Network Design Arc Set with Variable Upper Bounds

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In this paper we study the network design arc set with variable upper bounds. This set appears as a common substructure of many network design problems and is a relaxation of several fundamental mixed-integer sets studied earlier independently. In particular, the splittable flow arc set, the unsplittable flow arc set, the single node fixed-charge flow set, and the binary knapsack set are facial restrictions of the network design arc set with variable upper bounds. Here we describe families of strong valid inequalities that cut off all fractional extreme points of the continuous relaxation of the network design arc set with variable upper bounds. Interestingly, some of these inequalities are also new even for the aforementioned restrictions studied earlier. © 2007 Wiley Periodicals, Inc. NETWORKS, Vol. 50(1), 17–28 2007

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1. INTRODUCTION

We study the *network design arc set with variable upper bounds* defined as

$$\mathcal{P} = \left\{ x \in \mathbb{R}^N_+,\ y \in \mathbb{Z}_+,\ z \in \{0,1\}^N \ : \ \sum_{i \in N} a_i x_i \le a_0 + y,\ x \le z \right\},$$

where $a_i > 0$ for $i \in N$ and $a_0 \ge 0$. This set appears as a common substructure of many network design problems.

For a multicommodity network design problem with either fixed charges or combinatorial restrictions, x_i denotes the fraction of commodity i with demand a_i flowing along an arc with capacity $a_0 + y$. The binary variables z_i's are used for modeling combinatorial restrictions on the paths, such as cardinality restrictions, disjointness, etc., as well as applicable fixed charges. Alternatively, this model arises also when $a_0 + y$ is used to model a hub capacity with flow and fixed-charge variables (x_i, z_i) for each incoming arc $i \in N$ into the hub. We will refer to the following inequality as the *capacity constraint*:

$$\sum_{i \in N} a_i x_i \ \le \ a_0 + y. \tag{1}$$

An interesting feature of the set $\mathcal{P}$ is that it is a common relaxation of four fundamental sets that received significant attention in the literature. As such, $\mathcal{P}$ links these four sets that have been studied independently from each other. The first set is the *splittable flow arc set* [18]

$$\mathcal{Q} = \left\{ x \in \mathbb{R}^N_+,\ y \in \mathbb{Z}_+ \ : \ \sum_{i \in N} a_i x_i \ \le \ a_0 + y,\ x \ \le \ \mathbf{1} \right\},$$

which is obtained from $\mathcal{P}$ by restricting $z = \mathbf{1}$. The second relevant set is the *unsplittable flow arc set* [10]

$$\mathcal{R} = \left\{ y \in \mathbb{Z}_+,\ z \in \{0,1\}^N \ : \ \sum_{i \in N} a_i z_i \ \le \ a_0 + y \right\},$$

which is obtained from $\mathcal{P}$ by restricting $x = z$. The third set of interest is the *single node fixed-charge flow set* [22]

$$\mathcal{T} = \left\{ x \in \mathbb{R}^N_+,\ z \in \{0,1\}^N \ : \ \sum_{i \in N} a_i x_i \ \le \ a_0,\ x \ \le \ z \right\},$$

which is obtained from $\mathcal{P}$ by restricting $y = 0$. Finally, the fourth set is the *binary knapsack set* [7, 15, 25]

$$\mathcal{K} = \left\{ z \in \{0,1\}^N \ : \ \sum_{i \in N} a_i z_i \ \le \ a_0 \right\},$$

which is obtained from $\mathcal{P}$ by restricting $x = z$ and $y = 0$.

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The set $\mathcal{Q}$ is the simplest one among these four sets and an explicit linear description of its convex hull description is known (see Section 1.1). Optimization over the other sets is $\mathcal{NP}$-hard and only partial descriptions of the corresponding convex hulls are known.

Note that the convex hulls of $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{K}$ are faces of the convex hull of $\mathcal{P}$. Thus $\mathcal{P}$ has the characteristics of all these four sets and one can obtain strong inequalities for them from $\mathcal{P}$. We shall observe in the later sections that the seemingly unrelated inequalities given independently for $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{T}$, and $\mathcal{K}$ are just special cases of the valid inequalities for $\mathcal{P}$ when they are restricted to the appropriate faces of the convex hull of $\mathcal{P}$.

In the remainder of this section, we review some of the basic results known for the related sets $\mathcal{Q}, \mathcal{R}, \mathcal{T}$, and $\mathcal{K}$ so that we can show the connections between the inequalities for $\mathcal{P}$ and those known for the others. In Section 2, we describe basic polyhedral properties of $\mathcal{P}$. In Section 3, we give generalizations of the flow cover inequalities for $\mathcal{P}$ and discuss their strength as well as the fractional solutions cut off by them. In Section 4, we describe strong valid inequalities obtained through two consecutive applications of the mixed-integer rounding procedure [20]. It turns out that these inequalities are sufficient to cut off all fractional extreme points of the continuous relaxation of $\mathcal{P}$. Interestingly, some of the strong inequalities obtained for $\mathcal{P}$ are also new even for the aforementioned restrictions studied earlier.

Throughout, the convex hull and the continuous relaxation of a set are denoted by $\text{conv}(\cdot)$ and $\text{relax}(\cdot)$, respectively. For $v \in \mathbb{R}^N$, we define $v(S) = \sum_{i \in S} v_i$ for $S \subseteq N$. For $a \in \mathbb{R}$, we use $(a)^+$ to denote $\max\{a, 0\}$. We let $\hat{a} = (\lceil a(N) - a_0 \rceil)^+$ and $n = |N|$. We use e_i to denote the ith unit vector, $\mathbf{0}$ and $\mathbf{1}$ to denote a vector of zeros and ones, respectively.

1.1. Splittable Flow Arc Set

The splittable flow arc set $\mathcal{Q}$ is the relaxation of a multicommodity flow design problem for a single arc of the network. The *residual capacity inequalities* [6, 18]

$$\sum_{i \in S} a_i(1 - x_i) \geq \rho(\eta - y), \quad S \subseteq N, \tag{2}$$

where $\eta = \lceil a(S) - a_0 \rceil$ and $\rho = a(S) - a_0 - \lfloor a(S) - a_0 \rfloor$, are valid for $\mathcal{Q}$. For the slightly special case, where $a_0 = 0$, Magnanti et al. [18] show that adding all residual capacity inequalities to $\text{relax}(\mathcal{Q})$ gives a complete description of $\text{conv}(\mathcal{Q})$. Atamtürk and Rajan [6] give a polynomial separation algorithm for (2). In particular, they show that for a point $(x, y) \in \text{relax}(\mathcal{Q}) \backslash \mathcal{Q}$, a violated residual capacity inequality (2) is given by letting $S = \{i \in N : x_i > y - \lfloor y \rfloor\}$. Although stated in Ref. [6], a proof for convex hull description is not presented for $\mathcal{Q}$ when $a_0 \neq 0$. For completeness, we show below that the convex hull result for $\mathcal{Q}$ follows from Ref. [18].

Proposition 1. *Adding the residual capacity inequalities* (2) *to* $\text{relax}(\mathcal{Q})$ *gives* $\text{conv}(\mathcal{Q})$.

Proof. Given $\mathcal{Q}$, define the set

$$\mathcal{Q}_0 = \Bigg\{ x \in \mathbb{R}_+^N, x_0 \in \mathbb{R}_+, y_0 \in \mathbb{Z}_+ : \\ (\lceil a_0 \rceil - a_0)x_0 + \sum_{i \in N} a_i x_i \ \leq \ y_0,\ x \leq \mathbf{1},\ x_0 \leq 1 \Bigg\}.$$

From Ref. [18] adding the residual capacity inequalities to $\text{relax}(\mathcal{Q}_0)$ gives $\text{conv}(\mathcal{Q}_0)$. Now $X = \{(x, x_0, y_0) \in \text{conv}(\mathcal{Q}_0) : x_0 = 1\}$ is a face of $\text{conv}(\mathcal{Q}_0)$, and therefore is integral. This holds true after adding a lower bound $y_0 \geq \lceil a_0 \rceil$ on the only integer variable. Then projecting out variable x_0 and defining $y = y_0 - \lceil a_0 \rceil$ gives $\text{conv}(\mathcal{Q})$. Observe that residual capacity inequality $\sum_{i \in S} a_i(1 - x_i) + (\lceil a_i \rceil - a_0)(1 - x_0) \geq \rho_0(\eta_0 - y_0)$ for $\mathcal{Q}_0$ with $\eta_0 = \lceil a(S) + \lceil a_0 \rceil - a_0 \rceil$ and $\rho_0 = a(S) + \lceil a_0 \rceil - a_0 - \lfloor a(S) + \lceil a_0 \rceil - a_0 \rfloor$ equals (2) for $x_0 = 1$ and $y = y_0 - \lceil a_0 \rceil$ since $\eta_0 = \eta + \lceil a_0 \rceil$ and $\rho_0 = \rho$. ∎

1.2. Single Node Fixed-Charge Flow Set

The first polyhedral study of the single node fixed-charge flow set $\mathcal{T}$ is due to Padberg et al. [22]. Let $S \subseteq N$ be called a *cover* if $\lambda = a(S) - a_0 > 0$. For a cover S, the authors define the *flow cover inequality*

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+(1 - z_i) \leq a_0, \tag{3}$$

which is facet-defining for $\text{conv}(\mathcal{T})$ if $\lambda < \bar{a} = \max_{i \in S} a_i$. In the same paper they also show that the *augmented flow cover inequalities*

$$\sum_{i \in S \cup T} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+(1 - z_i) \\ \leq a_0 + \sum_{i \in T} (\bar{a} - \lambda) z_i, \tag{4}$$

where $T \subseteq \{i \in N \backslash S : a_i \leq \bar{a}\}$ define facets of *conv*$(\mathcal{T})$ under the same condition as well. Gu et al. [14] obtain generalizations of (4) through sequence independent lifting of (3). A complementary class of pack inequalities for $\mathcal{T}$ and their lifting are studied in Refs. [1, 23]. Flow sets with integer variable upper bounds are studied in Refs. [2, 9, 16].

1.3. Unsplittable Flow Arc Set

The unsplittable flow arc set $\mathcal{R}$ is studied first by Brockmüller et al. [10]. For $S \subseteq N$ they define the *c-strong inequalities*

$$\sum_{i \in S} \lceil a_i \rceil z_i + \sum_{i \in N \backslash S} \lfloor a_i \rfloor z_i \leq c_S + y, \tag{5}$$

where $c_S = \sum_{i \in S} \lceil a_i \rceil - \lceil a(S) - a_0 \rceil$. A set $S \subseteq N$ is called *maximal c-strong* if $c_{S \backslash \{i\}} = c_S$ for all $i \in S$ and

$c_{S\cup\{i\}} = c_S + 1$ for all $i \in N\backslash S$. Brockmüller et al. show that a c-strong inequality (5) is facet-defining for conv$(\mathcal{R})$ if and only if S is maximal c-strong. Atamtürk and Rajan [6] generalize (5) to *k-split c-strong inequalities*

$$\sum_{i\in S} \lceil ka_i \rceil z_i + \sum_{i\in N\backslash S} \lfloor ka_i \rfloor z_i \le c_S^k + ky, \tag{6}$$

where $c_S^k = \sum_{i\in S} \lceil ka_i \rceil - \lceil ka(S) - ka_0 \rceil$ for a positive integer k. Other strong inequalities obtained by lifting binary knapsack cover inequalities for $\mathcal{R}$ are described in Refs. [6,24].

1.4. Binary Knapsack Set

The binary knapsack set $\mathcal{K}$ is the most studied restriction of $\mathcal{P}$. The basic inequalities for $\mathcal{K}$ are the so-called cover inequalities: A set $S \subseteq N$ is called a *cover* if $\lambda = a(S) - b > 0$. For a cover S, the *cover inequality* [7,15,25]

$$\sum_{i\in S} x_i \le |S| - 1 \tag{7}$$

is valid for $\mathcal{K}$. Cover inequalities (7) from minimal covers define facets of the restriction conv$\{x \in \mathcal{K} : x_i = 0,\ i \in N\backslash S\}$ and they cut off all fractional extreme points of relax$(\mathcal{K})$. These inequalities typically need to be lifted in order to obtain facet-defining inequalities for conv$(\mathcal{K})$ [8,12,14,21,28,29].

2. BASIC PROPERTIES OF conv$(\mathcal{P})$

Note that optimizing a linear function over conv$(\mathcal{P})$ is $\mathcal{NP}$-hard as the binary knapsack polytope conv$(\mathcal{K})$ is a face of it. We state basic polyhedral properties of conv$(\mathcal{P})$. Observe that if $a(N) \le a_0$, then relax$(\mathcal{P})$ is integral and therefore conv$(\mathcal{P})$ equals relax$(\mathcal{P})$. This is because when the capacity constraint (1) is redundant, the remaining constraints defining $\mathcal{P}$ consist of only (variable) bound constraints. Proofs of the following results in this section can be found in Ref. [5].

Proposition 2. *The polyhedron* conv$(\mathcal{P})$ *is full-dimensional.*

Next are some simple results useful in characterizing the extreme points of relax$(\mathcal{P})$ and conv$(\mathcal{P})$.

Proposition 3. *Let (y, z, x) be an extreme point of* relax$(\mathcal{P})$.

1. *If $y > 0$, then $\sum_{i\in N} a_i x_i = a_0 + y$, and $x_i, z_i \in \{0, 1\}$ for all $i \in N$;*
2. *If $1 > x_k > 0$ for some $k \in N$, then $y = 0$, $x_i, z_i \in \{0, 1\}$ for all $i \in N\backslash\{k\}$ and $z_k \in \{x_k, 1\}$.*

Based on Proposition 3, we have the following characterization of the extreme points of relax$(\mathcal{P})$.

Corollary 1. *The point (y, z, x) is an extreme point of* relax$(\mathcal{P})$ *if and only if one of the following two cases holds:*

1. *There exist $S \subseteq T \subseteq N$ and $k \in S$ such that $a_k \ge \lambda = a(S) - a_0 > 0$ and*

$$x_i = \begin{cases} 1 & \text{if } i \in S\backslash\{k\} \\ 0 & \text{otherwise} \end{cases},$$

$$z_i = \begin{cases} 1 & \text{if } i \in T\backslash\{k\} \\ 0 & \text{otherwise} \end{cases},\quad y = 0, \text{ and}$$

either $x_k = z_k = 1 - \lambda/a_k$, or $x_k = 1 - \lambda/a_k$ and $z_k = 1$.

2. *There exist $S \subseteq T \subseteq N$ such that*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},\quad z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases},\quad y = (\lambda)^+.$$

In Sections 3 and 4 we present valid inequalities that cut off fractional extreme points of relax$(\mathcal{P})$. We next identify basic properties of the extreme points of conv$(\mathcal{P})$.

Proposition 4. *Let (y, z, x) be an extreme point of* conv$(\mathcal{P})$. *If $1 > x_k > 0$ for some $k \in N$, then*

1. *$\sum_{i\in N} a_i x_i = a_0 + y$, and $x_i \in \{0, 1\}$ for all $i \in N\backslash\{k\}$;*
2. *if $y > 0$, then either $a_k x_k < 1$ or $a_k x_k > a_k - 1$.*

Then we have the following characterization of the extreme points of conv$(\mathcal{P})$.

Corollary 2. *The point (y, z, x) is an extreme point of* conv$(\mathcal{P})$ *if and only if one of the following three cases holds:*

1. *There exist $S \subseteq T \subseteq N$ and $k \in S$ such that $\lambda = a(S) - a_0 > 0$ and $a_k \ge \rho$, where $\rho = \lambda - \lfloor \lambda \rfloor$, and*

$$x_i = \begin{cases} 1 & \text{if } i \in S\backslash\{k\} \\ 1 - \rho/a_k & \text{if } i = k \\ 0 & \text{otherwise} \end{cases},$$

$$z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases},\quad y = \lfloor \lambda \rfloor.$$

2. *There exist $S \subseteq T \subseteq N$ and $k \in N\backslash S$ such that $\lambda > 0$, $a_k \ge 1 - \rho$ and*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ (1 - \rho)/a_k & \text{if } i = k \\ 0 & \text{otherwise} \end{cases},$$

$$z_i = \begin{cases} 1 & \text{if } i \in T \cup \{k\} \\ 0 & \text{otherwise} \end{cases},\quad y = \lceil \lambda \rceil.$$

3. *There exist $S \subseteq T \subseteq N$ such that*

$$x_i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},\quad z_i = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{otherwise} \end{cases},\quad y = (\lceil \lambda \rceil)^+.$$

We next present some basic results on the facets of conv$(\mathcal{P})$.

Proposition 5. *Trivial facets of* conv($\mathcal{P}$).

1. *Inequalities* $0 \le x_k$, $x_k \le z_k$, $z_k \le 1$ *for all* $k \in N$ *are facet-defining for* conv($\mathcal{P}$).
2. *Inequality* $0 \le y$ *is facet-defining for* conv($\mathcal{P}$) *if and only if* $a_0 > 0$.
3. *The capacity inequality* (1) *is facet-defining for* conv($\mathcal{P}$) *if and only if*
   i. $a(N) - a_0 \ge \max\{1, \max_{i\in N} a_i\}$ *if* $a_0 > 0$,
   ii. *either* $|N| = 1$ *and* $a(N) = 1$, *or* $|N| > 1$ *and* $a(N) > 1$ *if* $a_0 = 0$.

Proposition 6. *For all nontrivial facet-defining inequalities* $\alpha x - \beta z - \gamma y \le \delta$ *of* conv($\mathcal{P}$) *the following statements are true:*

1. $\delta \ge 0$, $\beta \ge 0$, *and* $\gamma > 0$;
2. $\beta_i + \lceil a_i \rceil \gamma \ge \alpha_i \ge \beta_i$ *for all* $i \in N$;
3. $\exists i \in N$ *such that* $\alpha_i > \beta_i$;
4. $a(T) > a_0$ *for* $T = \{i \in N : \alpha_i > 0\}$.

3. FLOW COVER INEQUALITIES

In this section we describe valid inequalities for $\mathcal{P}$ that are based on flow the cover inequalities [22] given for the fixed-charge flow set $\mathcal{T}$. These inequalities are useful for cutting off a subset of the fractional extreme points of relax($\mathcal{P}$).

Flow cover inequalities can be derived by applying the mixed-integer rounding (MIR) procedure [20] to an appropriate relaxation of the set $\mathcal{T}$. We next review the basic idea behind the MIR inequalities.

Observation 1 ([27]). *If* $x + y \ge b$ *is a valid inequality for a mixed-integer set* $X \subseteq \{(x, y) \in \mathbb{R}_+ \times \mathbb{Z}\}$, *then the* MIR *inequality* $x \ge r(\lceil b \rceil - y)$, *where* $r = b - \lfloor b \rfloor$ *is also valid for* X.

3.1. Capacity Flow Cover Inequalities

We start with a simple application of the MIR procedure that help us generalize the flow cover inequalities.

Observation 2. *Consider a mixed-integer set*

$$Y^1 = \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{Z}_+^S : x + \sum_{i\in S} a_i y_i \ge b \right\},$$

with $b \ge 0$ *and* $a_i \ge 0$ *for all* $i \in S$. *It is possible to strengthen the inequality defining* Y^1 *as follows. Let* $\alpha \ge \max\{b, \bar{a}\}$ *and* $\bar{a} = \max_{i\in S} a_i$. *First, relax the inequality defining* Y^1 *by replacing all* $a_i > b$ *by* α *and divide the relaxed inequality by* α. *Then, invoke Observation 1 by treating all* y_i *with* $a_i \le b$ *as continuous to obtain*

$$x + \sum_{i\in S} \min\{a_i, b\}\, y_i \ge b.$$

For $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$, relax the capacity inequality as

$$a_0 + y \ge \sum_{i\in S} a_i x_i = \sum_{i\in S} a_i[1 - (1 - z_i) - (z_i - x_i)]$$

or equivalently

$$y + \sum_{i\in S} a_i(1 - z_i) + \sum_{i\in S} a_i(z_i - x_i) \ge \lambda. \tag{8}$$

Then by Observation 2, the *capacity flow cover inequality*

$$\min\{1, \lambda\}y + \sum_{i\in S} \min\{a_i, \lambda\}(1 - z_i) + \sum_{i\in S} a_i(z_i - x_i) \ge \lambda \tag{9}$$

is valid for $\mathcal{P}$. Inequality (9) can also be written as

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - \lambda)^+(1 - z_i) \le a_0 + \min\{1, \lambda\}y. \tag{10}$$

Remark 1. *Observe that for the single node fixed-charge flow set* $\mathcal{T}$, *the capacity flow cover inequality* (10) *reduces to the flow cover inequality* (3) *by letting* $y = 0$.

We next identify the conditions under which the capacity flow cover inequality (10) is facet-defining for conv($\mathcal{P}$). We study the cases when $a_0 = 0$ and $a_0 > 0$ separately, as the polyhedral structure of conv($\mathcal{P}$) depends on a_0.

Proposition 7. *Assume* $a_0 > 0$. *The capacity flow cover inequality* (9) *is facet-defining for* conv($\mathcal{P}$) *if and only if one of the following three conditions holds:* (i) $\lambda < \max_{i\in S}\{a_i\}$, (ii) $\lambda < 1$, *or* (iii) $S = N$.

Proof. *Necessity*. If $\lambda \ge \max_{i\in S}\{a_i\}$, $\lambda \ge 1$, and $S \ne N$, then inequality (9) becomes $\sum_{i\in S} a_i x_i \le a_0 + y$, which is implied by the capacity inequality (1) and $x_i \ge 0$, $i \in N\backslash S$.

Sufficiency. For a given $S \subseteq N$, we first write inequality (9) in canonical form as follows:

$$\min\{1, \lambda\}y + \sum_{i\in S\backslash S'} (a_i - \lambda)z_i - \sum_{i\in S} a_i x_i \ge \sum_{i\in S\backslash S'} a_i - a_0 - \lambda|S\backslash S'|,$$

where $S' = \{i \in S : a_i \le \lambda\}$. Let F be the face induced by inequality (9) and let $\alpha y + \beta z + \gamma x = \delta$ be satisfied by all points in F. We will show that any such equality is a multiple of the inequality that induces the face by generating pairs of points $p' = (y', z', x')$, and $p'' = (y'', z'', x'')$ and using the fact that $\alpha(y' - y'') + \beta(z' - z'') + \gamma(x' - x'') = 0$ if both points are in F. We first construct a point $p^1 = (y^1, z^1, x^1) \in \mathcal{P} \cap F$, where

$$y^1 = 0, \quad z_i^1 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},$$

$$x_i^1 = \begin{cases} 1 - \frac{\lambda}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Since $a_0, \lambda > 0$ by assumption, $a(S) > \lambda > 0$, and therefore $1 > x_k^1 > 0$ for all $k \in S$.

Let $s_k = (0, \mathbf{0}, (1/a_k)e_k)$. Since $1 > x_k^1 > 0$, there exists a small enough $\epsilon > 0$, such that $p^1 + \epsilon s_k - \epsilon s_j \in F$ for all $j, k \in S$. Therefore, $\gamma_k = -a_k\bar{\sigma}$ for all $k \in S$ for some fixed constant $\bar{\sigma} > 0$.

Next, for all $k \in S \backslash S'$, we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = 0, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S \backslash k \\ 0 & \text{otherwise} \end{cases}, \quad x_i^k = \begin{cases} 1 & \text{if } i \in S \backslash k \\ 0 & \text{otherwise} \end{cases}.$$

Using $p^1, q^k \in F$, we see that $\beta_k + (1 - \lambda/a(S))\gamma_k - \sum_{i \in S \backslash \{k\}} (\lambda/a(S))\gamma_i = 0$. Substituting $\gamma_k = -a_k\bar{\sigma}$ for all $k \in S$ and simplifying the equation gives $\beta_k = (a_k - \lambda)\bar{\sigma}$ for all $k \in S \backslash S'$ as desired.

Next, for all $k \in S'$ we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = 0, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S \backslash k \\ 0 & \text{otherwise} \end{cases},$$

$$x_i^k = \begin{cases} 1 - \frac{\lambda - a_k}{a(S) - a_k} & \text{if } i \in S \backslash k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $a(S \backslash \{k\}) \geq \lambda - a_k \geq 0$ for $k \in S'$. Since $\gamma x = \lambda$ for both $q^k, p^1 \in F$, we have $\beta_k = 0$ for all $k \in S'$.

Finally, we construct a point $p^2 = (y^2, z^2, x^2) \in F$, where

$$y^2 = 1, \quad z_i^2 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},$$

$$x_i^2 = \begin{cases} 1 - \frac{\lambda}{a(S)} + \frac{\min\{1, \lambda\}}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Using $p^1, p^2 \in F$, we conclude that $\alpha = \min\{1, \lambda\}\bar{\sigma}$ as desired.

If $S \neq S'$, then for all $k \in S \backslash S'$ the slack of the capacity inequality (1) for point q^k is $s = a_o + y - \sum_{i \in N} a_i x_i = a_k - \lambda > 0$. If, on the other hand, $S = S'$, then by assumption, we have $1 > \lambda$, and for p^2 (1) has a slack of $s = 1 - \lambda$. In either case, we have a point $p \in P$ with slack and we can perturb it to obtain points $p + t_i^1, p + t_i^2 \in F$, where $t_i^1 = (0, 1, 0)$ and $t_i^2 = (0, 1, (s/a_i)e_i)$, to show $\beta_i = \gamma_i = 0$ for all $i \notin S$.

Using p^1, for instance, we also have $\delta = \sum_{i \in S \backslash S'} a_i - a_0 - \lambda|S \backslash S'|$. We have therefore shown that inequality $\alpha y + \beta z + \gamma x = \delta$ is a multiple of the original inequality and the points defined earlier are affinely independent. As $(0, \mathbf{1}, \mathbf{0}) \in \mathcal{P} \backslash F$, F is a maximal proper face of $\text{conv}(\mathcal{P})$. ∎

Therefore, when $a_0 > 0$ the capacity flow cover inequality (9) is facet-defining under mild conditions. When $a_0 = 0$, however, inequality (9) defines a facet only when it reduces to the capacity inequality (1) or to the surrogate variable upper bound inequality (11).

Proposition 8. *Assume $a_0 = 0$. The capacity flow cover inequality* (9) *is facet-defining for* $\text{conv}(\mathcal{P})$ *if and only if one of the following three conditions holds:* (i) $|S| = 1$ *and* $a(S) < 1$, (ii) $S = N$ *and* $a(S) > 1$, *or* (iii) $|N| = 1$ *and* $a(S) = 1$.

Proof. *Necessity*. As $a_0 = 0$ we have $\lambda = a(S)$. If $a(S) \leq 1$ and $|S| > 1$, inequality (9) becomes $\sum_{i \in S} a_i x_i \leq a(S)y$, which is implied by individual inequalities $a_i(x_i - y) \leq 0$, $i \in S$. If $a(S) \geq 1$ and $S \neq N$, inequality (9) is implied by the capacity inequality (1) and $x_i \geq 0$, $i \in N \backslash S$.

Sufficiency. In the first case, inequality (9) reduces to $x_k \leq y$, $k \in N$. The following affinely independent points are clearly on the face: $(0, \mathbf{0}, \mathbf{0})$; $(0, e_i, \mathbf{0})$ for $i \in N$; $(1, \mathbf{1}, e_k)$; $(1, \mathbf{1}, \epsilon e_i + e_k)$ for $i \in N \backslash \{k\}$, where $0 < \epsilon \leq 1 - a_k$. In the other cases, inequality (9) is the capacity inequality (1) and the result follows from Proposition 5. ∎

Corollary 3. *If $a_0 = 0$ and $|N| > 1$, then the surrogate variable upper bound inequality*

$$x_i \leq y \tag{11}$$

is facet-defining for $\text{conv}(\mathcal{P})$ *if and only if $a_i < 1$.*

We next identify the fractional extreme points of $\text{relax}(\mathcal{P})$ that can be cut off using a capacity flow cover inequality.

Proposition 9. *Every fractional extreme point (x, y, z) of* $\text{relax}(\mathcal{P})$ *with $y < 1$ is cut off by a capacity flow cover inequality* (9).

Proof. Let $p = (x, y, z)$ be a fractional extreme point of $\text{relax}(\mathcal{P})$. By Corollary 1, if $y = 0$, there exist $S \subseteq N$ and $k \in S$ such that $a_k > \lambda > 0$. Then inequality (9) with such S is violated by p as

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) = a_0 + (a_k - \lambda)\lambda/a_k > a_0 = a_0 + \min\{1, \lambda\}y.$$

On the other hand, if $y > 0$, then there exist $S \subseteq N$ such that $y = \lambda \notin \mathbb{Z}$. If $\lambda < 1$, then inequality (9) with such S is violated by p as

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) = a_0 + \lambda > a_0 + \lambda^2 = a_0 + \min\{1, \lambda\}y.$$

∎

3.2. Lifting with Integer Capacity Variable

We next describe valid inequalities obtained by first fixing the value of the y variable, and then lifting the associated basic flow cover inequality. If the y variable is fixed to $v \in \mathbb{Z}_+$, then the resulting lifted inequality has the form

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) \le a_0 + v + \alpha(y-v), \quad (12)$$

where $S \subseteq N$ and $r = a(S) - v - a_0 > 0$.

Proposition 10. *Let $S \subseteq N$ be such that $r = a(S)-v-a_0 > 0$ for $v \in \mathbb{Z}_+$. Then the lifted capacity flow cover inequality* (12) *is valid for $\mathcal{P}$ if and only if*

1. $-\Phi(-1) \le \alpha$ *if $v = 0$;*
2. $-\Phi(-1) \le \alpha \le \Phi(1)$ *if $v > 0$, where Φ is defined as in* (13).

Moreover, (12) *defines a facet of* $\mathrm{conv}(\mathcal{P})$ *if α equals one of its bounds and $r < \max_{i\in S} a_i$.*

Proof. Inequality (12) is the flow cover inequality (3) for the restriction $\mathcal{P}(v) = \{(x,y,z) \in \mathcal{P} : y = v\}$ of $\mathcal{P}$ and it is valid for $\mathcal{P}(v)$ for any α. Then, as shown for lifting with integer variables in Ref. [26], (12) is valid for $\mathcal{P}$ if and only if $\underline{\alpha} \le \alpha \le \overline{\alpha}$, where

$$\underline{\alpha} = \max\left\{\frac{\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) - a_0 - v}{y - v} : (x,y,z) \in \mathcal{P}, y > v\right\}$$

and

$$\overline{\alpha} = \min\left\{\frac{a_0 + v - \sum_{i\in S} a_i x_i - \sum_{i\in S}(a_i - r)^+(1-z_i)}{v - y} : (x,y,z) \in \mathcal{P}, y < v\right\},$$

with $\overline{\alpha} = \infty$ if $v = 0$.

Without loss of generality, suppose $S = \{1,2,\ldots,|S|\}$ with $a_1 \ge a_2 \ge \cdots \ge a_{|S|}$. Let $p = \max\{i \in S : a_i > \lambda\}$, $A_i = \sum_{k=1}^{i} a_k$ for $i \in \{1,2,\ldots,p\}$, and $A_0 = 0$. It is shown in Ref. [14] that the lifting function

$$\Phi(a) = \min\left\{a_0 + v - \sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) : (x,y,z) \in \mathcal{P}(v-a)\right\}$$

can be stated as

$$\Phi(a) = \begin{cases} \max\{-r, a\} & \text{if } a \le 0, \\ ir & \text{if } A_i \le a \le A_{i+1} - r, \\ ir + (a - A_i) & \text{if } A_i - r \le a \le A_i, \\ pr + (a - A_p) & \text{if } A_p - r \le a_0 + v, \\ +\infty & \text{if } a > a_0 + v, \end{cases} \quad (13)$$

where $i \in \{0,1,\ldots,p-1\}$ and that Φ is superadditive on $[0, a_0 + v]$ and $(-\infty, 0]$, separately.

Then for $v > 0$ we have $\overline{\alpha} = \min_{a\in\mathbb{Z}, a>0} \frac{\Phi(a)}{a} = \Phi(1)$, where the last equation follows from superadditivity of Φ over $[0, a_0 + v]$. Similarly, $\underline{\alpha} = -\min_{a\in\mathbb{Z}, a<0} \frac{\Phi(a)}{a} = -\Phi(-1)$. Finally, if $r < \max_{i\in S} a_i$, inequality (12) is facet-defining for $\mathrm{conv}(\mathcal{P}(v))$ and in addition if $\alpha \in \{\underline{\alpha}, \overline{\alpha}\} < \infty$, the lifting is exact; hence, (12) defines a facet for $\mathrm{conv}(\mathcal{P})$. ∎

Note that $-\Phi(-1) = \min\{1, r\}$. Therefore, if we let $v = 0$, then

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) \le a_0 + \min\{1, r\}y \quad (14)$$

is valid for $\mathrm{conv}(\mathcal{P})$. This inequality is identical to the capacity flow cover inequality (9). Also notice that the facet sufficient condition of Proposition 10 is more restrictive than the condition of Proposition 7. Therefore, when $v = 0$, the lifted inequalities do not lead to new inequalities.

If $v > 0$, however, the resulting inequalities are new. First observe that $\min\{1, r\} \le \Phi(1)$ only if $\max_{i\in S} a_i \le 1$. So if $v > 0$, then

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) \le a_0 + v + r(y - v) \quad (15)$$

as well as

$$\sum_{i\in S} a_i x_i + \sum_{i\in S}(a_i - r)^+(1-z_i) \le a_0 + v + \Phi(1)(y - v) \quad (16)$$

are valid for $\mathrm{conv}(\mathcal{P})$ provided that $\max_{i\in S} a_i \le 1$. Inequalities (15) and (16) are facet-defining provided that $r < \max_{i\in S} a_i$. They are distinct only if $A_2 - r < 1$.

Recall that every fractional extreme point (x,y,z) of $\mathrm{relax}(\mathcal{P})$ with $y < 1$ is cut off by a capacity flow cover inequality (9). We next show that some of the remaining ones are cut off by the lifted capacity flow cover inequality (15).

Proposition 11. *Every fractional extreme point (x,y,z) of* $\mathrm{relax}(\mathcal{P})$ *with $y \ge 1$ is cut off by a lifted capacity flow cover inequality* (15) *with $v = \lfloor y \rfloor$ and $S = \{i \in N : x_i > 0\}$ provided that $a_i \le 1$ for all $i \in S$.*

Proof. By Corollary 1, if $y \geq 1$ for a fractional extreme point, then (i) $y \notin \mathbb{Z}_+$, (ii) $\sum_{i \in N} a_i x_i = a_0 + y$, and (iii) $x_i, z_i = 1 \in \{0, 1\}$ for all $i \in N$. Therefore,

$$\sum_{i \in S} a_i x_i + \sum_{i \in S} (a_i - r)^+ (1 - z_i) = a_0 + y$$
$$= a_0 + v + r > a_0 + v + r^2 = a_0 + v + r(y - v).$$

■

3.3. Augmented Capacity Flow Cover Inequalities

Next we give another application of the MIR procedure that helps us generalize the augmented flow cover inequalities (4).

Observation 3. *Consider a mixed-integer set*

$$Y^2 = \left\{ (x, y) \in \mathbb{R}_+ \times \mathbb{Z}_+^{S \cup T} : x + \sum_{i \in S} a_i y_i - \sum_{i \in T} a_i y_i \geq b \right\},$$

with $b \geq 0$ and $a_i \geq 0$ for all $i \in S \cup T$. For any $\alpha \geq \max\{b, \bar{a}\}$ and $\bar{a} = \max_{i \in S} a_i$, inequality

$$x + \sum_{i \in S} \min\{a_i, b\}\, y_i - \sum_{i \in T} \phi_\alpha(a_i, b) y_i \geq b, \qquad (17)$$

where $\phi_\alpha(a, b) = (b\lfloor a/\alpha \rfloor + (a - \alpha\lceil a/\alpha \rceil + b)^+)$, is valid for Y^2.

Derivation is similar to that of Observation 2 *except that the variables with negative coefficients must also be treated. Let $\rho_i = \alpha\lceil a_i/\alpha \rceil - a_i$. When relaxing the inequality defining Y^2, for $i \in T$, if $\rho_i > b$ we replace $-a_i$ with $-\alpha\lfloor a_i/\alpha \rfloor$. If $\rho_i \leq b$ we rewrite $-a_i$ as $-\alpha\lceil a_i/\alpha \rceil + \rho_i$. As in Observation* 2, *we divide the resulting inequality by α and apply the* MIR *procedure by* (i) *treating variable y_i, $i \in S$ as a continuous variable if $a_i < b$, and* (ii) *treating $(\rho_i/\alpha) y_i, i \in T$ as a continuous variable if $\rho_i < b$.*

We can augment inequality (9) to obtain new valid inequalities that have nonzero coefficient for variables $(x_i, z_i), i \in N \backslash S$. Let $S \subseteq N$ be such that $\lambda = a(S) - a_0 > 0$ and $T \subseteq N \backslash S$. Now let us relax the capacity constraint (1) as

$$a_0 + y \geq \sum_{i \in S} a_i [1 - (1 - z_i) - (z_i - x_i)]$$

or equivalently

$$\left[\sum_{i \in S \cup T} a_i (z_i - x_i) \right] + \left[y + \sum_{i \in S} a_i (1 - z_i) \right] - \left[\sum_{i \in T} a_i z_i \right] \geq \lambda. \qquad (18)$$

Then by Observation 3, for $\alpha_1 = \max\{1, \bar{a}, \lambda\}$ and $\bar{a} = \max_{i \in S}\{a_i\}$, inequality

$$\sum_{i \in S \cup T} a_i (z_i - x_i) + \min\{1, \lambda\} y + \sum_{i \in S} \min\{a_i, \lambda\}(1 - z_i) - \sum_{i \in T} \phi_{\alpha_1}(a_i, \lambda) z_i \geq \lambda \qquad (19)$$

is valid for $\mathcal{P}$. Inequality (19) can also be written as

$$\sum_{i \in S \cup T} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) - \sum_{i \in T} (a_i - \phi_{\alpha_1}(a_i, \lambda)) z_i \leq a_0 + \min\{1, \lambda\} y. \qquad (20)$$

Under certain conditions the coefficients of z_i for $i \in T$ coincide with the ones obtained through sequence independent lifting functions in Ref. [13] and inequality (20) defines a facet of conv($\mathcal{P}$) [5].

Similarly, treating variable y as a continuous variable in inequality (18) and applying Observation 3 with $\alpha_2 = \max\{\bar{a}, \lambda\}$ gives

$$\sum_{i \in S \cup T} a_i x_i + \sum_{i \in S} (a_i - \lambda)^+ (1 - z_i) - \sum_{i \in T} (a_i - \phi_{\alpha_2}(a_i, \lambda)) z_i \leq a_0 + y. \qquad (21)$$

Remark 2. *Observe that for the single node fixed-charge flow set $\mathcal{T}$, inequality* (21) *reduces to the flow cover inequality* (4) *by letting $y = 0$.*

4. MIXED INTEGER ROUNDING INEQUALITIES

In this section we describe new families of valid inequalities based on the application of MIR procedure on other valid inequalities for $\mathcal{P}$. The first family of inequalities presented below cut off all fractional extreme points of relax($\mathcal{P}$). In addition, all extreme points of conv($\mathcal{P}$) are extreme points of the polyhedron obtained by adding these inequalities to relax($\mathcal{P}$).

4.1. Capacity Flow-Cover-MIR Inequalities

Let $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$. Relaxing the capacity flow cover inequality (9) by skipping the coefficient reduction step for $i \in S' \subseteq S$ and increasing the coefficients of the $(1 - z_i)$ terms for $i \in S \backslash S'$, we obtain

$$\left[y + \sum_{i \in S \backslash S'} \min\{\eta, \lceil a_i \rceil\}(1 - z_i) + \sum_{i \in S'} \lfloor a_i \rfloor (1 - z_i) \right] + \left[\sum_{i \in S} a_i (z_i - x_i) + \sum_{i \in S'} r_i (1 - z_i) \right] \geq \lambda,$$

where $\eta = \lceil\lambda\rceil$ and $r_i = a_i - \lfloor a_i\rfloor$. Now applying to this inequality the MIR procedure, we obtain the *capacity flow cover MIR* (FC-MIR) *inequality*:

$$\sum_{i\in S} a_i(z_i - x_i) + \sum_{i\in S'} r_i(1 - z_i) \geq \rho\left(\eta - y - \sum_{i\in S\setminus S'} \min\{\eta, \lceil a_i\rceil\}(1 - z_i) - \sum_{i\in S'} \lfloor a_i\rfloor (1 - z_i)\right), \quad (22)$$

where $\rho = \lambda - \lfloor\lambda\rfloor$. Notice that as the capacity flow cover inequality (9) is itself obtained by the MIR procedure, inequality (22) is the result of two iterative applications of the MIR procedure.

Remark 3. *For the splittable flow arc set $\mathcal{Q}$, inequality* (22) *reduces to residual capacity inequality* (2) *by letting $z = \mathbf{1}$.*

Proposition 12. *FC-MIR inequality* (22) *is facet-defining for* $\text{conv}(\mathcal{P})$ *if and only if*

1. *$\eta > \lambda$, i.e., $\lambda \notin \mathbb{Z}$,*
2. *$a(S) > \rho$, i.e., either $a_0 > 0$ or $\eta > 1$,*
3. *$S' = \{i \in S : a_i < \lambda$ and $r_i < \rho\}$.*

Proof. *Necessity*. 1. If $\eta = \lambda$, then (22) is implied by $x_i \leq z_i$, $i \in S$ and $z_i \leq 1$, $i \in S'$. 2. If $a_0 = 0$ and $\eta = 1 \geq a_i$, then $a_i = r_i \leq \rho$ for all $i \in S$. Thus unless $S' = S$, inequality is weak. For $S' = S$, inequality becomes $\sum_{i\in S} a_i(1 - x_i) \geq a(S)(1 - y)$, which is implied by individual capacity flow cover inequalities (9) $x_i \leq y$, $i \in S$. 3. Let $S^* = \{i \in S : a_i < \lambda$ and $r_i < \rho\}$. If $S' \neq S^*$, then replacing S' with S^* gives a stronger inequality since $r_i + \rho\lfloor a_i\rfloor < \rho\min\{\eta, \lceil a_i\rceil\}$ for $i \in S^*$.

Sufficiency. For a given $S \subseteq N$, we first write (22) in canonical form as follows:

$$\rho y + \sum_{i\in S\setminus S'} \big(a_i - \rho\min\{\eta, \lceil a_i\rceil\}\big)z_i + \sum_{i\in S'} \Big(a_i - r_i - \rho\lfloor a_i\rfloor\Big)z_i - \sum_{i\in S} a_i x_i \geq \rho\left(\eta - \sum_{i\in S\setminus S'} \min\{\eta, \lceil a_i\rceil\} - \sum_{i\in S'} \lfloor a_i\rfloor\right) - \sum_{i\in S'} r_i$$

Let F be the face induced by (22) and let $\alpha y + \beta z + \gamma x = \delta$ be satisfied by all points in F. We start with constructing a point $p^1 = (y^1, z^1, x^1) \in F$ and show that F is not empty:

$$y^1 = \eta(S), \quad z_i^1 = x_i^1 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases},$$

where $\eta(S) = \lceil a(S) - a_0\rceil$. Let $t_k = (0, e_k, 0)$ and $s_k = (0, e_k, \epsilon e_k)$, where $\epsilon = (1 - \rho)/a_k$. Since for all $k \in N\setminus S$, both p^1 and $p^1 + t_k \in F$, we have $\beta_k = 0$ for all $k \in N\setminus S$. Similarly, $p^1 + t_k$ and $p^1 + s_k \in F$ implies that $\gamma_k = 0$ for all $k \in N\setminus S$.

Next, we construct $p^2 = (y^2, z^2, x^2) \in F$, where

$$y^2 = \eta(S) - 1, \quad z_i^2 = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}, \quad x_i^2 = \begin{cases} 1 - \frac{\rho}{a(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}.$$

Note that $1 > x_i^2 > 0$ for all $i \in S$. Let $t_k = (0, \mathbf{0}, (1/a_k)e_k)$. For each $i, j \in S$ and for a small enough $\epsilon > 0$, both p^2 and $p^2 + \epsilon t_i - \epsilon t_j \in F$, and therefore for some $\bar\sigma \in R$ we have $\gamma_k = -a_k\bar\sigma$ for all $k \in S$. Furthermore, $p^1, p^2 \in F$ implies that $\alpha = \rho\bar\sigma$.

We next observe that for any $u, v \in R$, if we let $u = \lceil u\rceil - 1 + r_u$, $v = \lceil v\rceil - 1 + r_v$ and $u + v = \lceil u + v\rceil - 1 + r_{uv}$, with $1 \geq r_u, r_v, r_{uv} > 0$, we have

(i) either $r_u + r_v > 1 \Leftrightarrow \lceil u + v\rceil = \lceil u\rceil + \lceil v\rceil \Leftrightarrow r_{uv} = r_u + r_v - 1 \leq \min\{r_u, r_v\}$,
(ii) or $r_u + r_v \leq 1 \Leftrightarrow \lceil u + v\rceil = \lceil u\rceil + \lceil v\rceil - 1 \Leftrightarrow r_{uv} = r_u + r_v > \max\{r_u, r_v\}$.

Let $l_k = (\min\{\eta(S), \lceil a_k\rceil\}, e_k, e_k)$. Since $r_k \geq \rho$ for all $k \in S\setminus S'$, we have $\eta(S\setminus k) = \eta(S) - \lceil a_k\rceil \leq \eta(S) - \min\{\eta(S), \lceil a_k\rceil\}$ and therefore $p^1 - l_k \in P$. Using $p^1, p^1 - l_k \in F$, we obtain the equation $\min\{\eta(S), \lceil a_k\rceil\}\alpha + \beta_k + \gamma_k = 0$ implying $\beta_k = (a_k - \rho\min\{\eta(S), \lceil a_k\rceil\})\bar\sigma$ for all $k \in S\setminus S'$.

Finally, for all $k \in S'$ we construct a point $q^k = (y^k, z^k, x^k)$, where

$$y^k = \eta(S) - \lfloor a_k\rfloor - 1, \quad z_i^k = \begin{cases} 1 & \text{if } i \in S\setminus k \\ 0 & \text{otherwise} \end{cases}, \quad x_i^k = \begin{cases} 1 - \frac{\rho - r_k}{a(S\setminus k)} & \text{if } i \in S\setminus k \\ 0 & \text{otherwise} \end{cases}.$$

Note that $r_k < \rho$ for all $k \in S'$, implying $\eta(S\setminus k) = \eta(S) - \lfloor a_k\rfloor$ and $r(S\setminus k) = \rho - r_k$. Therefore,

$$\sum_{i\in S} a_i x_i^k = a(S\setminus k)\left(1 - \frac{\rho - r_k}{a(S\setminus k)}\right) = a(S\setminus k) - r(S\setminus k)$$

and $q^k \in P$. Since both p^2, $q^k \in F$, we have

$$\begin{aligned} 0 &= \lfloor a_k\rfloor \rho\bar\sigma + \beta_k - a(S)\bar\sigma\left(1 - \frac{\rho}{a(S)}\right) + a(S\setminus k)\bar\sigma\left(1 - \frac{\rho - r_k}{a(S\setminus k)}\right) \\ &= \lfloor a_k\rfloor \rho\bar\sigma + \beta_k - a(S)\bar\sigma + \rho\bar\sigma + a(S\setminus k)\bar\sigma - \rho\bar\sigma + r_k\bar\sigma \\ &= \lfloor a_k\rfloor \rho\bar\sigma + \beta_k - a_k\bar\sigma + r_k\bar\sigma \end{aligned}$$

implying $\beta_k = \big(a_k - r_k - \rho\lfloor a_k\rfloor\big)\bar\sigma$ for all $k \in S'$, as desired.

We have therefore shown that inequality $\alpha y + \beta z + \gamma x = \delta$ is a multiple of the original inequality, and the points defined earlier are affinely independent. As $(\hat{a}+1,\mathbf{1},\mathbf{1}) \in \mathcal{P}\backslash F$, F is a maximal proper face of conv$(\mathcal{P})$. ∎

Observe that if $\lambda \le 1$, we have $\eta = 1$ and $\rho = \lambda$. Then by Proposition 12 facet-defining inequalities (22) satisfy $a_i < \lambda < 1$ for all $i \in S'$, in which case they are equivalent to capacity flow cover inequalities (9). Therefore, inequalities (22) are of particular interest if $\lambda > 1$ as they differ from inequalities (9) in that case.

Moreover, recall the lifted capacity flow cover inequality (15) with $v \in \mathbb{Z}_+$ and $r = a(S) - v - a_0 > 0$, which is valid and facet-defining provided that $r \le a_i \le 1$ for all $i \in S$. Notice that, under this condition, (i) $v = \lfloor a(S) - a_0 \rfloor = \eta - 1$, (ii) $r = \rho$, and (iii) $r_i = a_i$ for all $i \in S$. In this case FC-MIR inequality (22) becomes

$$\sum_{i\in S} a_i(z_i - x_i) + \sum_{i\in S'} a_i(1 - z_i) \ge r(v+1) - ry - r\sum_{i\in S\backslash S'}(1 - z_i)$$

or, equivalently,

$$\sum_{i\in S} a_i + \sum_{i\in S} a_i(z_i - 1) - \sum_{i\in S} a_i x_i + \sum_{i\in S}\min\{r, a_i\}(1 - z_i) \ge rv + r - ry,$$

which is identical to inequality (15) as $v + a_0 = a(S) - r$. Therefore, facet-defining lifted capacity flow cover inequalities form a subclass of FC-MIR inequalities.

We next show that all fractional extreme points of relax$(\mathcal{P})$ violate an FC-MIR inequality.

Proposition 13. *Every fractional extreme point of* relax$(\mathcal{P})$ *is cut off by an FC-MIR inequality* (22).

Proof. Let $p = (x, y, z)$ be a fractional extreme point of relax$(\mathcal{P})$. By Corollary 1, if $y = 0$, there exist $S \subseteq N$ and $k \in S$ such that $a_k > \lambda > 0$. Consider the inequality (22) with such S and $S' = \emptyset$ and let rhs denote its right-hand side value for this point. This inequality is violated by p as

$$\sum_{i\in S} a_i(z_i - x_i) = 0 < \rho\eta(1 - \lambda/a_k) = \text{rhs}.$$

On the other hand, if $y > 0$, there exist $S \subseteq N$ such that $y = \lambda \notin \mathbb{Z}$. Then inequality (22) with such S and $S' = \emptyset$ violates (x, y, z) as

$$\sum_{i\in S} a_i(z_i - x_i) = 0 < \rho(\eta - \lambda) = \text{rhs}.$$

∎

The following proposition complements Proposition 13.

Proposition 14. *If the capacity inequality* (1) *is facet-defining for* conv$(\mathcal{P})$*, then all extreme points of* conv$(\mathcal{P})$ *are extreme points of the polyhedron obtained by adding all FC-MIR inequalities* (22) *and surrogate variable upper bound inequalities* (11) *to* relax$(\mathcal{P})$.

Proof. Consider the extreme points defined in Corollary 2. Any point in the first case is the intersection of the following $2n+1$ facets: capacity inequality (1), MIR inequality (22) with S, $x_i \ge 0$ for $i \in N\backslash S$, $x_i \le z_i$ for $i \in S\backslash\{k\}$ $(x_i = z_i = 1)$, $z_i \le 1$ for $i \in T$, $x_i \le z_i$ for $i \in N\backslash T$ $(x_i = z_i = 0)$. We may assume that $a_k > \rho$, since otherwise case 1 reduces to case 3. Then FC-MIR inequality (22) is facet-defining because when $a_0 = 0$, the property $a_k > \rho$ implies that $a(S) > 1$.

Any point in the second case is the intersection of the following $2n + 1$ facets: capacity inequality (1), FC-MIR inequality (22) with S, $x_i \ge 0$ for $i \in N\backslash(S \cup \{k\})$, $x_i \le z_i$ for $i \in S$ $(x_i = z_i = 1)$, $z_i \le 1$ for $i \in T \cup \{k\}$, $x_i \le z_i$ for $i \in N\backslash(T \cup \{k\})$ $(x_i = z_i = 0)$. In this case, if FC-MIR inequality (22) is not facet-defining (i.e., $a_0 = 0$ and $a(S) < 1$), it is replaced with the surrogate variable upper bound inequality (11) for some $i \in S$, which is facet-defining as $a_i < 1$.

Finally, any point in the third case is the intersection of the facets defined by either $y \ge 0$ or FC-MIR inequality (22) with S, and $x_i \ge 0$ for $i \in N\backslash S$, $x_i \le z_i$ for $i \in S$ $(x_i = z_i = 1)$, $z_i \le 1$ for $i \in T$, $x_i \le z_i$ for $i \in N\backslash T$ $(x_i = z_i = 0)$. ∎

If the capacity inequality (1) is not facet-defining, then replacing it with the stronger capacity flow cover inequality (10) with $S = N$ in the first two cases again gives necessary $2n + 1$ facets.

4.2. MIR Inequalities

For $S \subseteq N$ such that $\lambda = a(S) - a_0 > 0$ and $T \subseteq N\backslash S$, let us relax the capacity constraint as follows:

$$a_0 + y \ge \sum_{i\in S} a_i x_i + \sum_{i\in T} a_i x_i \tag{23}$$

$$= \sum_{i\in S} a_i[1 - (1 - z_i) - (z_i - x_i)] - \sum_{i\in T} a_i[(z_i - x_i) + z_i]. \tag{24}$$

Let $S' \subseteq S$ and $T' \subseteq T$. We next relax inequality (24) as follows: (i) for $i \in S'$, we split the coefficient of $(1 - z_i)$ into $\lfloor a_i \rfloor$ and r_i; (ii) for $i \in S\backslash S'$, we round up the coefficient of $(1 - z_i)$; (iii) for $i \in T'$, we rewrite the coefficient of z_i as $\lceil a_i \rceil$ and $(r_i - 1)$, and (iv) for $i \in T\backslash T'$, we relax the coefficient of x_i to $\lfloor a_i \rfloor$ and add and subtract $(a_i - \lfloor a_i \rfloor)z_i$ to the inequality. Thus the resulting inequality is

$$\left[y + \sum_{i\in S\backslash S'} \lceil a_i \rceil (1 - z_i) + \sum_{i\in S'} \lfloor a_i \rfloor (1 - z_i) - \sum_{i\in T\backslash T'} \lfloor a_i \rfloor z_i - \sum_{i\in T'} \lceil a_i \rceil z_i \right]$$

$$+\left[\sum_{i\in S\cup T'} a_i(z_i-x_i)+\sum_{i\in T\setminus T'}\lfloor a_i\rfloor(z_i-x_i)\right.$$
$$\left.+\sum_{i\in S'} r_i(1-z_i)+\sum_{i\in T'}(1-r_i)z_i\right]\geq\lambda. \quad (25)$$

Applying the MIR procedure to (25) we obtain the valid inequality

$$\sum_{i\in S\cup T'} a_i(z_i-x_i)+\sum_{i\in T\setminus T'}\lfloor a_i\rfloor(z_i-x_i)$$
$$+\sum_{i\in S'} r_i(1-z_i)+\sum_{i\in T'}(1-r_i)z_i$$
$$\geq\rho\left(\eta-y-\sum_{i\in S\setminus S'}\lceil a_i\rceil(1-z_i)-\sum_{i\in S'}\lfloor a_i\rfloor(1-z_i)\right.$$
$$\left.+\sum_{i\in T'}\lceil a_i\rceil z_i+\sum_{i\in T\setminus T'}\lfloor a_i\rfloor z_i\right). \quad (26)$$

Proposition 15. *An MIR inequality* (26) *is facet-defining for* conv($\mathcal{P}$) *if*

1. $\eta>\lambda$, *i.e.,* $\lambda\notin\mathbb{Z}$,
2. $a(S)>\rho$, *i.e., either* $a_0>0$ *or* $\eta>1$,
3. $S\subseteq\{i\in N: a_i\leq\eta\}$,
4. $S'=\{i\in S: r_i<\rho\}$,
5. $T=T'\subseteq\{i\in T:(1-r_i)<\rho\}$.

Proof. We first rewrite inequality (26) as follows:

$$\rho y+\sum_{i\in S\setminus S'}(a_i-\rho\lceil a_i\rceil)z_i+\sum_{i\in S'}(a_i-r_i-\rho\lfloor a_i\rfloor)z_i$$
$$+\sum_{i\in T\setminus T'}(a_i-\rho\lfloor a_i\rfloor)z_i$$
$$+\sum_{i\in T'}(a_i+1-r_i-\rho\lceil a_i\rceil)z_i-\sum_{i\in S\cup T} a_ix_i$$
$$\geq\rho\left(\eta-\sum_{i\in S\setminus S'}\lceil a_i\rceil-\sum_{i\in S'}\lfloor a_i\rfloor\right)-\sum_{i\in S'} r_i.$$

For a given $S\subseteq N$, let F be the face induced by the valid inequality and assume that all $p\in F$ satisfy the equality $\alpha y+\beta z+\gamma x=\delta$. From the proof of Proposition 12 we have α,β_i,γ_i for all $i\in N\setminus T$ as desired.

Let $k\in T'$. Recall that, $r_k+\rho>1$ and therefore $\eta(S+k)=\eta(S)+\lceil a_k\rceil$. Consider $\bar{p}^1=p^1+\left(\lceil a_k\rceil-1, e_k, \frac{\lceil a_k\rceil-1+(1-\rho)}{a_k}e_k\right)$, and note that $\lceil a_k\rceil-1+(1-\rho)<a_k$. Since $p^1,\bar{p}^1\in F$, we have $(\lceil a_k\rceil-1)\rho\bar{\sigma}+\beta_k+\frac{\lceil a_k\rceil-\rho}{a_k}\gamma_k=0$.

Let $t_k=(0,0,(1/a_k)e_k)$ and $i\in S$. We have $\bar{p}^1,\bar{p}^1+\epsilon t_k-\epsilon t_i\in F$ for a small enough $\epsilon>0$, and therefore $\gamma_k=-a_k\bar{\sigma}$. Furthermore, when combined with above, we have $\beta_k=(\lceil a_k\rceil-\rho)\bar{\sigma}-(\lceil a_k\rceil-1)\rho\bar{\sigma}=(1-\rho)\lceil a_k\rceil\bar{\sigma}$, as desired. ∎

Remark 4. *For the unsplittable flow arc set $\mathcal{R}$ by letting $x=z$, the* MIR *inequalities* (26) *with $T=N\setminus S$ reduce to*

$$\sum_{i\in S'} r_i(1-z_i)+\sum_{i\in T'}(1-r_i)z_i$$
$$\geq\rho\left(\eta-y-\sum_{i\in S\setminus S'}\lceil a_i\rceil(1-z_i)-\sum_{i\in S'}\lfloor a_i\rfloor(1-z_i)\right.$$
$$\left.+\sum_{i\in T'}\lceil a_i\rceil z_i+\sum_{i\in N\setminus(S\cup T')}\lfloor a_i\rfloor z_i\right). \quad (27)$$

Observe if $x=z$, inequalities (26) *with $T=N\setminus S$ dominate all others with $T\subsetneq N\setminus S$; hence $T=N\setminus S$ in inequality* (27).

Furthermore, if $S'=T'=\emptyset$, inequality (27) *reduces to the c-strong inequality* (5). *Recall that a c-strong inequality is facet-defining for* conv($\mathcal{R}$) *only if S is maximal c-strong only if $r_i\geq\rho$ for all $i\in S$ and $r_i\leq 1-\rho$ for all $i\in N\setminus S$. Thus if S is not maximal c-strong, inequality* (27) *with $S'=\{i\in S: r_i<\rho\}$ and $T'=\{i\in T:(1-r_i)<\rho\}$ dominates the corresponding c-strong inequality.*

The following example illustrates the strength of (27) for conv($\mathcal{R}$). Let

$$\mathcal{R}=\left\{y\in\mathbb{Z}_+,\ z\in\{0,1\}^5\ :1x_1+0.5x_2+0.75x_3+0.75x_4+0.75x_5\ \leq\ y\right\}.$$

For $S=\{1,2\}$, which is not maximal c-strong, the c-strong inequality (5) is

$$x_1+x_2\leq y, \quad (28)$$

whereas the 2-split c-strong inequality (6) is

$$2x_1+x_2+x_3+x_4+x_5\leq 2y. \quad (29)$$

Inequality (27) with $S=\{1,2\}$, $S'=\emptyset$, and $T'=\{3,4,5\}$ ($\lambda=1.5,\eta=2,\rho=0.5$)

$$0.25x_3+0.25x_4+0.25x_5\geq 0.5(2-y-(1-x_1)-(1-x_2)), \quad (30)$$

which can also be stated as

$$x_1+x_2+0.5x_3+0.5x_4+0.5x_5\leq y$$

dominates both (28) and (29). It is easily checked that (30) is facet-defining for conv($\mathcal{R}$).

4.3. Scaled MIR Inequalities

For $S\subseteq N$ such that $\lambda=a(S)-a_0>0$ and $T\subseteq N\setminus S$, let us relax the capacity constraint as (24) and multiply the inequality with $\mu>0$ to obtain

$$\mu a_0+\mu y=\sum_{i\in S}\mu a_i[1-(1-z_i)-(z_i-x_i)]$$
$$-\sum_{i\in T}\mu a_i[(z_i-x_i)+z_i]. \quad (31)$$

For $S' \subseteq S$ and $T' \subseteq T$ applying the same type of relaxation as in Section 4.2, we obtain the intermediate valid inequality

$$\left[\lceil\mu\rceil y + \sum_{i\in S\setminus S'} \lceil \mu a_i\rceil (1-z_i) + \sum_{i\in S'} \lfloor \mu a_i\rfloor (1-z_i) - \sum_{i\in T\setminus T'} \lfloor \mu a_i\rfloor z_i - \sum_{i\in T'} \lceil \mu a_i\rceil z_i \right] + \left[\sum_{i\in S\cup T} \mu a_i(z_i - x_i) + \sum_{i\in S'} \bar r_i(1-z_i) + \sum_{i\in T'}(1-\bar r_i)z_i \right] \geq \mu\lambda, \quad (32)$$

where $\bar r_i = \mu a_i - \lfloor \mu a_i \rfloor$ for $i \in N$. Now applying the MIR procedure to (32) we obtain the *scaled MIR inequality*

$$\sum_{i\in S\cup T} \mu a_i(z_i - x_i) + \sum_{i\in S'} \bar r_i(1-z_i) + \sum_{i\in T'}(1-\bar r_i)z_i \geq \bar\rho \left(\bar\eta - \lceil\mu\rceil y - \sum_{i\in S\setminus S'} \lceil \mu a_i\rceil (1-z_i) - \sum_{i\in S'} \lfloor \mu a_i\rfloor (1-z_i) + \sum_{i\in T'} \lceil \mu a_i\rceil z_i + \sum_{i\in T\setminus T'} \lfloor \mu a_i\rfloor z_i \right), \quad (33)$$

where $\bar\eta = \lceil \mu\lambda \rceil$ and $\bar\rho = \mu\lambda - \lfloor \mu\lambda \rfloor$.

By simple comparison, one sees that choosing $S' = \{i \in S : \bar r_i < \bar\rho\}$ and $T' = \{i \in T : (1-\bar r_i) < \bar\rho\}$ in (33) leads to the strongest inequalities as inequalities for all other choices for S' and T' are implied by these and $\mathbf{0} \leq z \leq \mathbf{1}$.

Moreover, if $\mu - \lfloor\mu\rfloor < \bar\rho$, one can obtain a stronger inequality by not relaxing the term μy in inequality (31) to $\lceil\mu\rceil y$, but instead writing it as $\lfloor\mu\rfloor y + (\mu - \lfloor\mu\rfloor)y$ so that in the MIR procedure, the first part can be treated as an integer variable and the second part as a continuous variable. We do not write the resulting inequality explicitly to avoid repetition.

Remark 5. *If $1 \leq \lambda$, the capacity flow cover inequality* (9) *can be obtained by taking $\mu \geq \{1, \bar a, \lambda\}$ in inequality* (33), *where $\bar a = \max_{i\in S}\{a_i\}$. If $1 > \lambda$, then the strengthened version (mentioned in the earlier paragraph) of inequality* (33) *gives the capacity flow cover inequality.*

Clearly inequality (33) also subsumes the MIR inequality (26) by taking $\mu = 1$ and therefore it forms a superclass of all inequalities discussed in this paper except the FC-MIR inequality (22). When $\lceil a_i \rceil > \eta$ for some $i \in S$, the resulting FC-MIR inequality is different from (33).

We next show that scaled MIR inequalities (33) reduce to some well-known inequalities for the unsplittable flow set $\mathcal{R}$ and the binary knapsack set $\mathcal{K}$.

Remark 6. *For the unsplittable flow set $\mathcal{R}$ by letting $x = z$, $\mu = k \in \mathbb{Z}$, $S' = T' = \emptyset$, and $T = N\setminus S$, inequality* (33) *reduces to*

$$0 \geq \lceil k\lambda \rceil - ky - \sum_{i\in S} \lceil k a_i \rceil (1-z_i) + \sum_{i\in N\setminus S} \lfloor k a_i \rfloor z_i.$$

This is the k-split c-strong inequality (6), *which is shown to be facet-defining for* conv($\mathcal{R}$) *in Ref.* [6] *under certain conditions. Then from the observation earlier, if $S' = \{i \in S : \bar r_i < \bar\rho\}$ and $T' = \{i \in T : (1-\bar r_i) < \bar\rho\}$, inequality*

$$\sum_{i\in S'} \bar r_i(1-z_i) + \sum_{i\in T'}(1-\bar r_i)z_i \geq \bar\rho \left(\lceil k\lambda\rceil - ky - \sum_{i\in S\setminus S'} \lceil k a_i\rceil (1-z_i) - \sum_{i\in S'} \lfloor k a_i\rfloor (1-z_i) + \sum_{i\in T'} \lceil k a_i\rceil z_i + \sum_{i\in N\setminus(S\cup T')} \lfloor k a_i\rfloor z_i \right)$$

dominates the k-split c-strong inequality.

Remark 7. *For the binary knapsack set $\mathcal{K}$ by letting $x = z$ and $y = 0$, inequality* (32) *reduces to*

$$\sum_{i\in S'} \bar r_i(1-z_i) + \sum_{i\in T'}(1-\bar r_i)z_i \geq \bar\rho \left(\bar\eta - \sum_{i\in S\setminus S'} \lceil \mu a_i\rceil (1-z_i) - \sum_{i\in S'} \lfloor \mu a_i\rfloor (1-z_i) + \sum_{i\in T'} \lceil \mu a_i\rceil z_i + \sum_{i\in T\setminus T'} \lfloor \mu a_i\rfloor z_i \right). \quad (34)$$

For $S \subseteq N$ such that $\bar a = \max_{i\in S} a_i \geq \lambda$, letting $T = \emptyset$ and $\mu = 1/\bar a$, we obtain $\bar\eta = 1$, $\bar\rho = \lambda/\bar a$, and consequently

$$\sum_{i\in S'} a_i(1-z_i) \geq \lambda \left(1 - \sum_{i\in S\setminus S'} (1-z_i) \right),$$

where $S' = \{i \in S : a_i < \lambda\}$. Therefore, for a minimal cover S, i.e., for S such that $a_i \geq \lambda$ for all $i \in S$, this inequality reduces to the the knapsack cover inequality

$$\sum_{i\in S} z_i \leq |S| - 1.$$

Then for a minimal cover S, inequality (34) *gives the lifted knapsack cover inequality*

$$\sum_{i\in T'} (\bar a - a_i + \lfloor a_i/\bar a \rfloor \bar a) z_i \geq \lambda \left(1 - \sum_{i\in S}(1-z_i) + \sum_{i\in T'} \lceil a_i/\bar a \rceil z_i + \sum_{i\in T\setminus T'} \lfloor a_i/\bar a \rfloor z_i \right),$$

or equivalently

$$\sum_{i \in S} z_i + \sum_{i \in T} \frac{\phi_{\bar{a}}(a_i, \lambda)}{\lambda} z_i \leq |S| - 1,$$

where $\phi_\alpha(a, b) = (b\lfloor a/\alpha \rfloor + (a - \alpha\lceil a/\alpha \rceil + b)^+)$. *Since* $\phi(\cdot, \lambda) \geq 0$, *the strongest inequality is obtained by letting* $T = N \backslash S$. *This is the lifted knapsack cover inequality with* MIR *lifting function* [3, 4, 17].

5. CONCLUDING REMARKS

We studied the polyhedral structure of the network design arc set with variable upper bounds. This set is a common substructure of formulations of network design problems with multicommodity fixed charges and/or combinatorial restrictions.

Several fundamental sets studied earlier independently are facial restrictions of its convex hull. Therefore, valid inequalities for the network design arc set with variable upper bounds generalize the inequalities known for these sets. In this study we identified facets that cut off all fractional extreme points of the continuous relaxation of the network design arc set with variable upper bounds. Interestingly, some of these facets are new for the earlier studied restrictions as well.

We do not have any computational experience with the new inequalities yet. However, their special cases for the restrictions mentioned in the introduction have shown to be computationally effective in earlier studies. For computational evidence we refer the reader to Gu et al. [11] for lifted 0–1 knapsack cover inequalities, to Gu et al. [13] for lifted flow cover inequalities, to Atamtürk and Rajan [6] for residual capacity inequalities, to Brockmüller et al. [10] for c-strong inequalities, and to Atamtürk and Rajan [6] for k-split c-strong inequalities.

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