a valid superadditive lifting function, even though this function is slightly weaker than the one given in Theorem 11 of Gu et al. [5] for flow covers. Let (C^+, C^-) be a flow cover. If

$(\alpha_i, \beta_i) \in G_i$ for $i \in L^+$, then the following *lifted flow cover inequality* is valid for S :

$$\begin{aligned} & \sum_{i \in C^+} (y_i + (u_i - \lambda)^+(1 - x_i)) + \sum_{i \in L^+} ((\alpha_i + 1)y_i + \beta_i x_i) \\ & + \sum_{i \in C^-} \phi_{C^+ \cup L^-}(u_i, \lambda)(1 - x_i) - \sum_{i \in L^-} \min\{u_i, \lambda\}x_i + \sum_{i \in K} y_i \leq b + \sum_{i \in C^-} u_i. \end{aligned} \quad (7)$$

Now we compare the lifted flow pack and the lifted flow cover inequalities with the k -cover and k -reverse-cover inequalities [7] derived from mixed 0–1 knapsack relaxations of S . Observing that $\phi_{L^+ \cup C^-}(u, \mu) + \psi_{L^+ \cup C^-}(u) = u$ (ψ is a notation used in [7]), after algebraic manipulations, the k -cover inequality can be restated as

$$\begin{aligned} & \sum_{i \in C^+} (y_i - (u_i - \lambda)^+ x_i) + \sum_{i \in L^+} (y_i - (u_i - \phi_{C^+ \cup L^-}(u_i, \lambda))x_i) \\ & + \sum_{i \in C^-} \phi_{C^+ \cup L^-}(u_i, \lambda)(1 - x_i) - \sum_{i \in L^-} \min\{u_i, \lambda\}x_i - \sum_{i \in K} y_i \leq b + \sum_{i \in C^-} u_i \end{aligned} \quad (8)$$

and the k -reverse-cover inequality can be restated as

$$\begin{aligned} & \sum_{i \in C^+} (y_i + \phi_{L^+ \cup C^-}(u_i, \mu)(1 - x_i)) + \sum_{i \in L^+} (y_i - \min\{u_i, \mu\}x_i) \\ & + \sum_{i \in C^-} (u_i - \mu)^+(1 - x_i) - \sum_{i \in L^-} (u_i - \phi_{L^+ \cup C^-}(u_i, \mu))x_i - \sum_{i \in K} y_i \leq \sum_{i \in C^+} u_i. \end{aligned} \quad (9)$$

Lifted flow pack inequality (6) with $(\alpha_i, \beta_i) \in G_i^2$ for all $i \in L^+$ is at least as strong as k -reverse-cover inequality (9). To see this, observe that for $i \in L^+$, if $w_l - \mu < u_i \leq w_l$ or $u_i > w_r$, then $\beta_i = l\mu - w_l = \phi_{C^+ \cup L^-}(u_i, \lambda)$, and thus the inequalities are the same. Else if $u_i \leq w_r$ and $w_l < u_i < w_{l+1} - \mu$, then for $(\bar{x}, \bar{y}) \in S$, let $\zeta_1 = \alpha_i \bar{x}_i + \beta_i \bar{y}_i = \mu(l - u_i/u_i - w_l + \mu)\bar{x}_i + (\mu/u_i - w_l + \mu)\bar{y}_i$ and $\zeta_2 = -(u_i - \phi_{L^+ \cup C^-}(u_i, \mu))\bar{x}_i = (l\mu - u_i)\bar{x}_i$. Then $\zeta_1 - \zeta_2 = (1 - \mu/u_i - w_l + \mu)(u_i \bar{x}_i - \bar{y}_i) \geq 0$. Similarly, lifted flow cover inequality (7) is at least as strong as k -cover inequality (8) for S . Yet, it is still very interesting that one gets strong valid inequalities such as (8) and (9) for S from its much simpler mixed 0–1 knapsack relaxation.

Example. Suppose S is given by

$$y_1 + y_2 + y_3 + y_4 - y_5 - y_6 - y_7 \leq 3,$$

$$y_1 \leq 6x_1, \quad y_2 \leq 4x_2, \quad y_3 \leq 6x_3, \quad y_4 \leq 8x_4, \quad y_5 \leq 5x_5, \quad y_6 \leq 2x_6, \quad y_7 \leq 12x_7.$$

Let $C^+ = \{1\}$, $C^- = \{5\}$, $L^+ = \{2, 3\}$, $L^- = \{7\}$, and $K = \{6\}$. Then $\mu = 2$ and $y_2 - 2x_2 + y_3 - 2x_3 + 3(1 - x_5) - y_6 \leq 0$ is valid for $S^0 = \{(x, y) \in S: (x_1, y_1) = (1, 6), (x_7, y_7) = (0, 0)\}$. Note that $w_0 = 0$, $w_1 = 6$, $w_2 = 11$, $w_3 = 15$. Lifting it with x_1 , y_1 , x_7 , and y_7 , we obtain the following lifted flow pack inequalities:

$$y_1 + 2(1 - x_1) + y_2 - 2x_2 + y_3 - 2x_3 - 3(1 - x_5) - y_6 - y_7 - 0x_7 \leq 6,$$

$$y_1 + 2(1 - x_1) + y_2 - 2x_2 + y_3 - 2x_3 - 3(1 - x_5) - y_6 - \frac{3}{5}y_7 - \frac{8}{5}x_7 \leq 6,$$

$$y_1 + 2(1 - x_1) + y_2 - 2x_2 + y_3 - 2x_3 - 3(1 - x_5) - y_6 - \frac{1}{3}y_7 - 4x_7 \leq 6.$$

Alternatively, let $C^+ = \{1, 2\}$, $C^- = \{5\}$, $L^+ = \{4\}$, $L^- = \emptyset$, and $K = \{6, 7\}$. Then $\lambda = 2$ and $y_1 + 4(1 - x_4) + y_2 + 2(1 - x_2) - y_6 - y_7 \leq 8$ is valid for $S^0 = \{(x, y) \in S: (x_4, y_4) = (0, 0), (x_5, y_5) = (1, 5)\}$. Here, $w_0 = 0$, $w_1 = 6$, $w_2 = 10$. Lifting it with x_4 , y_4 , x_5 , and y_5 , we get the following lifted flow cover inequalities:

$$y_1 + 4(1 - x_1) + y_2 + 2(1 - x_2) + 0y_4 - 0x_4 - (1 - x_5) - y_6 - y_7 \leq 8,$$

$$y_1 + 4(1 - x_1) + y_2 + 2(1 - x_2) + \frac{1}{2}y_4 - 2x_4 - (1 - x_5) - y_6 - y_7 \leq 8.$$

3. Computational results

In this section, we present a summary of computational experiments on using the lifted flow pack inequalities (6) in a branch-and-cut algorithm for solving capacitated fixed-charge network flow problems. The branch-and-cut algorithm was implemented with MINTO [8] (version 3.0) using CPLEX (version 6.0) as the LP solver. All experiments were performed on a SUN Ultra 10 workstation. In addition to the lifted pack inequalities, we used the lifted flow cover inequalities that are automatically generated by MINTO. The lifted flow cover inequalities that MINTO generates are similar to inequality (7) and are described in Gu et al. [5].

In order to test the effectiveness of the lifted flow pack inequalities, we ran the branch-and-cut algorithm on a set of randomly generated capacitated fixed-charge network flow problems available at <http://ieor.berkeley.edu/~atamturk/data>. In Tables 1 and 2, we present a comparison of the performance of the branch-and-cut algorithm when run with only MINTO's lifted flow cover inequalities, by adding flow pack inequalities, and finally by adding lifted flow pack inequalities. Table 1 is for instance with 30 nodes and 50% arc density, whereas Table 2 is for instance with 60 nodes and 20% arc density. For this experiment, we also implemented a simple primal heuristic, which rounds fractional binary variables in LP solutions in the search tree to one, in order to construct feasible solutions quickly for pruning. We report the objective values of the initial LP relaxation after preprocessing (zinit), optimal integral solution (zopt), and the LP relaxation at the root node of the search tree after all cuts are added (zroot), the number of lifted flow cover cuts (lfcovs), the number of flow pack cuts (fpacks), the number of lifted flow pack cuts (lfpacks), the number of nodes evaluated (nodes), and the CPU time elapsed in seconds. All fractional entries are rounded to the nearest integer. A comparison of the entries in columns zinit and zopt shows that all of these problems have big duality gaps. LP relaxations at the root node (zroot) improve consistently with the addition of flow pack and lifted flow pack cuts. We also see significant reductions in the number of nodes explored and in the CPU time elapsed. Note that more lifted flow pack cuts are generated than flow pack cuts and that lifting flow pack cuts do improve the LP relaxations. The effect of lifting flow pack inequalities is more significant for the larger instances in Table 2. These computational results indicate that the flow pack inequalities and the lifted flow pack inequalities improve the performance of the branch-and-cut algorithm significantly for capacitated fixed-charge network flow problems.

Table 1
Computations with capacitated fixed-charge network flow problems: 30 nodes

Problem	zinit	zopt	Without fpacks				With fpacks				With lfpacks					
			zroot	lfcovs	nodes	time	zroot	lfcovs	fpacks	nodes	time	zroot	lfcovs	lfpacks	nodes	time
fc.30.1	120	307	259	458	9783	210	261	286	45	5434	122	261	309	186	6320	145
fc.30.2	152	325	273	341	4689	95	276	334	75	1807	38	280	230	171	1412	43
fc.30.3	98	294	235	252	3872	47	251	412	125	2193	68	254	245	159	1820	53
fc.30.4	100	763	712	261	3238	40	734	305	65	2213	35	738	205	104	1847	28
fc.30.5	158	301	274	146	215	4	277	172	36	145	4	277	128	60	149	4
fc.30.6	126	272	238	296	8759	105	241	276	64	3019	72	245	243	187	4357	93
fc.30.7	97	231	204	187	1471	14	207	167	30	458	8	212	176	98	554	10
fc.30.8	176	347	300	431	13623	270	305	381	72	4257	113	305	290	182	6063	147
fc.30.9	85	741	708	334	3172	56	709	257	43	1818	37	710	313	145	1793	40
fc.30.10	93	204	171	252	2991	32	173	249	66	1217	17	179	166	118	507	11

Table 2

Computations with capacitated fixed-charge network flow problems: 60 nodes

Problem	zinit	zopt	Without fpacks				With fpacks				With lfpacks				nodes	time
			zroot	lfcovs	nodes	time	zroot	lfcovs	lfpacks	nodes	time	zroot	lfcovs	lfpacks		
fc.60.1	171	487	394	612	79448	2062	413	751	225	23631	1122	415	468	350	21444	1054
fc.60.2	230	584	445	830	82050	1018	459	975	347	33580	1812	490	471	450	15070	634
fc.60.3	161	493	414	494	11124	300	420	698	194	15091	497	420	358	279	4929	150
fc.60.4	120	442	337	780	141882	4175	354	999	322	30956	1652	354	532	423	19386	960
fc.60.5	160	414	332	451	20436	309	336	587	155	9775	238	349	259	245	1734	39
fc.60.6	199	480	388	681	127285	3656	399	738	237	28929	1063	421	491	373	20553	939
fc.60.7	177	492	422	463	7450	184	426	497	101	5350	373	443	375	305	4090	176
fc.60.8	202	500	440	486	14360	353	453	671	159	7897	425	455	438	330	5239	289
fc.60.9	164	397	358	342	3620	54	365	312	47	1557	25	365	227	142	915	16
fc.60.10	180	913	823	820	44523	1982	827	759	196	8675	508	834	450	255	11912	534

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