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Two-Stage Robust Network Flow and Design Under Demand Uncertainty

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We describe a two-stage robust optimization approach for solving network flow and design problems with uncertain demand. In two-stage network optimization, one defers a subset of the flow decisions until after the realization of the uncertain demand. Availability of such a recourse action allows one to come up with less conservative solutions compared to single-stage optimization. However, this advantage often comes at a price: two-stage optimization is, in general, significantly harder than single-stage optimization.

For network flow and design under demand uncertainty, we give a characterization of the first-stage robust decisions with an exponential number of constraints and prove that the corresponding separation problem is $\mathcal{NP}$-hard even for a network flow problem on a bipartite graph. We show, however, that if the second-stage network topology is totally ordered or an arborescence, then the separation problem is tractable.

Unlike single-stage robust optimization under demand uncertainty, two-stage robust optimization allows one to control conservatism of the solutions by means of an allowed "budget for demand uncertainty." Using a budget of uncertainty, we provide an upper bound on the probability of infeasibility of a robust solution for a random demand vector.

We generalize the approach to multicommodity network flow and design, and give applications to lot-sizing and location-transportation problems. By projecting out second-stage flow variables, we define an upper bounding problem for the two-stage min-max-min optimization problem. Finally, we present computational results comparing the proposed two-stage robust optimization approach with single-stage robust optimization as well as scenario-based two-stage stochastic optimization.

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1. Introduction

We describe a two-stage robust optimization approach to network flow and design problems with uncertain demand. In this approach, the arcs of a network are partitioned into two sets: A (first stage) and B (second stage). In the first stage, all design variables and flow variables associated with arcs A need to be decided *before* the realization of demand uncertainty, whereas flow variables associated with arcs B are decided *after* observing the demand in the second stage.

Examples of two-stage network design problems with demand uncertainty include telecommunication, hub location, production, and distribution problems. Typically in such problems, one needs to make first-stage design and capacity allocation decisions before the realization of uncertain demand, whereas routing decisions are made after observing the demand in the second stage. In the stochastic programming literature, such problems are referred to as two-stage problems with recourse (Birge and Louveaux 1997).

Research on robust optimization has recently received renewed attention (Atamtürk 2006; Averbakh 2000, 2001; Ben-Tal and Nemirovski 1998, 2000; Bertsimas and Sim 2003, 2004, 2006; Bertsimas and Thiele 2006; Bienstock and Özbay 2005; Erdoğan and Iyengar 2006; El Ghaoui et al. 1998; Goldfarb and Iyengar 2003; Kouvelis and Yu 1997; Lim and Shanthikumar 2007; Ordóñez and Zhao 2004, to name a few). Most of this research is concentrated on convex robust optimization. In robust optimization, parameters of the models are described with an uncertainty set and one looks for a solution that is feasible with respect to all realizations in the chosen uncertainty set.

Our work is closely related to a recent paper by Ben-Tal et al. (2004), in which the authors study two-stage robust linear programming under the name adjustable robust linear programming. They show that two-stage robust linear programming is computationally intractable and propose a tractable alternative referred to as affinely-adjustable robust linear programming. In this scheme, second-stage decision variables are restricted to be affine functions of the uncertain parameters.

Here we do not follow the affinely adjustable robust optimization approach. Instead, we focus on two-stage robust network flow and design, and study the problem in detail by exploiting the underlying network structure.

The following related works have become available very recently. Erera et al. (2005) propose a two-stage robust optimization approach for container repositioning problems on time-expanded networks, in which recourse is done by a set of permitted recovery transformations to flow. Chen et al. (2006) give a two-stage robust optimization approach with a "deflected linear relationship" between second-stage decisions and uncertain data. Both of these studies are in the spirit of Ben-Tal et al. (2004) as they restrict the second-stage decisions. Thiele (2005) describes a Benders decomposition approach for robust linear programming with recourse.

The outline of this paper is as follows. In §2, we introduce the concepts in the context of network flows and give an explicit description of the first-stage robust decisions with an exponential number of constraints. In §3, we generalize the results to network design. In §4, we investigate the tractability of the corresponding separation problems. By using a "budget of demand uncertainty" we give an upper bound on the probability of infeasibility of the robust solution for a random demand vector. In §5, we present interesting special cases for which separation is tractable. In §6, based on the results in the previous sections, we introduce an upper bounding problem for the min-max-min optimization problem. In §7, we extend the approach to multiple commodities. In §8, we present a summary of computational experiments on a two-stage robust location-transportation problem with uncertain demand and compare the results with that of a single-stage robust optimization approach and a scenario-based two-stage stochastic programming approach. Finally, we conclude with §9.

Introductory Example. To motivate the paper, we start with a simple example.

Example 1. Consider the graph in Figure 1 with nodes $V = \{0, 1, 2\}$ and arcs $E = \{a, b, c\}$. Let x_a, x_b, and x_c denote the flow on the arcs. In addition, let arc a have an integer design variable y_a with 10 units of capacity; arcs b and c have no upper bound. For a demand vector $d \in \mathbb{R}^V$, the feasibility constraints can be stated as

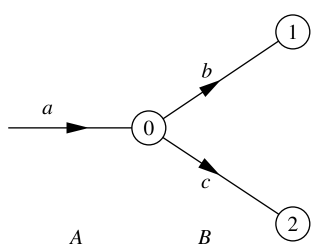
$$x_b \geqslant d_1; \quad x_c \geqslant d_2; \quad x_a \geqslant x_b + x_c + d_0;$$

$$10y_a \geqslant x_a; \quad x_a, x_b, x_c \in \mathbb{R}_+, \quad y_a \in \mathbb{Z}_+.$$

Now suppose that demand d is uncertain, but is known to belong to the set

$$\mathcal{U} = \{d \in \mathbb{R}^V\colon d_0 = 0,\ 0 \leqslant d_1 \leqslant 6,\ 0 \leqslant d_2 \leqslant 8,\ 3d_1 + 2d_2 \leqslant 19\}.$$

Figure 1. Two-stage robust network design.



A single-stage robust solution for this network is a solution vector (x_a, x_b, x_c, y_a) that is feasible for all $d \in \mathcal{U}$, which, then, requires that $x_b \geqslant 6$ and $x_c \geqslant 8$, and consequently $x_a \geqslant x_b + x_c + 0 \geqslant 14$; leading to a minimum integer capacity $y_a = 2$.

Let us now compare this solution with that of a two-stage version. Suppose that $A = \{a\}$ and $B = \{b, c\}$; that is, in the first stage we need to determine the capacity variable y_a and the flow x_a such that there exist feasible flows $x_b(d)$ and $x_c(d)$ in the second stage for any $d \in \mathcal{U}$. Then, because $d_1 + d_2 \leqslant 9$ for all $d \in \mathcal{U}$, it suffices to let $x_a \geqslant 9$ to guarantee feasible flows $x_b(d)$ and $x_c(d)$ in the second stage for any $d \in \mathcal{U}$, leading to a feasible capacity $y_a = 1$.

The 50% saving in installed capacity is the benefit due to robust decision making in two stages. It should be clear to the reader that one can generalize this example to one with k arcs in the second stage, for which the minimum single-stage robust capacity is k times that of the two-stage robust capacity.

2. Network Flow with Demand Uncertainty

Let $G = (V, E)$ be a directed graph with a node set V and an arc set E. For any demand vector $d \in \mathbb{R}^V$, the set of feasible network flow solutions for G can be stated as

$$\mathcal{P}_d := \{x \in \mathbb{R}^E_+\colon x(\delta^+(i)) - x(\delta^-(i)) \geqslant d_i \text{ for all } i \in V\},$$

where $\delta^+(S)$ and $\delta^-(S)$ denote the set of arcs into and out of the node set $S \subseteq V$, respectively, and $x(T)$ denotes $\sum_{a\in T} x_a$ for $T \subseteq E$, with $x(\varnothing) := 0$. For simplicity of notation, we refer to a singleton set $\{i\}$ by its unique element i. For $S \subseteq V$ and $d \in \mathbb{R}^V$, we denote $\sum_{a\in S} d_a$ as $d(S)$, with $d(\varnothing) := 0$.

Definition 1. Let $\mathcal{U} \subset \mathbb{R}^V$ be a compact set denoting the uncertain values for the demand. The *single-stage robust network flow set* $\mathcal{P}$ is the set of solutions that are feasible for *all* realizations of the demand in $\mathcal{U}$; that is, $\mathcal{P} := \bigcap_{d\in\mathcal{U}} \mathcal{P}_d$.

Optimization over the single-stage robust network flow set $\mathcal{P}$ typically produces solutions that are conservative. Let us now consider the two-stage approach.

Definition 2. For compact $\mathcal{U} \subset \mathbb{R}^V$ and a partitioning (A, B) of the arcs E of graph G, let the *first-stage robust network flow set* $\mathcal{P}(A)$ be the set of first-stage solutions x_A for which there exists some second-stage solution $x_B(d)$ such that $(x_A, x_B(d))$ belongs to P_d for all $d \in \mathcal{U}$; that is,

$$\mathcal{P}(A) := \bigcap_{d\in\mathcal{U}} \mathrm{Proj}_A\, \mathcal{P}_d,$$

$$\text{where } \mathrm{Proj}_A\, \mathcal{P}_d := \{x_A\colon (x_A, x_B) \in \mathcal{P}_d\}.$$

Crucial in Definition 2 is that the second-stage solution x_B is a function of the demand realization. Next, we give an explicit description of $\mathcal{P}(A)$ for $A \subseteq E$. Toward this end, for any $A \subseteq E$, let us define $\delta^+_A(S) = A \cap \delta^+(S)$ and $\delta^-_A(S) = A \cap \delta^-(S)$.

Definition 3. Given $A \subseteq E$, a subset S of nodes V is *weak* if $\delta^+(S) \subseteq A$.

Remark 1. For weak $S \subseteq V$, we have $\delta_A^+(S) = \delta^+(S)$, or equivalently, $\delta_B^+(S) = \varnothing$.

Theorem 1. *Let $\mathcal{P}(A)$ be defined as above and $x_A \in \mathbb{R}_+^A$. Then, $x_A \in \mathcal{P}(A)$ if and only if*

$$x_A(\delta_A^+(S)) - x_A(\delta_A^-(S)) \geqslant \zeta_S \quad \textit{for all weak } S \subseteq V, \qquad \textit{where } \zeta_S := \max\{d(S)\colon d \in \mathcal{U}\}. \quad (1)$$

Theorem 1 is a special case of Theorem 4 and its proof is given after Theorem 4. Accordingly, the first-stage robust network flow set $\mathcal{P}(A)$ can be stated explicitly as

$$\mathcal{P}(A) = \{x_A \in \mathbb{R}_+^A\colon x_A(\delta_A^+(S)) - x_A(\delta_A^-(S)) \geqslant \zeta_S \quad \text{for all weak } S \subseteq V\}$$

with exponentially many constraints (1). In §4, we discuss the separation problem of $\mathcal{P}(A)$ and its computational complexity.

Example 1 (Continued). For the graph in Figure 1 with $A = \{a\}$, the weak sets are $\varnothing$, $\{0\}$, $\{0, 1\}$, $\{0, 2\}$, and $\{0, 1, 2\}$. Therefore, the nondominated inequality (1) is $x_a \geqslant \zeta_{\{0,1,2\}} = 9$.

Observation 1. The right-hand-side of inequality (1) satisfies $\zeta_S \leqslant \sum_{i \in S} \zeta_i$ for $S \subseteq V$.

Remark 2. For the special case in which all arcs belong to the first stage, i.e., $A = E$ (single stage), any subset S of V is weak. Then, it follows from Observation 1 that the single-stage robust network flow solutions can be stated as

$$\mathcal{P} = \mathcal{P}(E) = \mathcal{P}_\zeta := \{x \in \mathbb{R}_+^E\colon x(\delta^+(i)) - x(\delta^-(i)) \geqslant \zeta_i \text{ for all } i \in V\}.$$

Thus, $\mathcal{P}$ is the network polyhedron $\mathcal{P}_d$ with $d = \zeta$ and it is a tractable set if there is a polynomial algorithm for computing ζ_i for all $i \in V$.

Remark 2 is consistent with Soyster's (1973) observation that when uncertainty is in the columns of a linear program, single-stage robust solutions must satisfy the worst realization for each entry of a column. We will elaborate on the conservativeness of the single-stage robust optimization for demand uncertainty later in Remark 3. The following elementary proposition underscores the value of recourse through the second-stage for avoiding conservative solutions of $\mathcal{P}$.

Proposition 2. *For $\mathcal{U} \subset \mathbb{R}^V$, we have $\operatorname{Proj}_A \mathcal{P} \subseteq \mathcal{P}(A)$ for $A \subseteq E$.*

Proof. Because $\mathcal{P} \subseteq \mathcal{P}_d$ for all $d \in \mathcal{U}$, if $(x_A, x_B) \in \mathcal{P}$, then $x_A \in \mathcal{P}(A)$. ☐

Although in general the set of first-stage robust solutions $\mathcal{P}(A)$ is a relaxation of the corresponding single-stage solution set $\operatorname{Proj}_A \mathcal{P}$, if uncertain demands at the nodes are independent, then there is no difference between the two, which is stated formally in the next proposition.

Proposition 3. *If $\mathcal{U} = \prod_{i \in V} \mathcal{U}_i$ for compact $\mathcal{U}_i \subset \mathbb{R}$, then $\mathcal{P}(A) = \operatorname{Proj}_A \mathcal{P}$ for $A \subseteq E$.*

Proof. Due to Proposition 2, it suffices to show that $\mathcal{P}(A) \subseteq \operatorname{Proj}_A \mathcal{P}$ for the special case. Because $\mathcal{U}$ is the cartesian product of individual demand uncertainties, we have $d = \zeta \in \mathcal{U}$. Then, for $x_A \in \mathcal{P}(A)$, $\exists (x_A, x_B) \in \mathcal{P}_\zeta = \mathcal{P}$. ☐

Consequently, two-stage robust optimization is interesting when there is a dependence among uncertain demands. In §4, we consider two such demand uncertainty sets with a constraint on the aggregate demand in addition to bounds on individual demand.

3. Network Design with Demand Uncertainty

In this section, we generalize the characterization of the first-stage robust decisions to network design problems. Incorporating side constraints on the first-stage variables x_A is handled simply by adding them to the formulation. However, side constraints on the second-stage variables x_B affect the constraints defining the projection.

Let u_{ij} denote the capacity of arc $(ij) \in E$ and $y_{ij} \in \mathbb{Z}_+$ the corresponding integer (design) variable used for modeling the fixed-charge of flow on this arc. For any demand vector $d \in \mathbb{R}^V$, the feasible network design set is

$$\mathcal{Q}_d := \{(x, y) \in \mathbb{R}_+^E \times \mathbb{Z}_+^E\colon x(\delta^+(i)) - x(\delta^-(i)) \geqslant d_i \quad \text{for all } i \in V,\ x \leqslant u \circ y\},$$

where $(u \circ y)_{ij} = u_{ij} y_{ij}$ for $(ij) \in E$. Therefore, the set of first-stage robust solutions is

$$\mathcal{Q}(A) := \bigcap_{d \in \mathcal{U}} \operatorname{Proj}_A \mathcal{Q}_d, \qquad \text{where } \operatorname{Proj}_A \mathcal{Q}_d := \{(x_A, y)\colon (x, y) \in \mathcal{Q}_d\}.$$

Theorem 4. *Let $\mathcal{Q}(A)$ be defined as above and $(x_A, y) \in \mathbb{R}_+^A \times \mathbb{Z}_+^E$ such that $x_A \leqslant u_A \circ y_A$. Then, $(x_A, y) \in \mathcal{Q}(A)$ if and only if*

$$x_A(\delta_A^+(S)) + (u_B \circ y_B)(\delta_B^+(S)) - x_A(\delta_A^-(S)) \geqslant \zeta_S \qquad \textit{for all } S \subseteq V. \quad (2)$$

Proof. For $(x_A, y) \in \mathbb{R}_+^A \times \mathbb{Z}_+^E$ satisfying $x_A \leqslant u_A \circ y_A$, $(x_A, y) \in \mathcal{Q}(A)$ if and only if for any $d \in \mathcal{U}$, $\exists x_B \in \mathbb{R}_+^B$ such that $(x_A, x_B, y) \in \mathcal{Q}_d$. It follows from the Farkas Lemma that for a given $d \in \mathcal{U}$ and (x_A, y), the system of nequalities

$$x_B(\delta_B^+(i)) - x_B(\delta^-(i)_B) \geqslant d_i - x_A(\delta_A^+(i)) + x_A(\delta_A^-(i)) \quad \text{for all } i \in V,$$

$$\mathbf{0} \leqslant x_B \leqslant u_B \circ y_B,$$

is feasible if and only if

$$\sum_{i\in V} v_i(d_i - x_A(\delta_A^+(i)) + x_A(\delta_A^-(i))) \leqslant \sum_{i\in B} w_i(u_B \circ y_B)_i \qquad (3)$$

holds for all $v \in \mathbb{R}^V$ and $w \in \mathbb{R}^B$ such that

$$v_j - v_i \leqslant w_{ij}, \quad (ij) \in B, \qquad (4)$$

$$v \geqslant \mathbf{0}, \quad w \geqslant \mathbf{0}. \qquad (5)$$

Note that (4) and (5) are the constraints of the dual of a network flow problem on $G(V, B)$—the subgraph induced by the second-stage arcs B and the extreme rays of the polyhedron defined by (4)–(5) correspond to the cuts of $G(V, B)$. Therefore, in (3) it suffices to consider (v, w) with v_j, $j \in V$ equals 1 if $j \in S$ and 0 otherwise; and w_{ij}, $(ij) \in B$ equals 1 if $(ij) \in \delta^+(S)$, and 0 otherwise for all $S \subseteq V$. Thus, for a given $d \in \mathcal{U}$, $(x_A, y) \in \operatorname{Proj}_A \mathcal{Q}_d$ if and only if

$$x_A(\delta_A^+(S)) + (u_B \circ y_B)(\delta_B^+(S)) - x_A(\delta_A^-(S)) \geqslant d(S) \quad \text{for all } S \subseteq V. \quad (6)$$

Taking the maximum of $d(S)$ over $\mathcal{U}$ gives the result. □

Theorem 4 provides an explicit description of $\mathcal{Q}(A)$. Note that there is no restriction on the sets S defining (2); therefore, they are not necessarily weak sets (Definition 3) as in the case of inequalities (1). Observe that one obtains Theorem 1 as a special case of Theorem 4 by letting $u = \infty$ and $y = \mathbf{1}$. Also, by letting $y = \mathbf{1}$, one obtains the special case of network flow with upper bounds.

For the case with $A = \varnothing$ and $y = \mathbf{1}$, inequalities (6) reduce to the so-called *Gale-Hoffman inequalities* (Gale 1957, Hoffman 1960),

$$u(\delta^+(S)) \geqslant d(S) \quad \text{for all } S \subseteq V, \qquad (7)$$

which characterize feasibility of a network flow problem with given demand $d \in \mathbb{R}^V$ and capacity $u \in \mathbb{R}^E$. By using exponentially-many Gale-Hoffman inequalities (7), Prékopa and Boros (1991) give lower and upper bounds on the feasibility probability of a network flow problem with random capacities and demands. Wallace and Wets (1993, p. 217) state that computing

$$\operatorname{Prob}(u(\delta^+(S)) \geqslant d(S) \text{ for all } S \subseteq V)$$

"would be possible—by checking the inequalities for a few million samples, for example—provided the number of inequalities is not too large, suggesting the elimination of redundant inequalities" and they give a connectedness criterion for characterizing the redundant inequalities among (7). Wallace and Wets (1989, 1995) describe enumeration algorithms with advanced preprocessing techniques to reduce the number of inequalities. They present computational results demonstrating that a large number of the inequalities can be eliminated by preprocessing. Nevertheless, it appears that except for small graphs, a separation approach rather than complete enumeration is necessary.

4. Separation Complexity and Uncertainty Set

In this section, we study the complexity of the separation problems for $\mathcal{P}(A)$ and $\mathcal{Q}(A)$. Given a point $(x_A, y) \in \mathbb{R}_+^A \times \mathbb{Z}_+^E$ such that $x_A \leqslant u_A \circ y_A$, the separation problem for $\mathcal{Q}(A)$, i.e., for inequalities (2), can be formulated as a bilinear mixed 0-1 program:

$$(\mathrm{SP}_{\mathcal{Q}}) \quad \varsigma = \min \sum_{i\in V} w_i z_i + \sum_{(ij)\in B} (u_{ij} y_{ij}) v_{ij} - \max \sum_{i\in V} d_i z_i$$
$$\text{s.t.} \quad z_j - z_i \leqslant v_{ij} \quad \text{for all } (ij) \in B,$$
$$z \in \{0,1\}^V,\ v \in \{0,1\}^B, \quad d \in \mathcal{U},$$

where $w_i = x_A(\delta_A^+(i)) - x_A(\delta_A^-(i))$ for $i \in V$. Here $z_i = 1$ if and only if $i \in S$; thus, $(ij) \in \delta^+(S)$ implies $v_{ij} = 1$. Because $u_{ij} y_{ij} \geqslant 0$, if $(ij) \notin \delta^+(S)$, there exists an optimal solution with $v_{ij} = 0$. Therefore, we have $\varsigma < 0$ if and only if inequality (2) corresponding to an optimal solution for $\mathrm{SP}_{\mathcal{Q}}$ is violated. The separation problem for the unbounded network flow case $\mathcal{P}(A)$ of §2 is obtained by setting $u = \infty$, $y = \mathbf{1}$, and consequently $v = \mathbf{0}$:

$$(\mathrm{SP}_{\mathcal{P}}) \quad \varsigma = \min \sum_{i\in V} w_i z_i - \max \sum_{i\in V} d_i z_i$$
$$\text{s.t.} \quad z_j - z_i \leqslant 0 \quad \text{for all } (ij) \in B,$$
$$z \in \{0,1\}^V, \quad d \in \mathcal{U}.$$

Observe that constraints $z_j \leqslant z_i$, $(ij) \in B$, ensure that feasible solutions of $\mathrm{SP}_{\mathcal{P}}$ correspond to weak sets of G according to Definition 3.

The computational complexity of these separation problems is a function of the structure of $G(V, B)$—the subgraph induced by the second-stage arcs B—as well as the demand uncertainty set $\mathcal{U}$. We consider two types of uncertainty sets:

(1) Cardinality-restricted uncertainty set (Bertsimas and Sim 2003):

$$\mathcal{U}_C := \left\{ d \in \mathbb{R}^V \colon \sum_{i\in \overline{V}} \lceil |d_i - \bar{d}_i| / h_i \rceil \leqslant \Gamma,\ \bar{d} - h \leqslant d \leqslant \bar{d} + h \right\},$$

where $\overline{V} = \{i \in V \colon h_i > 0\}$. Here Γ denotes the maximum number of demands that are allowed to differ from their midvalue $\bar{d}_i$.

(2) Budget uncertainty set:

$$\mathcal{U}_B := \left\{ d \in \mathbb{R}^V \colon \sum_{i\in V} \pi_i d_i \leqslant \pi_o,\ \bar{d} - h \leqslant d \leqslant \bar{d} + h \right\},$$

where $\bar{d} \pm h$ bound individual demands and the constraint $\pi d \leqslant \pi_o$ represents a joint "budget" for allowed uncertainty in the demand. To avoid trivial cases, we assume that $\pi_i h_i \neq 0$ for some $i \in V$.

Theorem 5. *The separation problem $SP_{\mathcal{P}}$ is $\mathcal{NP}$-hard for $\mathcal{U}_C$ for bipartite $G(V, B)$.*

Proof. Let (M, N) be the bipartitioning of the nodes of the second-stage graph $G(V, B)$. Consider the following case. Let $\bar{d}_i = h_i = 0$ for $i \in M$ and let there be an exogenous first-stage arc into each node $i \in M$, labeled as i as well. Nodes N are not incident to any first-stage arc. Then, the separation problem w.r.t. $\mathcal{U}_C$ can be stated as

$$\begin{aligned}
(\mathrm{SP}_{\mathcal{P}}\colon \mathcal{U}_C) \quad \varsigma = \min \quad & \sum_{i\in M} w_i z_i - \sum_{i\in N} (\bar{d}_i z_i + h_i y_i)\\
\text{s.t.} \quad & \sum_{i\in N} y_i \leqslant \Gamma, \quad y_i \leqslant z_i \ \text{ for all } i \in N,\\
& z_j \leqslant z_i \quad \text{for all } (ij) \in B,\\
& y \in \{0,1\}^N, \quad z \in \{0,1\}^{M\cup N}.
\end{aligned}$$

We prove the theorem by a reduction from the $\mathcal{NP}$-complete problem `CLIQUE` (Garey and Johnson 1979): Given an undirected graph $H = (U, F)$ and integer $k \leqslant n := |U|$, does H contain a clique of size k? Let $G = (M \cup N, B)$ be a bipartite graph with $M = U$, $N = F$, $B = \{(ie), (je)\colon (ij) := e \in F\}$ and define an instance of $\mathrm{SP}_{\mathcal{P}}\colon \mathcal{U}_C$ with parameters $w_i = 1$ for $i \in M$, $\bar{d}_j = 0$, $h_j = n^2$ for $j \in N$, and $\Gamma = \binom{k}{2}$.

Claim. *H contains a clique of size k if and only if the separation problem $SP_{\mathcal{P}}\colon \mathcal{U}_C$ for $G = (M \cup N, B)$ with the above defined parameters satisfies $\varsigma \leqslant k - n^2\binom{k}{2}$.*

Suppose that H has a clique $H' = (U', F')$ of size k, i.e., $|U'| = k$ and $|F'| = \binom{k}{2}$. Consider the solution (y, z) such that $z_i = 1$ for $i \in U'$ and $z_i = y_i = 1$ for $i \in F'$ and $y_i = 0$, $z_i = 0$, otherwise. Because this is a feasible solution for the separation problem with objective value $k - n^2\binom{k}{2}$, we have $\varsigma \leqslant k - n^2\binom{k}{2}$.

For the other direction, let (y, z) be a solution with objective value no more than $k - n^2\binom{k}{2}$. Then, $\sum_{i\in N} y_i = \Gamma$ holds, because otherwise $\sum_{i\in M} w_i z_i - \sum_{i\in N}(\bar{d}_i z_i + h_i y_i) \geqslant (1-\Gamma)n^2 > k - n^2\binom{k}{2}$. Consequently, it follows from the objective that $\sum_{i\in M} z_i \leqslant k$. Let $U' = \{i \in M\colon z_i = 1\}$ and $F' = \{i \in N\colon z_i = 1\}$. We have $|U'| \leqslant k$ and from the constraints $z_j \leqslant z_i$, $(ij) \in B$, $|F'| \leqslant \binom{k}{2}$. On the other hand, because $\sum_{i\in N} y_i = \Gamma$, it follows that $|F'| \geqslant \binom{k}{2}$, implying $|F'| = \binom{k}{2}$ and $|U'| = k$. Hence, H' is a clique of size k, as desired. □

Next, we show that the separation problem with respect to the cardinality-restricted uncertainty set $\mathcal{U}_C$ is a special case of the separation problem for the budget uncertainty set $\mathcal{U}_B$, which implies the $\mathcal{NP}$-hardness of the latter problem.

Corollary 6. *The separation problem $SP_{\mathcal{P}}$ is $\mathcal{NP}$-hard for $\mathcal{U}_B$ for bipartite $G(V, B)$.*

Proof. For convenience, letting $\tilde{d} = d - (\bar{d} - h)$ and $\tilde{\pi}_o = \pi_o - \pi(\bar{d} - h)$, we restate $\mathcal{U}_B$ as $\widetilde{\mathcal{U}}_B = \{\tilde{d} \in \mathbb{R}^V\colon \pi\tilde{d} \leqslant \tilde{\pi}_o,\ 0 \leqslant \tilde{d} \leqslant 2h\}$. Then, for the graph considered in the proof of Theorem 5, using $\tilde{y}_i = y_i/2h_i$, $i \in N$, the separation problem w.r.t. $\widetilde{\mathcal{U}}_B$ can be written as

$$\begin{aligned}
(\mathrm{SP}_{\mathcal{P}}\colon \widetilde{\mathcal{U}}_B) \quad \varsigma = \min \quad & \sum_{i\in M} w_i z_i - \sum_{i\in N} ((\bar{d}_i - h_i) z_i + 2h_i \tilde{y}_i)\\
\text{s.t.} \quad & \sum_{i\in N} (2h_i\pi_i)\tilde{y}_i \leqslant \tilde{\pi}_o, \quad \tilde{y}_i \leqslant z_i \quad \text{for all } i \in N,\\
& z_j \leqslant z_i \quad \text{for all } (ij) \in B,\\
& \tilde{y} \in \mathbb{R}^N_+, \quad z \in \{0,1\}^{M\cup N}.
\end{aligned}$$

Observe that $\mathrm{SP}_{\mathcal{P}}\colon \widetilde{\mathcal{U}}_B$ reduces to the case described in the proof of Theorem 5 when $w = \mathbf{1}$, $\tilde{\pi}_o = \Gamma$, $\bar{d}_i = h_i$, $h_i = n^2/2$, and $\pi_i = 1/(2h_i)$ for $i \in N$, because $\tilde{y}$ is integral for the extreme points. □

Corollary 7. *The separation problem $SP_{\mathcal{Q}}$ is $\mathcal{NP}$-hard for uncertainty sets $\mathcal{U}_B$ and $\mathcal{U}_C$.*

Controlling Conservatism. In two-stage robust optimization under demand uncertainty, the conservativeness of the robust solutions can be controlled by adjusting the parameters Γ and (π, π_o) of the respective uncertainty sets $\mathcal{U}_B$ and $\mathcal{U}_C$. For instance, in the case of $\mathcal{U}_B$, if $\pi = \mathbf{1}'$, the budget constraint simply becomes an upper bound π_o on the sum of individual demands. By letting $\pi_o(\alpha) = \sum_{i\in V}(\bar{d}_i + \alpha h_i)$ for $0 < \alpha < 1$, one avoids overly conservative solutions that assume the largest demand value $d_i = \bar{d}_i + h_i$ for each node $i \in V$.

Remark 3. The ability to control conservatism of two-stage robust solutions by a parameterized uncertainty set is a major advantage of the two-stage robust approach over its single-stage counterpart. For the choice of budget set $\mathcal{U}_B$ above, assuming that $\bar{d}_i + h_i \leqslant \bar{d}(V)$ for $i \in V$, the single-stage robust set $\mathcal{P}$ equals $\mathcal{P}_\zeta$ with $\zeta_i = \bar{d}_i + h_i = \max\{d_i\colon d \in \mathcal{U}_B(\alpha)\}$ for any $0 < \alpha < 1$. Similarly, in the case of $\mathcal{U}_C$, the single-stage robust set $\mathcal{P}$ remains unchanged as $\mathcal{P}_{\bar{d}+h}$ for all $\Gamma \geqslant 1$.

For the budget uncertainty set $\mathcal{U}_B$, the next theorem gives an upper bound on the probability of infeasibility for a robust solution $(x_A, y) \in \mathcal{Q}(A)$ if the demand is a bounded, symmetric, independent random vector.

Theorem 8. *Let d be a symmetric and independent random vector with mean $\bar{d}$ and support $[\bar{d}_i - h_i, \bar{d}_i + h_i]$*

$(h_i > 0)$ *for* $i \in V$. *If* $(x_A, y) \in \mathcal{Q}(A)$ *w.r.t.* $\mathcal{U}_B$ *with* $\pi\bar{d} < \pi_o$, *then*

$$\text{Prob}(\nexists x_B\colon (x_A, x_B, y) \in \mathcal{Q}_d) \leqslant \exp\left(-\frac{(\pi_o - \pi\bar{d})^2}{2\|\pi \circ h\|^2}\right).$$

PROOF. Let $\xi_i = (d_i - \bar{d}_i)/h_i$ for $i \in V$; thus, $\xi_i \in [-1, +1]$ with mean zero. Then,

$$\text{Prob}(\nexists x_B\colon (x_A, x_B, y) \in \mathcal{Q}_d) \leqslant \text{Prob}(d \notin \mathcal{U}_B) = \text{Prob}\left(\sum_{i\in V}(\pi_i h_i)\xi_i > \pi_o - \pi\bar{d}\right).$$

Bounding this probability using Markov's inequality is standard (e.g., Ben-Tal and Nemirovski 2000, Bertsimas and Sim 2004). For $\gamma > 0$,

$$\text{Prob}\left(\sum_{i\in V}(\pi_i h_i)\xi_i > \pi_o - \pi\bar{d}\right)$$

$$\leqslant \frac{\mathrm{E}[\exp(\gamma\sum_{i\in V}(\pi_i h_i)\xi_i)]}{\exp(\gamma(\pi_o - \pi\bar{d}))} \quad \text{(Markov's inequality)}$$

$$= \frac{\prod_{i\in V} 2\int_0^1 \sum_{k=0}^{\infty}((\gamma\pi_i h_i t)^{2k}/(2k)!)\,dF_{\xi_i}(t)}{\exp(\gamma(\pi_o - \pi\bar{d}))} \quad \text{(indep., sym.)}$$

$$= \frac{\prod_{i\in V}\sum_{k=0}^{\infty}((\gamma\pi_i h_i)^{2k}/(2k)!)(2\int_0^1 t^{2k}\,dF_{\xi_i}(t))}{\exp(\gamma(\pi_o - \pi\bar{d}))}$$

$$\leqslant \frac{\prod_{i\in V}\sum_{k=0}^{\infty}(\gamma\pi_i h_i)^{2k}/(2k)!}{\exp(\gamma(\pi_o - \pi\bar{d}))} \quad \text{(symmetric extremal dist.)}$$

$$\leqslant \frac{\prod_{i\in V}\exp(\gamma^2\pi_i^2 h_i^2/2)}{\exp(\gamma(\pi_o - \pi\bar{d}))} = \exp\left(\frac{\gamma^2}{2}\|\pi \circ h\|^2 - \gamma(\pi_o - \pi\bar{d})\right),$$

where the exponent is a convex function in γ, minimized at $\gamma^* = (\pi_o - \pi\bar{d})/\|\pi \circ h\|^2 > 0$. Evaluating the upper bound at $\gamma = \gamma^*$ gives the result. $\square$

REMARK 4. For $\pi_o = \pi(\bar{d} + \alpha h)$ with $0 < \alpha \leqslant 1$, we have

$$\exp\left(-\frac{(\pi_o - \pi\bar{d})^2}{2\|\pi \circ h\|^2}\right) = \exp\left(-\frac{\alpha^2(\sum_{i\in V}\pi_i h_i)^2}{2(\sum_{i\in V}\pi_i^2 h_i^2)}\right).$$

As an example, by letting $\pi_i = 1/h_i$ for $h_i > 0$ and $n = |\{i \in V\colon h_i > 0\}|$, we obtain

$$\text{Prob}(\nexists x_B\colon (x_A, x_B, y) \in \mathcal{Q}_d) \leqslant \exp\left(-\frac{\alpha^2}{2}n\right).$$

The upper bound on infeasibility probability improves rapidly as the number of nodes with uncertain demand increases. For example, if $\alpha = 0.5$, the probability of infeasibility of a robust solution is at most 0.1353 for $n = 16$, 0.0183 for $n = 32$, and 0.0003 for $n = 64$.

REMARK 5. Note that the probability bound given above is independent of the first-stage arcs A. In particular, for $A = \varnothing$, Theorem 8 gives an upper bound on the probability that a network with arc capacities $u_{ij}y_{ij}$, $(ij) \in E$, satisfying $y \in \mathcal{Q}(\varnothing)$ will not meet a random demand vector as defined above.

5. Polynomial Special Cases

Although the separation complexity for $\mathcal{P}(A)$ and $\mathcal{Q}(A)$ is $\mathcal{NP}$-hard in general, there are interesting special cases that are computationally tractable. In this section, we explore such cases by considering special network topologies.

5.1. Totally Ordered Graphs

An acyclic graph is *totally ordered* if its node set V can be labeled as $1, \ldots, |V|$ such that for any $i < j$, there exists a directed path from i to j. If the graph $G(V, B)$—induced by the arcs corresponding to the second-stage decisions x_B—is totally ordered, then $G(V, E)$ has $|V| + 1$ weak sets. This can be seen by observing that for any weak subset S, if $i \in S$, then $k \in S$ for $k \leqslant i$. Therefore, the weak sets are the $|V|$ nested sets $S_i = \{1, 2, \ldots i\}$ for $i \in V$ and $\varnothing$. Therefore, the robust set $\mathcal{P}(A)$ is tractable in this case if $\mathcal{U}$ is tractable.

5.2. Arborescences

Another interesting case arises when $G(V, B)$ is an arborescence. In this case, the number of weak sets of $G(V, E)$ is exponential in $|V|$; nevertheless, the separation problem $\text{SP}_{\mathcal{P}}$ can be solved in polynomial time by dynamic programming for the cardinality-restricted uncertainty set $\mathcal{U}_C$.

THEOREM 9. *The separation problem* $\text{SP}_{\mathcal{P}}$ *can be solved in* $O(|V|\binom{\Gamma+2}{2})$ *for the cardinality-restricted uncertainty set* $\mathcal{U}_C$.

PROOF. The proof is by a simple modification of the dynamic program given in Theorem 3 of Faigle and Kern (1994) for cardinality-constrained optimization of ideals on an arborescence, whose complexity bound is improved here by a careful calculation.

Let $v_1, v_2, \ldots, v_{|V|}$ be the nodes of the arborescence $G(V, B)$, v_1 being the root node. Let T_k denote the subtree rooted at node v_k; therefore, $T_1 = G(V, B)$. Define $w(\gamma, H)$ as the optimal value of $\text{SP}_{\mathcal{P}}$ with graph H and cardinality restriction $\Gamma = \gamma$. Then, the optimal value of the separation problem with graph $G(V, B)$ and cardinality restriction Γ is $\omega(\Gamma, T_1)$, which is computed as follows.

Consider the children $v_{k_1}, \ldots, v_{k_t}$ of some node v_k, and for $1 \leqslant i \leqslant t$ and $0 \leqslant \gamma \leqslant \Gamma$, define

$$\phi_k(\gamma, i) := \min\left\{\sum_{j=1}^{i}\omega(\gamma_j, T'_{k_j})\colon \sum_{j=1}^{i}\gamma_j \leqslant \gamma,\ T'_{k_j} \text{ is subtree of } T_{k_j} \text{ rooted at } v_{k_j}\right\}.$$

Then, it follows that

$$\phi_k(\gamma, i) = \begin{cases} \omega(\gamma, T_{k_1}), & i = 1, \\ \min\{\phi_k(\gamma', i-1) + \omega(\gamma - \gamma', T_{k_i})\colon 0 \leqslant \gamma' \leqslant \gamma\}, & 2 \leqslant i \leqslant \gamma, \end{cases}$$

and $\omega(\gamma, T_k) = w_k - \bar{d}_k + \min\{-h_k + \phi_k(\gamma - 1, t), \phi_k(\gamma, t)\}$. Thus, given $\omega(\gamma, T_{k_i})$ for the children nodes v_{k_i} of v_k, we can compute $\omega(\gamma, T_k)$ by a simple recursion.

Starting from the leaf nodes, ω and ϕ are computed recursively up to the root node v_1. Given $\phi_k(\gamma', i-1)$ and $\omega(\gamma - \gamma', T_{k_i})$ for $0 \leqslant \gamma' \leqslant \gamma$, computing $\phi_k(\gamma, i)$ takes $\gamma + 1$ steps; thus, $\omega_k(\gamma, T_0)$ is computed in $t(\gamma + 1)$ steps, where t is the number of children of v_k. Hence, all $\omega_k(\gamma, T_0)$ for $0 \leqslant \gamma \leqslant \Gamma$ are computed in $t\binom{\Gamma+2}{2}$ steps. Summing over all nodes in V, we see that the overall complexity is $O\big(|V|\binom{\Gamma+2}{2}\big)$. $\square$

5.3. Lot-Sizing Problems

Consider the lot-sizing problem of an item subject to uncertain demand $d \in \mathcal{U}$ over a finite discrete horizon of n periods. For $i = 1, \ldots, n$, let x_i denote the amount of production/order in period i, and I_i the inventory at the end of the period. We let the production/order variables be the first-stage decisions that are to be fixed before the demand is observed. Therefore, capacities and fixed charges on them can be readily incorporated. On the other hand, inventory levels will be a function of the demand realization in the second stage.

5.3.1. Unbounded Inventories. First, we discuss the situation in which there is no inventory storage capacity. Because $G(V, B)$—induced by the inventory arcs—is a simple directed path, it is a totally ordered graph. Then, from §5.1, there are n constraints of the form (1) given by the n nested subsets (Figure 2). Therefore, the robust production set is tractable for tractable $\mathcal{U}$. Note that the n inequalities (1) have the same structure as the constraints of the aggregate production formulation for the nominal lot-sizing problem. Therefore, the robust lot-sizing problem can be solved as a nominal lot-sizing problem by simply adjusting the demands. This result was shown earlier by Bertsimas and Thiele (2006).

5.3.2. Bounded Inventories. Next, we consider the lot-sizing problem with storage capacities. In this case, there are exponentially many sets to consider for defining $\mathcal{Q}(A)$. However, a closer examination of the inequalities shows that only $\binom{n+1}{2}$ of inequalities (2) are nondominated. The other inequalities are implied by these (Figure 3).

Figure 2. Weak cuts for lot sizing with unbounded inventory.

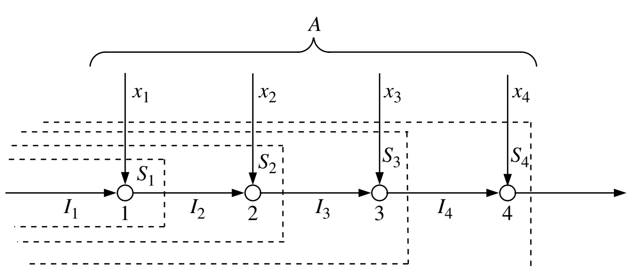
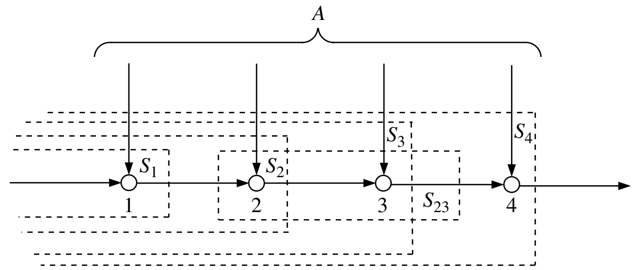


Figure 3. Cuts for the lot-sizing problem with bounded inventory.



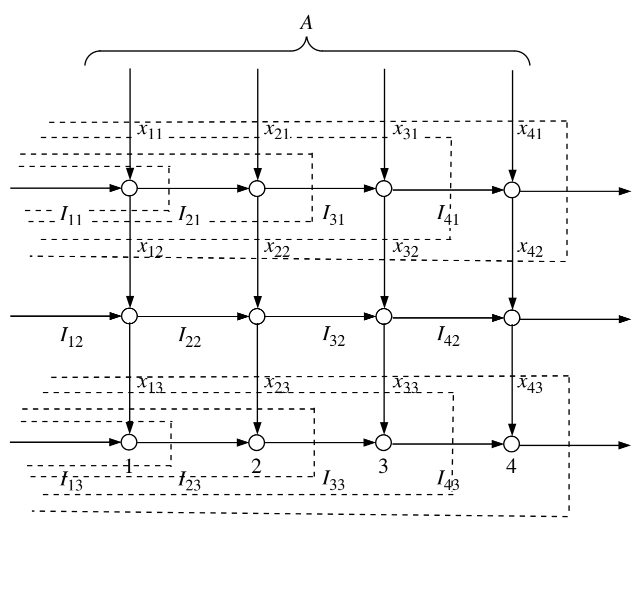
Theorem 10. *For the robust lot-sizing problem with inventory capacities, $\mathcal{Q}(A)$ is described by $\binom{n+1}{2}$ inequalities* (2) *defined by the interval sets $S_{ij} = \{i, i+1, \ldots, j\}$ for $1 \leqslant i \leqslant j \leqslant n$.*

Theorem 10 is stated and proved more generally in the next subsection.

5.3.3. Multilevel Lot Sizing. The results in previous subsections can be extended to multilevel lot-sizing problems. Consider an r-level lot-sizing problem with n time periods, as depicted in Figure 4. Let the production/order decisions x_{ik}, $1 \leqslant i \leqslant n$, $1 \leqslant k \leqslant r$, be the first-stage decisions that need to be fixed before the demand at the final level r is observed. The inventory decisions I_{ik} are a function of the demand realization. For the multilevel lot-sizing problem, in the unbounded inventory case, $\mathcal{P}(A)$ is described by rn inequalities (1), whereas, in the bounded inventory case, $\mathcal{Q}(A)$ is described by $r\binom{n+1}{2}$ inequalities (2). We show the result for the latter general case below.

Theorem 11. *For the robust multilevel lot-sizing problem with inventory capacities, $\mathcal{Q}(A)$ is described by $r\binom{n+1}{2}$ inequalities* (2) *defined by the interval sets $S^k_{ij} = \{i, i+1, \ldots, j\}$ for $1 \leqslant i \leqslant j \leqslant n$ and $1 \leqslant k \leqslant r$.*

Figure 4. Weak cuts for the multilevel lot-sizing problem with unbounded inventory.



Proof. Let V^k denote the level k $(k=1,2,\ldots,r)$ nodes of the multilevel lot-sizing problem, and for $S\subseteq V$ let $S^k = S\cap V^k$; therefore, $S=\bigcup_{k=1}^r S^k$. First, we show that inequality (2) for S is dominated by inequalities (2) for S^k, $1\leqslant k\leqslant r$. For notational simplicity, let $S^0=S^{r+1}=\varnothing$ and let u_{ik} denote the upper bound on the inventory variable I_{ik}. Inequality (2) for $S\subseteq V$ can be written as

$$\sum_{k=1}^{r}\left(\sum_{i\in S^k} x_{ik}-\sum_{i\in S^k} x_{i(k+1)}+\sum_{i\notin S^k,\, i+1\in S^k} u_{ik}\right)\geqslant \zeta_S. \tag{8}$$

Because $\sum_{k=1}^r \zeta_{S^k}\geqslant \zeta_S$, inequalities

$$\sum_{i\in S^k} x_{ik}-\sum_{i\in S^k} x_{i(k+1)}+\sum_{i\notin S^k,\, i+1\in S^k} u_{ik}\geqslant \zeta_{S^k},\quad k=1,2,\ldots,r, \tag{9}$$

collectively dominate (8).

If S^k is not an interval set as in the statement of the theorem, then there exists $s\in V^k\backslash S^k$ such that $S_-^k=\{i\in S^k\colon i<s\}$ and $S_+^k=S^k\backslash S_-^k$ are nonempty. In this case, because inequality (9) can be written as

$$\left(\sum_{i\in S_-^k} x_{ik}-\sum_{i\in S_-^k} x_{i(k+1)}+\sum_{i\notin S_-^k,\, i+1\in S_-^k} u_{ik}\right) + \left(\sum_{i\in S_+^k} x_{ik}-\sum_{i\in S_+^k} x_{i(k+1)}+\sum_{i\notin S_+^k,\, i+1\in S_+^k} u_{ik}\right)\geqslant \zeta_{S^k}$$

and $\zeta_{S_-^k}+\zeta_{S_+^k}\geqslant \zeta_{S^k}$, it is dominated by inequalities (2) for S_-^k and S_+^k. Hence, it suffices to consider only the $r\binom{n+1}{2}$ inequalities (2) defined by the interval sets $S_{ij}^k=\{i,i+1,\ldots,j\}$ for $1\leqslant i\leqslant j\leqslant n$, $1\leqslant k\leqslant r$, for describing $\mathscr{Q}(A)$. □

6. Optimization

In this section, we discuss the two-stage robust optimization problem. We are interested in minimizing the worst objective (absolute robustness; see Kouvelis and Yu 1997) in two stages. Therefore, given the capacity cost $f\geqslant 0$ and flow cost $c\geqslant 0$, the relevant optimization problem to solve is

$$\text{(ARO)}\quad \varrho=\min_{x_A,y}\Big\{c_A x_A+fy + \Big\{\max_{d\in\mathscr{U}}\Big\{\min_{x_B} c_B x_B\colon (x,y)\in Q_d\Big\}\Big\}\Big\}.$$

In principle, the second-stage objective component $c_B x_B$ can be brought into the constraints by introducing an auxiliary variable and then x_B can be projected out from the formulation as before. However, doing so destroys the network structure of the second-stage constraints because the extremal dual values used in the projection do not correspond to the cuts of the graph as in the case $c_B=0$ (Theorem 4). Therefore, to maintain the network structure, as an alternative, we propose to minimize an upper bound by introducing auxiliary variables $z_B\in\mathbb{R}_+^B$ such that $x_B\leqslant z_B\leqslant u_B\circ y_B$ and replacing the second-stage objective $c_B x_B$ with $c_B z_B$. Projecting out x_B as in Theorem 4, by treating z_B as the upper bound on x_B, we obtain

$$\begin{aligned}\text{(ROB)}\quad \hat{\varrho}=\min\ & c_A x_A+c_B z_B+fy\\ \text{s.t.}\ & x_A(\delta_A^+(S))+z_B(\delta_B^+(S))-x_A(\delta_A^-(S))\geqslant \zeta_S \quad \text{for all } S\subseteq V,\\ & x_A\leqslant u_A\circ y_A,\quad z_B\leqslant u_B\circ y_B,\\ & x\in\mathbb{R}_+^A,\quad z\in\mathbb{R}_+^B,\quad y\in\mathbb{Z}_+^E.\end{aligned}$$

Because $c_B\geqslant 0$ and $x_B\leqslant z_B$ for all $d\in\mathscr{U}$, we have $\varrho\leqslant\hat{\varrho}$. Unless $c_B=0$ or $\mathscr{P}(A)=\text{Proj}_A\mathscr{P}$, there is typically a gap between ϱ and $\hat{\varrho}$. However, simulation results in §8 on a robust location-transportation problem, where demand is generated randomly according to uniform distribution, show that ROB is a reasonable alternative to ARO for finding robust solutions. Interestingly, ROB also offers an alternative to two-stage stochastic programming as it may be easier to solve than a large-scale stochastic programming formulation with many scenarios. The conservativeness of the robust solutions of ROB can be controlled by enlarging or shrinking the demand uncertainty set $\mathscr{U}$ as discussed in §4. We elaborate on this issue in §8.

7. Multicommodity Network Design

In this section, we generalize the earlier results to multicommodity network flow and design. Given a set K of commodities with demand $d^k\in\mathbb{R}^V$, $k\in K$, the feasible set of solutions for the multicommodity network design problem can be written as

$$\mathscr{R}_d:=\Big\{(x,y)\in\mathbb{R}_+^{E\times K}\times\mathbb{Z}_+^E\colon x^k(\delta^+(i))-x^k(\delta^-(i))\geqslant d_i^k \quad \text{for } i\in V, k\in K;\ \sum_{k\in K} x^k\leqslant u\circ y\Big\},$$

using x^k for the flow of commodity $k\in K$ and y for the joint capacity decision.

Now let $\mathscr{U}^k\subset\mathbb{R}^V$ be a compact demand uncertainty set for commodity $k\in K$ and $\mathscr{U}=\prod_{k\in K}\mathscr{U}^k$. Then, the single-stage robust solution set is $\mathscr{R}=\bigcap_{d\in\mathscr{U}}\mathscr{R}_d$. For the two-stage approach with multiple commodities, it is reasonable to define stages in two different ways. The first scheme follows the previous sections, where the arcs of the network are partitioned into first and second stages. In the second scheme, the commodities with known and unknown demand define the stages.

7.1. Arc-Based Stages

The arc-based partitioning of the stages is appropriate for situations in which the flow of commodities on a subset of the arcs can be decided after the realization of the demands. For instance, in production planning, once the tactical decisions on production levels of families of products are determined, inventories are a function of the observed demand.

In transportation applications, once decisions for long-haul transportation among the hubs of a network are made subject to uncertain demand, local deliveries from the hubs can be decided after observing the demands.

Let (A, B) be a partitioning of the arcs E as before. For a commodity $k \in K$, flow variables x^k_{ij}, $(ij) \in A$, are the first-stage variables and x^k_{ij}, $(ij) \in B$, are the second-stage variables. Introducing auxiliary variables z^k_B, $k \in K$, and rewriting $\mathcal{R}_d$ as

$$x^k_B(\delta^+_B(i)) - x^k_B(\delta^-_B(i)) \geqslant d^k_i - x^k_A(\delta^+_A(i)) + x^k_A(\delta^-_A(i)) \quad \text{for all } i \in V, k \in K, \tag{10}$$

$$\mathbf{0} \leqslant x^k_B \leqslant z^k_B \quad \text{for all } k \in K, \tag{11}$$

$$\mathbf{0} \leqslant \sum_{k\in K} x^k_A \leqslant u_A \circ y_A, \quad \mathbf{0} \leqslant \sum_{k\in K} z^k_B \leqslant u_B \circ y_B, \tag{12}$$

allows us to project out x^k_B for each commodity $k \in K$ as in Theorem 4. This leads to the arc-based two-stage robust multicommodity network design problem:

$$\begin{aligned}
\text{(ROB-A)} \quad \min \quad & \sum_{k\in K} (c^k_A x^k_A + c^k_B z^k_B) + fy \\
\text{s.t.} \quad & x^k_A(\delta^+_A(S)) + z^k_B(\delta^+_B(S)) - x^k_A(\delta^-_A(S)) \geqslant \zeta^k_S \\
& \qquad \text{for all } S \subseteq V, k \in K, \\
& \sum_{k\in K} x^k_A \leqslant u_A \circ y_A, \quad \sum_{k\in K} z^k_B \leqslant u_B \circ y_B, \\
& x \in \mathbb{R}^{A\times K}_+, \quad z \in \mathbb{R}^{B\times K}_+, \quad y \in \mathbb{Z}^E_+,
\end{aligned}$$

where $\zeta^k_S = \max\{d^k(S)\colon d \in \mathcal{U}\}$ for $k \in K$. Because the structure of the constraints of ROB-A is the same as ROB for each commodity, the corresponding separation problems have the same properties and computational complexity as in the single-commodity case.

7.2. Commodity-Based Stages

In this scheme, a commodity partitioning defines the stages. For a partitioning (K_1, K_2) of the commodity set K, we let x^k_{ij}, $(ij) \in E$, $k \in K_1$, be the first-stage decisions, whereas x^k_{ij}, $(ij) \in E$, $k \in K_2$, are the second-stage decisions. Defining stages using a commodity partitioning is appropriate for situations in which commodities with known and unknown demand share a joint capacity, and capacity and flow decisions for commodities with known demand are made in the first stage, whereas the others are deferred until their demand becomes known. A typical application arises in transportation of commodities with high- and low-priority classes. Flow decisions of low-priority commodities with uncertain demand can be deferred, whereas high-priority commodities need to be routed in advance; see, for example, Bertsimas and Simchi-Levi (1996).

Letting $A = \varnothing$, $B = E$ in inequalities (10)–(12) and projecting out only x^k_E for $k \in K_2$ gives the commodity-based two-stage multicommodity robust network design problem:

$$\begin{aligned}
\text{(ROB-K)} \quad \min \quad & \sum_{k\in K_1} c^k x^k + \sum_{k\in K_2} c^k z^k + fy \\
\text{s.t.} \quad & x^k(\delta^+(i)) - x^k(\delta^-(i)) \geqslant d^k_i \\
& \qquad \text{for all } i \in V, k \in K_1, \\
& z^k(\delta^+(S)) \geqslant \zeta^k_S \quad \text{for all } S \subseteq V, k \in K_2, \\
& \sum_{k\in K_1} x^k + \sum_{k\in K_2} z^k \leqslant u \circ y, \\
& x \in \mathbb{R}^{E\times K_1}_+, \quad z \in \mathbb{R}^{E\times K_2}_+, \quad y \in \mathbb{Z}^E_+.
\end{aligned}$$

Naturally, ROB-A and ROB-K can be generalized to accommodate situations in which a combination of both arc-based as well as commodity-based stages is desired.

8. Application and Computations: Robust Location-Transportation

In this section, we apply the developed two-stage robust optimization framework to a location-transportation problem with uncertain demand and present a summary of computational experiments. The purpose of the experiments is threefold:

(1) to understand the computational difficulty of solving ROB with a cutting-plane method that solves the separation problem $\text{SP}_{\mathcal{Q}}$ (the proof of Theorem 5 with $\bar{d}_i = h_i = 0$ for $i \in M$ implies that $\text{SP}_{\mathcal{Q}}$ remains $\mathcal{NP}$-hard for the location-transportation problem);

(2) to understand the effect of adjusting the demand uncertainty set on solution quality and time; and

(3) to compare the solution quality and time of two-stage robust optimization with those of single-stage robust optimization and two-stage stochastic programming.

Let us begin by describing the location-transportation application. Let M be a set of potential facilities to serve a set N of customers. The demand for each customer is uncertain and lies within an interval $[\bar{d}_j - h_j, \bar{d}_j + h_j]$ for $j \in N$. The first-stage decisions that need to be made before observing the demands are the facilities to open (y) and their supply levels (w). Transportation decisions (x) are made after observing the demand in the second stage. Let the facility capacities be c_i, $i \in M$, and transportation capacities be u_{ij}, $(ij) \in B$, where B is the set of transportation arcs from the candidate facilities to the customers. The fixed and variable costs of supplying from facility $i \in M$ are denoted by f_i and b_i, respectively, and the variable transportation cost from facility $i \in M$ to customer $j \in N$ is denoted by t_{ij}. For a given demand $d \in \mathbb{R}^N$, the *nominal location-transportation problem* is formulated as

$$\begin{aligned}
\min \quad & bw + fy + tx \\
\text{s.t.} \quad & x(\delta^+(j)) \geqslant d_j, \quad j \in N, \\
& w_i - x(\delta^-(i)) \geqslant 0, \quad i \in M, \\
& w \leqslant c \circ y, \quad x \leqslant u, \\
& w \in \mathbb{R}^M_+, \quad x \in \mathbb{R}^B_+, \quad y \in \{0,1\}^M.
\end{aligned}$$

Table 1. Comparison of the two-stage robust, one-stage robust, and stochastic solutions.

	Γ	$\hat{\varrho}$	CPU (sec.)	Constr. (13)	Exp. obj. val.	Max. obj. val.
Stochastic prog.	—	—	508	—	1,795	2,869
Two-stage robust	2	2,090	275	177	1,815	2,033
Two-stage robust	4	2,181	133	132	1,816	2,033
Two-stage robust	6	2,258	94	131	1,889	2,049
Two-stage robust	8	2,312	59	120	1,920	2,062
Two-stage robust	10	2,350	49	122	1,969	2,078
Two-stage robust	15	2,408	25	103	1,965	2,070
Two-stage robust	20	2,418	11	80	2,042	2,144
Two-stage robust	25	2,419	5	53	2,049	2,152
One-stage robust	30	2,419	0	—	2,054	2,152

Then, the corresponding *two-stage robust counterpart* (ROB) for the location-transportation problem can be written as

$$
\begin{aligned}
\hat{\varrho} = \min\ & bw + fy + tz \\
\text{s.t.}\ & w(\delta^+(S)) + z(M\setminus S; T) \geqslant \max_{d\in\mathcal{U}} d(T), \\
& \qquad\qquad S\subseteq M, T\subseteq N, \qquad (13) \\
& w \leqslant c\circ y, \quad z \leqslant u, \\
& w\in\mathbb{R}^M_+, \quad z\in\mathbb{R}^B_+, \quad y\in\{0,1\}^M,
\end{aligned}
$$

where $z(M\setminus S; T)$ denotes $\sum_{i\in M\setminus S, j\in T} z_{ij}$.

In Table 1, we summarize the results of the computational experiments for location-transportation instances with $|M|=20$, $|N|=30$, $\bar{d}_j\in[0,20]$, and $h_j\in[0,\bar{d}_j]$ for $j\in N$. For two-stage robust optimization, we use the cardinality uncertainty set $\mathcal{U}_C$ parameterized with Γ. The first three columns of the table show that as Γ increases, $\mathcal{U}_C$ is relaxed, and consequently the objective $\hat{\varrho}$ increases monotonically. When $\Gamma=30$, we arrive at the case in which demand at the nodes are independent, that is, $\mathcal{U}_C=\{d\in\mathbb{R}^N\colon \bar{d}-h\leqslant d\leqslant \bar{d}+h\}$. Recall from Proposition 3 that, in this case, the two-stage robust problem is equivalent to the single-stage robust problem with $d_j=\zeta_j=\bar{d}_j+h_j$ for $j\in N$. Also by Remark 3, the single-stage solutions are the same for all $\Gamma\geqslant 1$ for $\mathcal{U}_C$. In Column 4 of the table, we see that the solution time for two-stage robust optimization is negatively correlated with Γ, which is partially explained in Column 5 by the number of constraints of the form (13) added during the computations.

To compare two-stage robust optimization with two-stage stochastic optimization, we sampled 200 scenarios from independent and uniformly distributed demand $d_j\sim U(\bar{d}_j-h_j, \bar{d}_j+h_j)$, $j\in N$. The objective of the stochastic program is to minimize the sum of the first-stage cost and the *expected* second-stage cost using the discrete approximation of the uniform distribution for the demand.

The first row of Table 1 summarizes the results for the stochastic programming approach. First, note that the solution times for robust optimization compare favorably with stochastic programming. In Column 6 of the table, we report the expected objective value for the robust solution for the same distribution. In other words, for each Γ, this column is the sum of the first-stage robust solution objective and the expectation of the second-stage cost, given the robust solution in the first stage. Interestingly, for small values of Γ, the expected objective value realized by the robust solution is very close to the minimum obtained by the stochastic programming approach. Even though robust optimization does not aim to minimize the expected cost, adjusting the uncertainty set $\mathcal{U}_C$ appropriately to avoid conservative solutions allowed us to obtain reasonable solutions with respect to the expectation criterion as well.

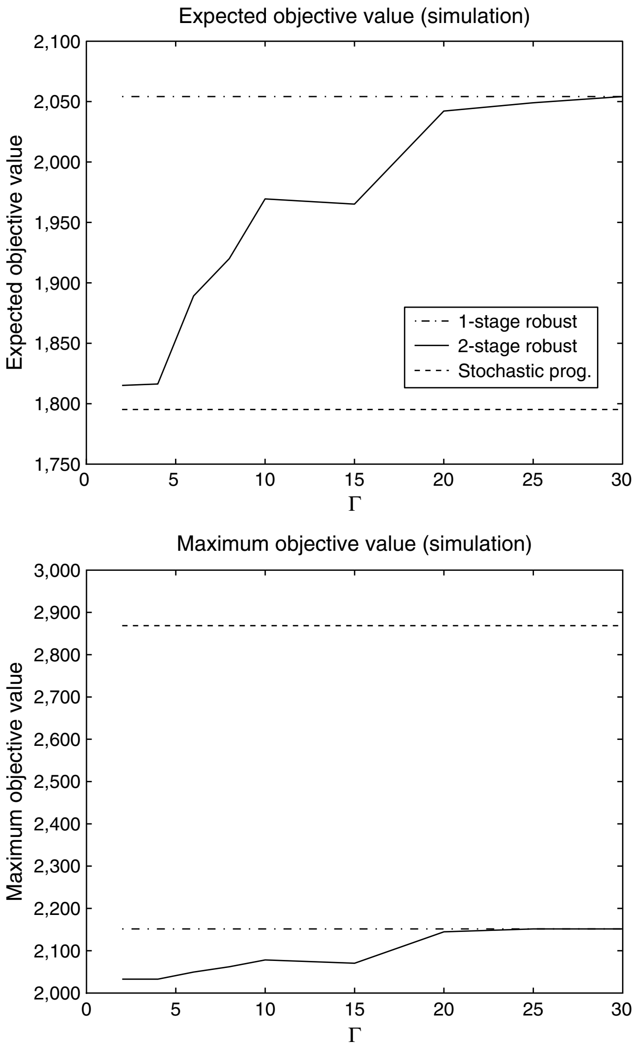
In Column 7 of the table, we report the maximum (worst) objective value realized by the same scenarios with the stochastic programming solution, the two-stage robust solutions, and the single-stage robust solution. Not surprisingly, the stochastic programming first-stage solution, which is a minimizer of the expected cost, leads to very costly overall solutions in some scenarios. For the robust solutions, we observe the effect of adjusting the uncertainty set in the realized maximum objective values. As Γ decreases, the realized maximum objective decreases, but the likelihood of some scenarios being infeasible increases (we did not come across any infeasible scenario even with $\Gamma=2$ in our experiments).

The expectation and maximum objective values realized by the demand scenarios for the two-stage robust, single-stage robust, and stochastic programming solutions are plotted in Figure 5 for ease of comparison. The plot for the robust maximum objective may not be monotonically increasing as ROB minimizes an upper bound ($\hat{\varrho}$) on the maximum objective. Nevertheless, the charts indicate that two-stage robust optimization offers an interesting trade-off between the scenario-based stochastic programming and the single-stage robust optimization when compared with maximum as well as expected objective criteria.

9. Conclusion

We describe a two-stage robust optimization approach for network flow and network design problems with demand uncertainty. The proposed methodology is applicable to a host of practical telecommunication, location, production, and distribution network design problems, in which design

Figure 5. Comparison of the two-stage robust, one-stage robust, and stochastic solutions.



decisions must be made before observing the demand and some of the flow-routing decisions can be deferred until after the demand is observed.

We gave an explicit description of the first-stage decisions which may involve, in addition to design variables, a subset of the flow variables. We studied the complexity of the corresponding separation problems and identified interesting tractable cases. We also generalized the approach to multicommodity flows.

Unlike single-stage robust optimization, the two-stage robust optimization approach allows one to control the conservatism of the solutions through a parameterized budget uncertainty set for the aggregate demand. We showed by using an allowed budget for uncertain demand that it is possible to give an upper bound on the probability of infeasibility of the robust solution for a random demand vector.

Our computational experiments for a location-transportation problem indicate that the proposed two-stage robust optimization approach offers an interesting trade-off between scenario-based stochastic programming and the more conservative single-stage robust optimization. The robust optimization approach does not suffer from the requirement for generating a large number of demand scenarios; therefore, the corresponding problems are solved relatively fast in practice (even though the subproblem is $\mathcal{NP}$-hard). On the other hand, the two-stage robust optimization approach allows one to come up with solutions that are not as conservative as the ones from the single-stage robust optimization approach for demand uncertainty.

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