Capacity Acquisition, Subcontracting, and Lot Sizing

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The fundamental question encountered in acquiring capacity to meet nonstationary demand over a multiperiod horizon is how to balance the trade-off between having insufficient capacity in some periods and excess capacity in others. In the former situation, part of the demand is subcontracted while, in the latter, capacity that has been paid for is rendered idle. Capacity and subcontracting decisions arise in many economic activities ranging from production capacity planning in semiconductor fabs to leasing communication networks, from transportation contracts to staffing of call centers. In this paper, we investigate the trade-offs between acquiring capacity, subcontracting, production, and holding inventory to satisfy nonstationary demand over a finite horizon. We present capacity acquisition models with holding and without holding inventory and identify *forecast-robust* properties of the models that restrict the dependence of optimal capacity decisions on the demand forecasts. We develop algorithms for numerous practical cost structures involving variable and fixed charges and prove that they all have polynomial time complexity. For models with inventory, we solve a sequence of *constant capacity lot-sizing and subcontracting subproblems*, which is also of independent interest.

(Algorithms; Extreme Points of Polyhedra; Nonspeculative Costs)

1. Introduction

Subcontracting orders to third parties has become a prevalent practice for improving an enterprise's agility. According to Day (1956), subcontracting refers to

the procurement of an item or service that a firm is normally capable of economic production in its own facilities and which requires the prime contractor to make specifications available to the supplier.

Firms subcontract orders to improve their response to demand that is highly variable. However, when to subcontract and how much to subcontract are not always easy-to-answer questions. Subcontracting decisions are intimately related to capacity acquisition decisions.

One traditional strategy of coping with the variability of demand is producing to build up inventory during periods with low demand and meeting the demand in excess of production capacity from inventory. However, the rise in the value of products makes it prohibitive to keep large quantities of inventory, and therefore the practice of subcontracting orders emerges as an effective instrument for reducing the capital investment in inventories.

Nonetheless, the benefits of subcontracting usually come at a high marginal cost compared with in-house production cost. This makes the *make-or-buy* decision challenging since otherwise one would subcontract all of the demand. An alternative way of meeting variable demand is acquiring a large enough

Figure 1 Trading Off Capacity, Subcontracting, Production, and Inventory Costs

capacity that will alleviate the need for subcontracting and holding excessive inventories. But then this capacity, which is typically very expensive, remains idle most of the time. Thus, the cost of capacity acquisition must be balanced with the savings in subcontracts and inventories.

Once capacity has been fixed for a time horizon, one can determine production levels, inventory build-ups, and subcontracting to minimize the relevant costs. In Figure 1, we illustrate producing to build up inventory during low-demand periods and subcontracting orders that cannot be met within capacity and from inventory. If early in-house production and holding inventory for a number of time periods cost more than subcontracting, then it is reasonable to subcontract part of the excess demand. Clearly, optimal quantity of subcontracts in each period is a function of the available production capacity, the demand pattern, and the costs associated with subcontracting, production, and holding inventory.

Therefore, whenever a firm has to answer a make-or-buy question, it faces a four-way trade-off between capacity allocation, subcontracting, production, and inventory. We study this important trade-off and develop analytical models and algorithms for simultaneously optimizing the interrelated capacity level, subcontracting, production, and inventory decisions to satisfy nonstationary demand over a finite horizon. Bringing considerations of subcontracting, production, and inventory costs into capacity decisions contrasts with the common practice of firms that

determines capacity level at the strategic level without taking into consideration tactical and operational concerns. As discussed in the sequel, such practice results in excessive inventories or underutilization of capacity.

1.1. Applications

The scope of our research is not limited to manufacturing alone because capacity acquisition, subcontracting, and inventory decisions arise in many economic activities ranging from production capacity planning in semiconductor fabs to leasing communication networks, and from transportation contracts to staffing of call centers. We provide a few applications below that serve as motivations for undertaking this study.

Manufacturing. In manufacturing, capacity and production/inventory decisions are traditionally made separately and in a hierarchical manner. Capacity is decided by top management and then production and inventory decisions are delegated to lower levels of the organization. However, in a recent OR practice article, Bradley and Arntzen (1999) argue that simultaneous consideration of capacity, production, and inventory decisions yields superior financial results, by means of two case studies, one performed at an electronics firm, the other at an office supplies firm. They state that although maximizing capacity utilization is a popular policy among executives and managers, it is not always the best financial strategy and can lead to inappropriate balance of inventory

and capacity:

A junior level buyer-planner with some on-the-job training can routinely purchase raw material inventories of equal or greater value than an acquisition of equipment. Similarly, production planners can construct a production schedule that results in WIP or FGI of the same magnitude as investment in new capacity with little or no managerial scrutiny. So, it is not surprising that large inventories and gross imbalances between production capacity and inventory are observed in practice. Although inventory levels were monitored on a daily basis in the plants we studied and high levels of inventory were criticized, plant managers had no alternative but to maintain high inventory levels in order to attain service levels. While various capital review boards successfully restricted investment on capacity, there was no analogous control on inventory (Bradley and Arntzen 1999, p. 796).

Logistics. Balancing the trade-off between capacity and subcontracting costs is a very important issue in logistics. In transportation contracts between a shipper and a carrier involving regular shipments of large quantities, the shipper is typically guaranteed a constant capacity. For instance, a certain volume of capacity is made available to the shipper each day over the duration of the contract. Deliveries in excess of the agreed upon capacity are charged at a higher rate as they require special handling either by the contracted carrier or another carrier. Therefore, excess deliveries may be treated as subcontracted demand.

The logistics capacity cost is usually a concave function of the volume that reflects quantity discounts. Since backlogs, i.e., orders delivered later than their due dates, may be viewed as carrying inventory from future periods rather than earlier periods, one faces the same trade-off between acquiring capacity, subcontracting, and backlogs as in the manufacturing setting. Contracting for logistics delivery capacity at a high-tech company was the original motivation for our study.

Telecommunication. The problem of balancing the trade-off between acquiring capacity and subcontracting arises in telecommunication as well as in other service industries where holding inventory is not an option. High costs associated with building,

maintaining, and upgrading telecommunication networks force many firms to lease capacity from network providers for their communication needs rather than build their own network. Commonly, the network provider allocates a fixed capacity to the lessee for the duration of the contract for which the lessee pays a fixed cost whether the capacity is fully utilized or not. A variable cost is charged for the demand beyond the contracted capacity, which may be viewed as subcontracted demand. The fixed capacity cost is, usually, a concave function of capacity that exhibits a substantial economy of scale.

Call Centers. In operating call centers the major cost factor is the size of the workforce. Hiring new employees entails training costs to build skills. Once the number of operators to employ has been decided, their salary has to be paid even if the anticipated call demand does not materialize. On days when capacity is tight it is possible to employ operators on an overtime basis. There is, however, a limit to the amount of overtime—up to 50% of regular time (Gans and Zhou 1999). Demand in excess of overtime capacity is routed to other call centers, hence is subcontracted. Hiring decisions are usually determined on a monthly basis. This decision is guided by detailed forecasts on the anticipated demand.

Terminology. In this paper, we use established terminology from manufacturing, such as *production* and *inventory*, although the scope of our study is not limited to manufacturing per se. While *capacity cost* may mean the cost of purchasing equipment or of setting up a production line in the context of manufacturing, it refers to the fixed cost incurred to acquire a guaranteed capacity in transportation and telecommunication contracts. Similarly, in the context of transportation contracts, production (subcontracting) cost refers to the variable cost incurred to ship part of an order within (in excess of) the guaranteed capacity, whereas the inventory cost refers to the back-order cost.

1.2. Relevant Literature

Extensive research has been done on certain aspects of the topics we study here. However, in the available literature the decisions on capacity acquisition,

subcontracting, production, and inventory are studied independently. We categorize the relevant literature into three groups according to their subject areas:

1. The first category pertains to production and inventory planning. Most of the research on production and inventory planning for nonstationary demand has concentrated on either uncapacitated models (e.g., Wagner and Whitin 1958, Zangwill 1966, Federgruen and Tzur 1991, Wagelmans et al. 1992, Aggarwal and Park 1993) or models with given capacities (e.g., Florian and Klein 1971, Baker et al. 1978, Florian et al. 1980, Bitran and Yanasse 1982, Van Hoesel and Wagelmans 1996) rather than treating capacity as a decision variable. Pochet and Wolsey (1993) study a lot-sizing problem in which capacity is an integer multiple of some basic batch size. In two case studies, Bradley and Arntzen (1999) demonstrate that firms can benefit tremendously by optimizing their capacity decisions and production/inventory decisions simultaneously rather than making first the capacity decision and then the inventory/production decisions.

2. The subject matter of the second category is capacity planning. Capacity expansion and contraction models in the literature focus on dynamic adjustment of capacity based on the costs of increasing and decreasing capacity. Research in this category include Love (1973), Luss (1982), Lee and Luss (1987), Li and Tirupati (1994). These models do not address subcontracting, production, or inventory decisions in capacity planning.

3. The third category addresses subcontracting. Kamien and Li (1990) present a game-theoretic multiperiod subcontracting model with given fixed capacities. Van Mieghem (1999) studies coordinating capacity, subcontracting, and production decisions for a single-period problem. Multiperiod-capacity allocation and subcontracting problems have been addressed for cellular manufacturing and for flexible manufacturing systems. The typical approach taken in this field has been to formulate the problems as integer programs and to develop practical heuristics to solve these formulations (e.g., Lee et al. 1997, Logendran and Puvanunt 1997, Logendran and Ramakrishna 1997).

1.3. Summary of Contributions

To the best of our knowledge this is the first paper that considers the trade-offs between capacity acquisition, subcontracting, production, and inventory decisions to meet nonstationary demand over a multiperiod horizon. We provide analytical models, structural results on the optimal solutions, and algorithms that simultaneously optimize these interrelated decisions. Moreover, we identify the *forecast-robust* features of the capacity acquisition models that restrict the dependence of optimal capacity decisions on demand forecasts.

In many cases, production and subcontracting costs involve fixed charges, such as machine set-up costs and order transaction costs. Formally, in time period t , the cost of producing x_t units $\bar{p}_t(x_t)$ equals $f_t + p_t x_t$ if $x_t > 0$ and 0 if $x_t = 0$, the cost of subcontracting y_t units $\bar{s}_t(y_t)$ equals $g_t + s_t y_t$ if $y_t > 0$ and 0 if $y_t = 0$. If the *fixed-charge* components f_t and g_t are zero, then production and subcontracting costs are *linear* functions. Inventory holding cost is assumed to be linear in quantity and h_t per unit in period t . Commonly, production and subcontracting have a *nonspeculative cost structure*, i.e., they satisfy $p_t + h_t \geq p_{t+1}$, $f_t \geq f_{t+1}$, $s_t + h_t \geq s_{t+1}$, and $g_t \geq g_{t+1}$ for all t . Notice that no relationship between production cost and subcontracting cost is assumed. The costs are nonspeculative, for instance, if variable and fixed charges are nonincreasing over time and holding cost is nonnegative. Nonspeculative cost structure arises naturally when time value of money is considered in multiperiod models.

We develop algorithms for the capacity acquisition and subcontracting problems with nonspeculative production and subcontracting costs with and without fixed charges as well as with (general) concave production and subcontracting costs. Unless it is stated otherwise, we assume that capacity acquisition cost is a concave function of capacity.

We show that when holding inventory is allowed, an optimal solution to capacity acquisition and subcontracting problem can be found by a search on capacity, which amounts to solving a polynomial number of constant capacity lot-sizing and subcontracting subproblems. This subproblem is interesting on its own right and has not been studied in the literature before. It generalizes the classical lot-sizing problem, since it incorporates subcontracting decisions into

Table 1 Computational Complexity of the Algorithms

Production and Subcontracting Cost Structure	Constant Capacity	Continuous Capacity with Concave Cost	Discrete Capacity with Concave Cost
Nonspeculative linear production Nonspeculative linear subcontracting	$O(n)$	$O(n^3)$	$O(n \min\{m, n^2\})$
Nonspeculative linear production costs Nonspeculative fixed-charge subcontracting	$O(n^2)$	$O(n^4)$	$O(n^2 \min\{m, n^2\})$
Nonspeculative fixed-charge production Nonspeculative linear subcontracting	$O(n^2)$	$O(n^5)$	$O(n^2 \min\{m, n^3\})$
Nonspeculative fixed-charge production Nonspeculative fixed-charge subcontracting	$O(n^3)$	$O(n^6)$	$O(n^3 \min\{m, n^3\})$
Concave production concave subcontracting	$O(n^5)$	$O(n^8)$	$O(n^5 \min\{m, n^3\})$

production planning. The subproblem and the overall capacity acquisition and subcontracting problem are solved in polynomial time for all of the above-mentioned cost structures.

In this paper we present *continuous capacity* models, in which capacity can be acquired at any nonnegative level. However, our results are also applicable for *discrete capacity* problems, that is, when capacity availability is restricted to a discrete set. The complexity of the algorithms presented in this paper is summarized in Table 1. Here, n denotes the number of time periods and m denotes the number of capacity levels for the discrete capacity case.

1.4. Outline

In §2, we present two models for capacity acquisition and subcontracting and discuss properties of their optimal solutions that motivate the algorithms in §3. Specifically, in §2.1, we give a capacity acquisition and subcontracting model with no inventory holding. We provide a closed-form expression of the optimal capacity when the cost functions are linear and discuss the sensitivity of optimal capacity level to forecast errors. We then present results on optimal capacity level when capacity cost is a piecewise-linear convex, a piecewise-linear concave, and, finally, a general concave function of capacity. In §2.2, we introduce a capacity acquisition and subcontracting model that allows holding inventory between time periods. We show that an optimal solution to this model can be found by solving a polynomial number of constant capacity lot-sizing and subcontracting subproblems.

In §3, we describe algorithms for solving the constant capacity lot-sizing and subcontracting subproblem with the cost structures described in Table 1. We conclude with future research directions in §4.

2. Capacity Decision Models

2.1. Capacity Acquisition and Subcontracting Without Inventory

As described for a telecommunication application in §1, holding inventory is not an option when demand in a time period must be satisfied in that period. For such problems, we first present a model with linear costs for finding an optimal capacity and optimal subcontracting levels. Later, we extend this model to incorporate more general cost functions.

Given a stream of demand forecasts $d_1, d_2, \dots, d_n$ for n time periods, we assume that demand for each period can be satisfied either through production in that period or by subcontracting in the same period. In the continuous capacity model, a constant production capacity can be acquired at a positive unit cost of nv for the n time periods. Let the unit production cost be p , the unit subcontracting cost be s . Let x_t denote the production quantity, y_t denote subcontracting quantity in time period t , and z denote the acquired capacity for production. Thus, the vector $(x, y, z) \in R^{2n+1}$ represents a solution to the problem. The Capacity-Subcontracting (CS) problem of minimizing total cost

is formulated as

$$\begin{aligned} \min \quad & nvz + \sum_{t=1}^n (px_t + sy_t) \\ \text{s.t.:} \quad & x_t + y_t = d_t \quad t = 1, \dots, n, \quad (1) \\ & x_t - z \leq 0 \quad t = 1, \dots, n, \quad (2) \\ & x_t, y_t \geq 0 \quad t = 1, \dots, n. \quad (3) \end{aligned}$$

CS can be solved efficiently by linear programming. We now characterize an optional solution explicitly and analyze the sensitivity of optimal capacity level to changes in the demand data. Without loss of generality, let d_t be positive and indexed in *nondecreasing* order. Also, we assume that $v + p < s$, since otherwise the problem has a trivial solution, in which capacity is zero and all of the demand is subcontracted in every time period.

PROPOSITION 1. *The i th-smallest demand, d_i with $i = \lceil (1 - (v/(s-p)))n \rceil$, is an optimal capacity level for CS.*

PROOF. Since $v + p < s$, for an optimal solution (x, y, z) , we have $z \geq d_1$; also as $p < s$, we have $x_t = \min\{d_t, z\}$ and $y_t = \max\{0, d_t - z\}$. Therefore for $d_t \leq z \leq d_{t+1}$ with $1 \leq t \leq n-1$, the optimal value as a function of the capacity is $\zeta(z) = (nv - (n-t)(s-p))z + pd_{1t} + sd_{(t+1)n}$, where $d_{jk} = \sum_{t=j}^k d_t$. So ζ is a piecewise-linear convex function of capacity over $[d_1, d_n]$. Observe that the derivative $\zeta' = (nv - (n-i)(s-p)) = 0$ for $i = (1 - (v/(s-p)))n$. If i is integral, then any capacity level in the interval $[d_i, d_{i+1}]$ is optimal. Otherwise, ζ' is negative in the interval $(d_{[i]}, d_{[i]})$ and positive in $(d_{[i]}, d_{[i]+1})$, thus the unique optimal capacity level is $d_{[i]}$. $\square$

Hence, given the sorted demand data, an optimal solution is found in constant time $O(1)$.

COROLLARY 2. *If capacity availability is restricted to a discrete set $\{c_1, c_2, \dots, c_m\}$ indexed in increasing order, then either c_j or c_{j+1} satisfying $c_j \leq d_{[i]} \leq c_{j+1}$ is optimal for $j = 1, 2, \dots, m-1$.*

The i th-Order Statistic and the Property of Forecast-Robustness of Optimal Capacity. Proposition 1 shows that an optimal capacity of CS is given by an i th-order statistic of the demand forecasts, where the *critical index* i equals $\lceil (1 - (v/(s-p)))n \rceil$.

Figure 2 Capacity Level Remains Optimal Under Large Forecast Errors

This property has important implications on using forecast data as it focuses attention on a certain aspect of the demand forecast. Consider the forecast and demand plot in Figure 2 and let the cost parameters (v, s, p) equal $(4, 10, 3)$, so that the critical index i equals 3. Given the large errors in the forecasts, at first sight there is little reason to expect that the optimal capacity level found using the forecasts would remain optimal for the realized demand shown in Figure 2. Indeed, the optimal capacity level remains unchanged as the third-order statistic of the forecast and of the actual demand realization coincide. Changes in the demand data will obviously affect the subcontracting and production decisions and hence the optimal value. However, the optimal capacity is much more robust compared with the other decisions. Since capacity decisions—in practice—are much harder to change compared with the production and subcontracting decisions, robustness of the optimal capacity is an appealing property of this model.

The current practice of forecasting demands over a multiperiod horizon places an emphasis on the variability of the demand over the entire horizon. However, since optimal capacity is an i th-order statistic of the demand, the variability of the entire demand stream has no effect whatsoever on the optimal capacity. Thus, the extent to which the demands that exceed the optimal capacity level vary, and the period in

which they occur, are irrelevant to the capacity decision. To acquire a forecast-robust capacity level, one would only need to estimate the i th-order statistic of the demand, which can be done more accurately than forecasting the demand for each period.

Data with Seasonality. Knowing that the optimal capacity is an i th-order statistic of the demand is particularly helpful when faced with data with seasonality. Suppose that in a 12-period cycle, 5 periods are classified as low season, 4 as midseason, and 3 as high season. If the critical index $\lceil (1 - (v/(s-p)))n \rceil$ is, say, 4, then it is better to forecast the low-season data as accurately as possible since the demand realization in midseasons and high seasons will most likely have no effect on the optimal capacity.

An intuitive explanation for this property of optimal capacity is that if the average marginal capacity cost v is close to the marginal net subcontracting cost $s - p$, then we cannot afford low utilization and optimal capacity will be low. Therefore, large errors in midseasons and high seasons will not influence the capacity decision. Conversely, if the average marginal capacity cost is much smaller than the marginal net subcontracting cost, then low-capacity utilization is preferable to avoid subcontracting and to ensure high optimal capacity. Therefore, large forecast errors in small and midseason periods will not influence the capacity decision.

Piecewise-Linear Convex Capacity Cost. In some cases the capacity cost presents diseconomies of scale. The previous discussion with linear capacity costs easily extends to the case with piecewise-linear convex capacity cost. Let the capacity cost $v(z)$ be a piecewise-linear convex function with slope v_j when capacity z is between breakpoints b_j and b_{j+1} with $b_j < b_{j+1}$ and $v_j < v_{j+1}$ for $j = 1, 2, \dots, k-1$, $b_1 = 0$, and $b_{k+1} = \infty$. Then, the optimal-value function $\zeta(z)$ is still piecewise-linear convex, but this time an optimal solution may also be one of the breakpoints. More precisely, if we define the critical index for interval j as $i(j) = \lceil n(1 - (v_j/(s-p))) \rceil$, we see that $i(1) \geq i(2) \geq \dots \geq i(k)$; that is, the cheaper the average marginal capacity cost, the higher the optimal capacity. Since $\zeta(z)$ is a piecewise convex function with breakpoints b_j and d_j , we have the following result.

PROPOSITION 3. *If capacity cost is a piecewise-linear convex function with breakpoints $b_1, b_2, \dots, b_{k+1}$, then the minimizer of ζ , i.e., the optimal capacity, is either the demand point $d_{i(j)}$ with $b_{j-1} \leq d_{i(j)} \leq b_j$ or the breakpoint b_j with $d_{i(j+1)} \leq b_j \leq d_{i(j)}$. If capacity availability is restricted to a discrete set $\{c_1, c_2, \dots, c_m\}$, then optimal capacity is either the largest capacity smaller than the continuous optimal capacity or the smallest capacity larger than the continuous optimal capacity.*

Since for each interval j , $d_{i(j)}$ is found in $O(1)$, an optimal capacity level can be found in a single pass along the breakpoints of the capacity cost function, thus in $O(k)$ complexity. However, since $i(1) \geq i(2) \geq \dots \geq i(k)$, the reader can verify that this complexity can be improved to $O(\log k)$ by binary search.

Piecewise-Linear Concave Capacity Cost. In many situations the capacity cost exhibits economies of scale. If the capacity-cost $v(z)$ is a piecewise-linear concave function with slope v_j when capacity z is between the breakpoints b_j and b_{j+1} with $b_j < b_{j+1}$ and $v_j > v_{j+1}$ for $j = 1, 2, \dots, k$ and $b_1 = 0$ and $b_{k+1} = \infty$, then the optimal-value function $\zeta(z)$ may have multiple local minima. However, $\zeta(z)$ is convex between breakpoints b_j and b_{j+1} . Therefore from the above discussion, we have the following characterization of local minima of ζ .

PROPOSITION 4. *Capacity z is a local minimum of ζ if and only if $z = d_{i(j)}$ and $b_j \leq d_{i(j)} \leq b_{j+1}$ for some $1 \leq j \leq k$.*

Evaluating the objective function of CS requires $O(n)$ for each capacity level, thus a naive way of computing the optimal capacity can be done in $O(nk)$ complexity. However, as the objective function is linear in production and subcontracting, the objective value of CS for all k capacity levels (actually also for all n demand points) can be computed in a total of $O(n)$ complexity by updating the objective value in increasing order of capacity levels. Therefore, an optimal solution to the capacity acquisition and subcontracting problem with piecewise-linear concave capacity cost can be found in $O(k + n)$ complexity.

General Concave Costs. Next we consider general concave cost functions for capacity, production, and subcontracting, which may also be time dependent. If a minimization problem of a concave function over a polyhedron has an optimal solution, then there exists an optimal solution among extreme points of the polyhedron. Therefore, when capacity, subcontracting, and production costs are nondecreasing concave functions, then there exists an optimal solution that is an extreme point solution of CS. A characterization of extreme-point solutions of CS reveals that it is sufficient to consider only the demands as candidates for optimal capacity level.

PROPOSITION 5. *If capacity, subcontracting, and production costs are nondecreasing concave functions, then the optimal capacity level is either zero or one of the n demand points d_t , $t = 1, 2, \dots, n$.*

PROOF. Since the cost functions are nondecreasing, there exists an optimal solution. An extreme-point solution of (1)–(3) is defined by $2n + 1$ linearly independent equations, $n + 1$ of which must be among (2)–(3). If $x_t - z = 0$ and $x_t = 0$ for some t , then z is zero. Otherwise let $\mathcal{J} = \{t : x_t = 0\}$. So $y_t = d_t > 0$ and $x_t < z$ for all $t \in \mathcal{J}$. Then the remaining $n + 1 - |\mathcal{J}|$ equations must come from $x_t - z \leq 0$ and $y_t \geq 0$ for periods with $x_t > 0$. Since there are $2(n - |\mathcal{J}|)$ such inequalities, there exists a period t with $x_t - z = 0$ and $y_t = d_t - x_t = 0$, thus $z = d_t$. $\square$

Assuming that each cost function can be evaluated in constant time, the objective value can be evaluated in $O(n)$ for each demand point and therefore, by Proposition 5, an optimal solution to capacity acquisition and subcontracting problem with concave objective function can be found in $O(n^2)$. A careful reader will observe that if subcontracting and production cost functions are *time-invariant*, i.e., $\bar{p}_t(x_t) = \bar{p}(x_t)$ and $\bar{s}_t(y_t) = \bar{s}(y_t)$, then the objective function can be evaluated in a total of $O(n)$ complexity for all n capacity levels. Hence, CS is solved in $O(n)$ if capacity, subcontracting, and production costs are nondecreasing time-invariant concave functions.

2.2. Capacity Acquisition and Subcontracting with Inventory

In this section we consider a more general model, where the demand in each period can be satisfied

either through production or by subcontracting in that period or from inventory. Let the capacity acquisition cost function $\bar{v}(\cdot)$ be a concave nondecreasing function of capacity that captures the economies of scale observed in practice. Let the production cost $\bar{p}_t(\cdot)$ and the subcontracting cost $\bar{s}_t(\cdot)$ be also concave nondecreasing functions. Let i_t be the decision variable denoting the inventory level at the end of time period t and $h_t (\geq 0)$ be the unit inventory holding cost in time period t . Then the capacity acquisition and subcontracting problem with inventory (CSI) can be stated as

$$\min \bar{v}(z) + \sum_{t=1}^n (\bar{p}_t(x_t) + \bar{s}_t(y_t) + h_t i_t)$$

$$\text{s.t.: } i_{t-1} + x_t + y_t - i_t = d_t \quad t = 1, 2, \dots, n, \quad (4)$$

$$x_t - z \leq 0 \quad t = 1, 2, \dots, n, \quad (5)$$

$$x_t, y_t, i_t \geq 0 \quad t = 1, 2, \dots, n, \quad (6)$$

$$i_0 = i_n = 0.$$

CSI is a nonconvex optimization problem and, in general, has multiple local minima. CSI is feasible, because y_t and z are not bounded from above. Since the feasible region of CSI is closed, the objective function is concave and nondecreasing in all components, and the variables are bounded from below, there exists an optimal solution to CSI. Furthermore, since CSI is a minimization problem of a concave function over a polyhedron with an extreme point (for instance $(i, x, y, z) = (0, 0, d, 0)$), there exists an optimal solution that is an extreme point of the feasible region of CSI. Therefore, when searching for an optimal solution, it is sufficient to consider only extreme point solutions of CSI. Even though CSI has exponentially many extreme points, we show that the number of distinct capacity levels in all extreme points of CSI is bounded above by $O(n^3)$. The notion of a block (or regeneration interval) will be useful for explaining this result.

DEFINITION 1. Given a feasible solution (i, x, y, z) to CSI, a *block* $[b, k]$ is a sequence of consecutive periods $b, b + 1, \dots, k$ with $i_{b-1} = i_k = 0$ and $i_t > 0$ for $t = b, b + 1, \dots, k - 1$.

PROPOSITION 6. *In every extreme-point solution of CSI with positive capacity, there exists a block in which there is no subcontracting and all of the production levels are either zero or at capacity.*

PROOF. Since CSI has $3n$ variables (excluding the fixed variables i_0 and i_n), an extreme-point solution of CSI is defined by $3n$ linearly independent inequalities binding at this point. As demand equalities (4) are linearly independent, the remaining $2n$ inequalities are among (5)–(6). Any feasible solution of CSI is associated with a partition into blocks, possibly a single block. Let (i, x, y, z) be an extreme-point solution with $z > 0$ and $\mathcal{B}$ be the set of blocks associated with (i, x, y, z) . The part of the constraints for block $[b_j, k_j]$, $j \in \mathcal{B}$ consists of

$$i_{t-1} + x_t + y_t - i_t = d_t \quad t = b_j, b_j + 1, \dots, k_j, \quad (7)$$

$$x_t - z \leq 0 \quad t = b_j, b_j + 1, \dots, k_j, \quad (8)$$

$$x_t \geq 0 \quad t = b_j, b_j + 1, \dots, k_j, \quad (9)$$

$$y_t \geq 0 \quad t = b_j, b_j + 1, \dots, k_j, \quad (10)$$

$$i_t \geq 0 \quad t = b_j, b_j + 1, \dots, k_j. \quad (11)$$

Since $i_{k_j} = 0$, $j \in \mathcal{B}$, together with the demand equalities (4), are linearly independent and $i_t > 0$ for $t = b_j, b_j + 1, \dots, k_j - 1$ for all $j \in \mathcal{B}$, the remaining $2n - |\mathcal{B}| + 1$ inequalities must come among Constraints (8)–(10). As $z > 0$, at most $k_j - b_j + 1$ of (8)–(9) are binding in block j . Therefore at most $2(k_j - b_j + 1)$ of Inequalities (8)–(10) are binding in each block $j \in \mathcal{B}$. Suppose the proposition is false. Then there is no block where this maximum is achieved. Thus, the total number of Inequalities (8)–(10) binding in all blocks is bounded above by $\sum_{j \in \mathcal{B}} (2k_j - 2b_j + 1) < 2n - |\mathcal{B}| + 1$. This contradicts with the assumption that (i, x, y, z) is an extreme point of CSI. $\square$

COROLLARY 7. *There are at most $n \binom{n}{2} + n + 1$ distinct capacity levels in all extreme points of CSI.*

PROOF. Since all the demand in a block $[b, k]$ described in Proposition 6 is satisfied by an integer number of time periods q with full-capacity production, the capacity level equals $z = 1/q \sum_{t=b}^k d_t$ and in the worst case may attain n distinct values. Also there are $\binom{n}{2}$ blocks given by pairs (b, k) of time periods for $1 \leq b < k \leq n$ and n singleton blocks, thus the number of distinct capacity levels in all extreme points of CSI is bounded above by $n \binom{n}{2} + n + 1$, including the zero capacity. $\square$

Corollary 7 restricts the search for an optimal capacity level to a discrete set whose cardinality is bounded above by $O(n^3)$. Therefore, CSI with concave capacity-acquisition, production, and subcontracting costs can be solved in $O(n^3 T)$, where T is the complexity of solving the problem for a given capacity. We call the subproblem obtained by fixing the capacity level in CSI the *constant capacity lot-sizing and subcontracting problem*. In §3, we give polynomial-time algorithms for solving this subproblem under numerous practical assumptions on the structure of production and subcontracting costs.

Nonspeculative Linear Production Costs. In a commonly occurring special case of the capacity acquisition and subcontracting problem, the production costs are nonspeculative linear as defined in §1. Subcontracting and capacity acquisition costs are general concave functions. The nonspeculative linear structure of the production cost allows us to further eliminate solutions of CSI from consideration. Since there is no incentive to produce early and keep inventory, in an optimal solution, one produces at full capacity in every time period with positive incoming inventory. In this case, we can strengthen Proposition 6 as follows.

PROPOSITION 8. *Under nonspeculative linear production costs, if there exists an optimal solution to CSI with positive capacity, then there also exists an optimal solution with a block in which there is no subcontracting and all of the production levels are at capacity.*

COROLLARY 9. *Under nonspeculative linear production costs, there exists an optimal capacity level among at most $n \binom{n}{2} + n + 1$ distinct values.*

By Corollary 9, it is sufficient to consider at most $n \binom{n}{2} + n + 1$ distinct values, given by the averages of the block demands and zero, as candidates for an optimal capacity. This reduces the number of capacity levels to consider by a factor of n , compared with the case where production costs are not nonspeculative linear. Thus CSI can be solved in $O(n^2 T)$, where T is the complexity of solving a constant capacity lot-sizing and subcontracting subproblem. In §§3.1 and 3.3, we discuss how to solve this subproblem for nonspeculative

linear production costs and subcontracting costs without and with fixed charges.

The second consequence of Proposition 8 is on the robustness of optimal capacity. Since optimal capacity is the average of the demand estimates of a block, its variance is at most the variance of demand estimates.

3. Constant Capacity Lot-Sizing and Subcontracting Problem

In §2.2, we showed that CSI reduces to solving a polynomial number of constant capacity lot-sizing and subcontracting subproblems. This subproblem is the restriction of CSI obtained by fixing the capacity variable to a positive value c . As the subproblem incorporates the subcontracting decisions to production and inventory holding decisions, it generalizes the lot-sizing problem. The Constant Capacity (CC) lot-sizing and subcontracting problem is formulated as

$$\begin{aligned} \min \quad & \sum_{t=1}^n (\bar{p}_t(x_t) + \bar{s}_t(y_t) + h_t i_t) \\ \text{s.t.:} \quad & i_{t-1} + x_t + y_t - i_t = d_t \quad t = 1, 2, \dots, n, \\ & x_t \leq c \quad t = 1, 2, \dots, n, \\ & x_t, y_t, i_t \geq 0 \quad t = 1, 2, \dots, n, \\ & i_0 = i_n = 0. \end{aligned}$$

CC can also be modeled as a network flow problem with concave costs. The underlying network of CC is depicted in Figure 3. In this figure, d is a dummy source node connected to production node p and subcontracting node s . The flow on arc (p, t) represents the production quantity in period t , whereas the flow on arc (s, t) represents the quantity subcontracted in period t . Inventory carried from period t to period $t+1$ is represented by the flow on arc $(t, t+1)$. Production arcs (p, t) , $1 \leq t \leq n$ have capacity c , whereas the rest of the arcs are uncapacitated. It is well known that the set of basic variables in an extreme point of a network flow problem corresponds to a spanning tree of its network. Therefore, the extreme points of CC can be characterized as follows.

Figure 3 Network Model of the Lot-Sizing and Subcontracting Problem

PROPOSITION 10. A feasible point of (i, x, y) of CC is an extreme point if and only if every block of (i, x, y) has either at most one period t with $0 < x_t < c$ (fractional production) or at most one period t with $y_t > 0$ (subcontracting), but not both.

3.1. Nonspeculative Linear Production and Subcontracting Costs

In this section, we discuss how to solve the constant capacity lot-sizing and subcontracting problem (CC) with nonspeculative linear costs. Here, production and subcontracting cost functions are of the form $\bar{p}_t(x_t) = p_t x_t$, $\bar{s}_t(y_t) = s_t y_t$, respectively, and satisfy $p_t + h_t \geq p_{t+1}$ and $s_t + h_t \geq s_{t+1}$. Note that no relationship between production and subcontracting cost is assumed.

Under nonspeculative linear costs, it is sufficient to consider only solutions in which, for any time period t , (a) the outgoing inventory is positive only if the subcontracting level is zero, because $s_t + h_t \geq s_{t+1}$ (and there is no fixed charge for subcontracting, i.e., $g_{t+1} = 0$), and (b) the incoming inventory is positive only if the production level is at capacity, because of (a) and $p_t + h_t \geq p_{t+1}$ (and there is no fixed charge for production, i.e., $f_{t+1} = 0$).

PROPOSITION 11. Under nonspeculative linear production and subcontracting costs, there exists an optimal

solution (i, x, y) of CC that satisfies

- (a) $i_t y_t = 0$ for all $t = 1, 2, \dots, n$, and
- (b) $i_{t-1}(c - x_t) = 0$ for all $t = 1, 2, \dots, n$.

Consequently, the search for an optimal solution is restricted to solutions in which subcontracting occurs only in the last period of a block and all but the first period of a block have production at full capacity. Below we state an algorithm that solves the constant capacity lot-sizing and subcontracting problem with nonspeculative linear costs by finding a minimum-cost solution among the solutions that satisfy Proposition 11. Algorithm 1 finds an optimal trade-off between producing early to carry inventory and subcontracting in each period by a single forward pass along the time periods. In this algorithm, we keep a stack S to store the time periods with excess capacity. Using a stack is appropriate when dealing with nonspeculative costs because, for each time period, it facilitates access to the closest period with excess capacity in constant time. Algorithm 1 uses $\text{Push}(S, t)$ operation to insert element t into stack S and $\text{Pop}(S)$ to return and delete the last inserted element from S . If S is empty, then $\text{Pop}(S)$ returns 0. $\text{Purge}(S)$ operation empties stack S . For details of stack operations refer to Cormen et al. (1990). Throughout this paper, we let $H_t = \sum_{i=0}^{t-1} h_i$ for $t = 1, 2, \dots, n$, with $h_0 = 0$. Observe that all H_t for $t = 1, 2, \dots, n$ can be computed in $O(n)$. Also note that, given x and y , inventory levels i_t for $t = 1, 2, \dots, n$ can be computed in $O(n)$ as well.

Algorithm 1 Solving CC with nonspeculative linear costs in a single forward pass.

```

1: Let  $S \leftarrow \emptyset$ .
2: for  $t = 1$  to  $n$  do
3:    $x_t \leftarrow 0$ .  $\text{Push}(S, t)$ .
4:   while  $d_t > 0$  do
5:      $k \leftarrow \text{Pop}(S)$ .  $\backslash\backslash$   $k = 0$  if  $S$  is empty.
6:     if  $k = 0$  or  $p_k + H_t - H_k \geq s_t$  then
7:        $y_t \leftarrow d_t$ ;  $d_t \leftarrow 0$ .  $\text{Purge}(S)$ .  $\backslash\backslash$  Subcontract.
8:     else
9:        $\Delta \leftarrow \min\{c - x_k, d_t\}$ ;  $x_k \leftarrow x_k + \Delta$ ;  $d_t \leftarrow d_t - \Delta$ .
        $\backslash\backslash$  Produce.
10:    if  $x_k < c$  then
11:       $\text{Push}(S, k)$ .
12:    end if
13:  end while
14: end for
15: end for

```

THEOREM 12. Algorithm 1 solves the constant capacity lot-sizing and subcontracting problem (CC) with nonspeculative linear production and subcontracting costs in $O(n)$.

PROOF. We first argue that the solution (i, x, y) produced by Algorithm 1 is indeed optimal. Solution (i, x, y) satisfies Proposition 11, because (a) after a subcontracting period (e.g., $y_t > 0$), the stack S is purged and future demand is met in future periods, and (b) production in an early period is considered only when the production in the current period is at full capacity. Also notice that since at each iteration either Line 7 or Line 9, but not both, is executed, (i, x, y) satisfies Proposition 10.

For contradiction, suppose (i, x, y) is not optimal. Let $(\bar{i}, \bar{x}, \bar{y})$ be an optimal solution that satisfies Propositions 10 and 11 and differs from (i, x, y) in the smallest number of periods among all optimal solutions. Let t be the earliest time period that (i, x, y) differs from $(\bar{i}, \bar{x}, \bar{y})$. Consider the following cases:

Case 1. Suppose $x_t = \bar{x}_t$. Then we must have $y_t \neq \bar{y}_t$; say, $y_t < \bar{y}_t$. By Proposition 11, t is the last period of a block in $(\bar{i}, \bar{x}, \bar{y})$. Then $\bar{i}_t = 0$, but this means there must be a period earlier than t where the solutions differ, which is a contradiction with the choice of t .

Case 2. Suppose $x_t < \bar{x}_t$. Then t must be the first period of a block in (i, x, y) , since otherwise, by Proposition 11, $x_t = \bar{x}_t = c$. If t is not the first period of a block in $(\bar{i}, \bar{x}, \bar{y})$, then $\bar{x}_t = c$ and there exists a period earlier than t , where the solutions differ, which contradicts the choice of t . Therefore, t is the first period of a block for both solutions. Consider the earliest period $t' \geq t$, where the solutions differ other than x_t and $\bar{x}_t$ (if $y_t \neq \bar{y}_t$, then $t' = t$). From feasibility of (i, x, y) , t' is in the same block with t in $(\bar{i}, \bar{x}, \bar{y})$. Suppose $y_{t'} \neq \bar{y}_{t'}$. If $y_{t'} < \bar{y}_{t'}$, then by Proposition 11, for $(\bar{i}, \bar{x}, \bar{y})$, t' is the last period of the block that t is in. However, this contradicts the feasibility of (i, x, y) . Therefore, we must have $y_{t'} > \bar{y}_{t'} \geq 0$. From Line 6 of Algorithm 1, $p_t + H_{t'} - H_t \geq s_t$ holds. If $p_t + H_{t'} - H_t > s_t$, the solution $(\bar{i}, \bar{x}, \bar{y})$ can be improved by producing less in t and subcontracting more in t' , which contradicts its optimality. If $p_t + H_{t'} - H_t = s_t$, by reducing $\bar{x}_t$ to x_t and subcontracting $x_t - \bar{x}_t$ more in t' , we obtain another optimal solution that differs from (i, x, y) in smaller number of periods than $(\bar{i}, \bar{x}, \bar{y})$ does, which contradicts the choice of $(\bar{i}, \bar{x}, \bar{y})$.

Now suppose $y_{t'} = \bar{y}_{t'}$, so $x_{t'} \neq \bar{x}_{t'}$. Observe that in this case $y_{t'} = \bar{y}_{t'} = 0$, since otherwise $x_{t'} = \bar{x}_{t'} = c$. Because t' is in the same block with t in $(\bar{i}, \bar{x}, \bar{y})$, we have $\bar{x}_{t'} = c$ and $x_{t'} < c$. Let k be the last period of the block that t' is in for $(\bar{i}, \bar{x}, \bar{y})$. By feasibility of (i, x, y) , we have $t' < k$. From Proposition 11, we have $i_{t'-1} = 0$. Since also $\bar{x}_j = c$ for all $j = t', t' + 1, \dots, k$, again by feasibility of (i, x, y) , there exists $y_\ell > 0$ for $t' < \ell \leq k$. So, from Line 6 of Algorithm 1, $p_{t'} + H_\ell - H_{t'} \geq s_\ell$ holds, but this contradicts the choice of $(\bar{i}, \bar{x}, \bar{y})$.

Case 3. Suppose $x_t > \bar{x}_t$. Argument is symmetric to the one presented in Case 2.

To see that the running time of the algorithm is linear in n , observe that each time period is pushed into the stack and removed from the stack at most once, with the exception of push back into stack in Line 11. However, a time period k is pushed back into the stack only when $x_k < c$. This happens only when d_t becomes zero, in which case the iteration counter t is increased. So the number of push backs in Line 11 is at most n throughout the algorithm. $\square$

COROLLARY 13. *There exists an $O(n^3)$ algorithm to solve the capacity acquisition and subcontracting problem (CSI) with nonspeculative linear production and subcontracting costs.*

PROOF. Follows from Theorem 12 and Corollary 9. $\square$

3.2. Nonspeculative Fixed-Charge Production, Nonspeculative Linear Subcontracting Costs

Here we consider the case where the production cost has a fixed-charge component, that is, the cost of producing x_t units $\bar{p}_t(x_t)$ equals $f_t + p_t x_t$ if $x_t > 0$ and 0 if $x_t = 0$. We let the cost of subcontracting y_t units $\bar{s}_t(y_t) = s_t y_t$ and assume that the production and subcontracting costs are nonspeculative, i.e., $p_t + h_t \geq p_{t+1}$, $f_t \geq f_{t+1}$ and $s_t + h_t \geq s_{t+1}$.

In the presence of fixed charges, CC is not a linear program anymore. Due to fixed charges of production, it may not be profitable to produce in every time period with positive demand. Since the objective function is concave, there exists an extreme-point optimal solution of CC. Therefore, due to Proposition 10, we consider only solutions in which each block has either a single fractional production or a single

subcontracting period, but not both. Using the non-speculative structure of the objective function, it is possible to further eliminate solutions from consideration. There exists an optimal solution in which, for any time period t , (a) the outgoing inventory is positive only if the subcontracting level is zero, because $s_t + h_t \geq s_{t+1}$ (and there is no fixed charge for subcontracting, i.e., $g_{t+1} = 0$) and (b) the production level is fractional only if the incoming inventory is zero, because of (a) and $p_t + h_t \geq p_{t+1}$. In other words, it is sufficient to consider solutions in which subcontracting occurs in the last period of a block and fractional production occurs in the first period of a block. These observations are formalized in Proposition 14.

PROPOSITION 14. *Under nonspeculative fixed-charge production costs and nonspeculative linear subcontracting costs, there exists an optimal solution (i, x, y) of CC that satisfies*

- (a) $i_t y_t = 0$ for all $t = 1, 2, \dots, n$, and
- (b) $i_{t-1} x_t (c - x_t) = 0$ for all $t = 1, 2, \dots, n$.

Since a feasible solution of CC can be decomposed into blocks and the cost of the feasible solution is simply the sum of the costs associated with these blocks, to find an optimal solution to CC, we can use the dynamic programming recursion originally suggested by Florian and Klein (1971) for the capacitated lot-sizing problem.

Let v_{bk} be the value of an optimal solution for block $[b, k]$, which we refer to as the *cost of block $[b, k]$* hereafter, and let v_k be the optimal value of CC restricted to periods $\{1, 2, \dots, k\}$. An optimal solution to CC can be found by computing v_n using the dynamic programming (DP) recursion

$$v_0 = 0,$$

$$v_k = \min_{1 \leq b \leq k} \{v_{b-1} + v_{bk}\}, \quad k = 1, 2, \dots, n.$$

The complexity of DP is $O(n^2)$ plus the complexity of computing the costs of all blocks. In the remainder of this section, we show that computing all block costs can also be done in $O(n^2)$, which leads to the conclusion that the constant capacity lot-sizing and subcontracting problem with nonspeculative production costs with fixed charges and nonspeculative linear subcontracting costs can be solved in $O(n^2)$.

We next discuss the computation of v_{bk} , the cost of block $[b, k]$. Let v_{bk}^p be the cost of the block under the assumption that there is no subcontracting period in this block and v_{bk}^s be the cost of the block under the assumption that there is a subcontracting period in this block. Proposition 10 implies that in the former case, there is at most one fractional production period in the block, whereas in the latter case there is no fractional production period but a single subcontracting period. Clearly, $v_{bk} = \min\{v_{bk}^p, v_{bk}^s\}$.

We first explain how to compute v_{bk}^p . Since in this case there is no subcontracting in block $[b, k]$, the block cost is computed as in the lot-sizing problem. For a lot-sizing problem with linear holding costs, demand for any time period t can be assumed to satisfy $d_t \leq c$ (Bitran and Yanasse 1982). Otherwise, since in every feasible solution at least $d_t - c$ units of inventory are carried from period $t - 1$ to t , one can construct an equivalent instance by repeating inductively: For any t such that $d_t > c$, increase d_{t-1} by $d_t - c$ and add the constant term $h_{t-1}(d_t - c)$ to the objective. If $d_1 > c$, then the problem is infeasible. Notice that this argument does not hold when demand can be satisfied also by subcontracting.

For the lot-sizing problem with nonspeculative production costs with fixed charges, it is sufficient to consider solutions in which, for any time period t , (a) the production level is fractional only if the incoming inventory is zero, because $p_{t-1} + h_{t-1} \geq p_t$, and (b) the production level is positive only if the incoming inventory is sufficient to satisfy the demand in that period, because $d_t \leq c$, $p_t + h_t \geq p_{t+1}$, and $f_t \geq f_{t+1}$.

PROPOSITION 15. (Bitran and Yanasse 1982) *A constant capacity lot-sizing problem with nonspeculative production costs has an optimal solution that satisfies*

- (a) $i_{t-1}x_t(c - x_t) = 0$ for all $t = 1, 2, \dots, n$, and
- (b) $i_{t-1} < d_t$ for all $t = 1, 2, \dots, n$ such that $i_{t-1}x_t > 0$.

Indeed, Bitran and Yanasse (1982) show the above described structure of optimal solutions for a special case of nonspeculative costs, where fixed and variable charges of production are nonincreasing over time and holding cost is nonnegative. However, their argument is valid for nonspeculative production costs as well.

Proposition 15 implies that an *inventory minimal* solution is optimal for block $[b, k]$. That is,

$$x_t = \begin{cases} \epsilon & \text{if } t = b, \\ c & \text{if } i_{t-1} < d_t \text{ and } i_t = i_{t-1} + x_t - d_t, \\ 0 & \text{otherwise,} \end{cases} \quad (12)$$

where $\epsilon = d_{bk} - qc$ is the fractional production with $0 \leq \epsilon < c$ and q integer. The cost of a block under the assumption that there is no subcontracting in the block can be computed in $O(n)$ steps by building an inventory minimal solution (12) in a forward pass along the block. Since there are $O(n^2)$ possible blocks, computing all block costs can be done in $O(n^3)$. The next proposition leads to a more efficient computation of the block costs.

PROPOSITION 16. *Let (x^a, i^a) and (x^b, i^b) be inventory minimal solutions for blocks $[a, k]$ and $[b, k]$ with $a < b$, respectively. Then $x_t^a = x_t^b$ and $i_{t-1}^a = i_{t-1}^b$ for all $t = b + 1, b + 2, \dots, k$.*

PROOF. Let $\epsilon_a = d_{ak} - cq_a$ with $0 \leq \epsilon_a < c$ and q_a integer, and $\epsilon_b = d_{bk} - cq_b$ with $0 \leq \epsilon_b < c$ and q_b integer. Thus, ϵ_a is the fractional production in period a for block $[a, k]$ and ϵ_b is the fractional production in period b for block $[b, k]$. To prove the result, it is enough to show that $i_{b-1}^a = \epsilon_b$. Observe that $d_{a(b-1)} = \epsilon_a - \epsilon_b + c(q_a - q_b)$ and since $d_{ak} \geq d_{bk}$, we have $q_a \geq q_b$. Consider the following two cases:

Case 1. ($\epsilon_a \geq \epsilon_b$). In this case, $c(q_a - q_b) \leq d_{a(b-1)} < c(q_a - q_b + 1)$. Since $d_{a(b-1)} - c(q_a - q_b) = \epsilon_a - \epsilon_b \leq \epsilon_a$, in addition to ϵ_a , one requires $q_a - q_b$ full-production periods to meet the demand $d_{a(b-1)}$ in an inventory minimal solution for block $[a, k]$. Thus $i_{b-1}^a = c(q_a - q_b) + \epsilon_a - d_{a(b-1)} = \epsilon_b$.

Case 2. ($\epsilon_a < \epsilon_b$). Here, $c(q_a - q_b - 1) < d_{a(b-1)} < c(q_a - q_b)$. Since $d_{a(b-1)} - c(q_a - q_b - 1) = c + \epsilon_a - \epsilon_b > \epsilon_a$, in addition to ϵ_a , one requires again $q_a - q_b$ full-production periods to cover $d_{a(b-1)}$ in an inventory minimal solution for block $[a, k]$. Hence $i_{b-1}^a = \epsilon_b$ in this case as well. $\square$

According to Proposition 16, given an inventory minimal solution for block $[b, k]$, an inventory minimal solution for block $[b - 1, k]$ can be computed in constant time, because optimal production decisions remain invariant for periods $b + 1, b + 2, \dots, k$.

Algorithm 2 computes all v_{bk}^p with $1 \leq b \leq k$ for fixed k in $O(n)$ in a single backward pass along the time periods. Note that in practice we may stop computations if $i_{b+1} = 0$, since in this case optimal solution corresponding to v_{jk} for $j \leq b$ consists of more than one block. In Algorithm 2, stack S keeps the periods for which production decisions need to be updated in the next iteration. Thus the DP algorithm that uses Algorithm 2 for computing block costs solves the constant capacity lot-sizing problem with nonspeculative costs in $O(n^2)$. This is an improvement over Bitran and Yanasse's (1982) algorithm by a factor of n .

Now we consider subcontracting decisions within a block and compute v_{bk}^s , the cost of block $[b, k]$ under the assumption that there is a subcontracting period in this block. Recall that due to Proposition 10, we only need to consider solutions with no fractional production period and a single subcontracting period. Moreover, from Proposition 14, it is sufficient to consider solutions with subcontracting in the last period of block $[b, k]$. Then, from Bitran and Yanasse (1982) we may assume that $d_t \leq c$ for all $t = b, b+1, \dots, k-1$ and since the production cost is nonspeculative with fixed charges, we construct an inventory minimal solution (12) to cover the demands for periods $b, b+1, \dots, k-1$. Demand of period k can be satisfied either by subcontracting or by full production, possibly in early periods, which is decided by checking if $f_t + (p_t + H_k - H_t)c < s_k c$ for each $t, b \leq t \leq k-1$ with no production in the inventory minimal solution. Thus a solution with subcontracting is found in $O(n)$ per block and, hence, all v_{bk}^s is computed in $O(n^3)$. One can, however, reduce this complexity to $O(n^2)$ by using an optimal solution for block $[b, k]$ to compute an optimal solution for block $[b, k+1]$. Algorithm 3 computes all v_{bk}^s with $b \leq k \leq n$ for fixed b in $O(n)$ in a single forward pass along the time periods.

THEOREM 17. *The dynamic programming (DP) algorithm, which uses Algorithms 2 and 3 for computing the block costs, solves the constant capacity (CC) lot-sizing and subcontracting problem with nonspeculative fixed-charge production costs and nonspeculative linear subcontracting costs in $O(n^2)$.*

COROLLARY 18. *There exists an $O(n^5)$ algorithm to solve the capacity acquisition and subcontracting problem*

(CSI) with nonspeculative production costs and nonspeculative linear subcontracting costs.

PROOF. Follows from Theorem 17 and Corollary 7. $\square$

Algorithm 2 Computing all v_{bk}^p with $1 \leq b \leq k$ in a single backward pass.

```

1: Let  $S \leftarrow \emptyset$ ,  $cost \leftarrow 0$ .
2: for  $b = k$  to 1 do
3:    $\epsilon \leftarrow d_{bk} - \lfloor d_{bk}/c \rfloor c$ .
4:   if  $d_b < \epsilon$  or  $\epsilon = 0$  then
5:      $x_b \leftarrow \epsilon$  ( $x_b \leftarrow c$  if  $\epsilon = 0$ );  $i_b \leftarrow x_b - d_b$ .
        $\backslash \backslash$  Fractional production.
6:    $cost \leftarrow cost + f_b + p_b x_b + h_b i_b$ .
7:   while  $S \neq \emptyset$  do
8:      $t \leftarrow \text{Pop}(S)$ .
9:     if  $i_{t-1} < d_t$  then
10:       $x_t \leftarrow c$ ;  $i_t \leftarrow i_{t-1} + c - d_t$ .  $\backslash \backslash$  Produce at capacity.
11:       $cost \leftarrow cost + f_t + p_t c + h_t i_t$ .
12:     else
13:       $x_t \leftarrow 0$ ;  $i_t \leftarrow i_{t-1} - d_t$ .  $\backslash \backslash$  No production.
14:       $cost \leftarrow cost + h_t i_t$ .
15:     end if
16:   end while
17:    $v_{bk}^p \leftarrow cost$ .
18:    $cost \leftarrow cost - f_b - p_b x_b - h_b i_b$ .
19: else
20:    $v_{bk}^p \leftarrow \infty$ .
21: end if
22:    $\text{Push}(S, b)$ .
23: end for
```

3.3. Nonspeculative Linear Production Nonspeculative Fixed-Charge Subcontracting Costs

In this section, we consider the constant capacity lot-sizing and subcontracting problem with linear production costs and subcontracting costs with fixed charges, that is, $p_t(x_t) = p_t x_t$ and $s_t(y_t)$ equals $g_t + s_t y_t$ if $y_t > 0$ and equals 0 if $y_t = 0$. We assume that both production and subcontracting costs are nonspeculative, so that they satisfy $p_t + h_t \geq p_{t+1}$, $s_t + h_t \geq s_{t+1}$, and $g_t \geq g_{t+1}$.

Since subcontracting cost has a fixed-charge component, in some periods it may be economical to subcontract the demand for more than one period to avoid frequent fixed charges. Thus $i_t y_t = 0$ does not hold for optimal solutions in this section. However, since $p_t + h_t \geq p_{t+1}$ (and there is no fixed charge for production, i.e., $f_{t+1} = 0$), it is sufficient to consider solutions in which, if in one period both production and outgoing

inventory are positive, production equals capacity in the next period.

Algorithm 3 Computing all v_{bk}^s with $b \leq k \leq n$ in a single forward pass.

```

1: Let  $S \leftarrow \emptyset$ ,  $cost \leftarrow 0$ ,  $i_{b-1} = 0$ .
2: for  $k = b$  to  $n$  do
3:    $y_k = d_k - i_{k-1}$ ;
4:   if  $y_k < 0$  then
5:      $v_{bk}^s \leftarrow \infty$ .
6:   else
7:      $cost \leftarrow cost + s_k y_k$ .
8:   end if
9:   Push( $S$ ,  $k$ ).
10:  while  $S \neq \emptyset$  and  $c \leq y_k$  do
11:     $t \leftarrow \text{Pop}(S)$ .
12:    if  $f_t + (p_t + H_t - H_t)c < s_k c$  then
13:       $x_t \leftarrow c$ ;  $y_k \leftarrow y_k - c$ .  $\backslash\backslash$  Move  $c$  units from
        subcontracting to production if profitable.
14:       $cost \leftarrow cost + f_t + (p_t + H_t - H_t - s_k)c$ .
15:    else
16:       $v_{bk}^s \leftarrow cost$ .
17:      Push( $S$ ,  $t$ ). Break loop.
18:    end if
19:  end while
20:  while  $S \neq \emptyset$  and  $0 < y_k$  do
21:     $t \leftarrow \text{Pop}(S)$ .
22:     $x_t \leftarrow c$ ;  $y_k \leftarrow y_k - c$ .  $\backslash\backslash$  Move  $c$  units from
        subcontracting to production for feasibility.
23:     $cost \leftarrow cost + f_t + (p_t + H_t - H_t - s_k)c$ .
24:  end while
25:   $i_k = -y_k$ ;  $cost \leftarrow cost + i_k(h_k - s_k)$ .
    $\backslash\backslash$  If  $i_k \leq 0$ , then  $v_{bt}^s \leftarrow \infty$  for all  $k < t \leq n$ .
26: end for

```

PROPOSITION 19. Under nonspeculative linear production costs, there exists an optimal solution (i, x, y) of CC that satisfies $x_t i_t (c - x_{t+1}) = 0$ for all $t = 1, 2, \dots, n-1$.

Thus, in a block any period with production is followed by periods with full-capacity production. If there is no subcontracting in a block, then except for the first period of the block, which may possibly have a fractional production, all periods have production at full capacity. Then, v_{bk}^p , the cost of blocks $[b, k]$ $b = 1, 2, \dots, k$ under the assumption that there is no subcontracting period in them, can be computed with a single backward pass algorithm similar to Algorithm 2, except that one produces in every time period. As in Algorithm 2, this can be done in $O(n)$ for all $1 \leq b \leq k$ for fixed k . Thus all v_{bk}^p values are computed in $O(n^2)$.

Next, we describe how to compute v_{bk}^s , the cost of block $[b, k]$ under the assumption that there is a subcontracting period in this block. By Propositions 10 and 19, it is sufficient to consider solutions in which the first production in the block (which is not necessarily in the first period of the block) is at full capacity and all subsequent periods in the block also have full-capacity production. Consider the following two cases:

Case 1. There is no production in the first period of the block. Then from feasibility, the subcontracting is done in the first period of block $[b, k]$. Let $t^* = t^*(b)$ be the earliest time period that satisfies $b \leq t^* \leq k$ and $s_b + H_{t^*} - H_b > p_{t^*}$. Then, the block is structured as follows: We subcontract in period b . Periods b through $t^* - 1$ ($\geq b$) have no production and periods t^* through k have production at full capacity. For $t^* \leq t$, let the $d(t^*, t)$ be the deficit in period t after producing at capacity in all periods $t^*, t^* + 1, \dots, t$. Thus $d(t^*, t) = d_{t^*t} - (t - t^* + 1)c$. To meet the demand of the block, subcontracting quantity in period b must equal $d_{bt^*-1} + d(t^*, k)$. Observe that all such solutions for blocks $[b, k]$ for $k = b, b+1, \dots, n$ can be found with a single forward pass along the time periods in $O(n)$. Also notice that if $d(t^*, k) < \max_{t^* \leq t \leq k} \{d(t^*, t)\}$, then this solution is infeasible.

Case 2. There is production in the first period of the block. Then, by Propositions 10 and 19 there is full production in all periods of the block. Observe that to feasibly supply the deficit of block $[b, k]$ by subcontracting, the subcontracting quantity must be exactly $d(b, k)$. Furthermore, since subcontracting costs are nonspeculative, subcontracting is done in the earliest period t^* in this block with $d(b, t^*) > 0$ in an optimal solution. Clearly, the cost of all such blocks $[b, k]$ $k = b, b+1, \dots, n$ for fixed b can be computed with a single forward pass in $O(n)$. As before, the described solution is feasible if and only if $d(b, k) = \max_{b \leq t \leq k} \{d(b, t)\}$. So v_{bk}^s is the minimum of the objective values of the solutions found in these two cases. Hence, all $v_{bk} = \min\{v_{bk}^p, v_{bk}^s\}$ can be computed in $O(n^2)$.

THEOREM 20. The dynamic programming (DP) algorithm solves the constant capacity (CC) lot-sizing and subcontracting problem with nonspeculative linear production

costs and nonspeculative fixed-charge subcontracting costs in $O(n^2)$.

COROLLARY 21. *There exists an $O(n^4)$ algorithm to solve the capacity acquisition and subcontracting problem (CSI) with nonspeculative linear production costs and nonspeculative fixed-charge subcontracting costs in $O(n^4)$.*

PROOF. Follows from Theorem 20 and Corollary 9. $\square$

3.4. Nonspeculative Fixed-Charge Production and Subcontracting Costs

Here, we consider the more general case of the lot-sizing and subcontracting problem in which both production and subcontracting costs have fixed charges. So $\bar{p}_t(x_t)$ equals $f_t + p_t x_t$ if $x_t > 0$ and 0 if $x_t = 0$, and $\bar{s}_t(y_t)$ equals $g_t + s_t y_t$ if $y_t > 0$ and 0 if $y_t = 0$. Production and subcontracting costs are nonspeculative, thus $p_t + h_t \geq p_{t+1}$, $f_t \geq f_{t+1}$, $s_t + h_t \geq s_{t+1}$, and $g_t \geq g_{t+1}$. We also assume that subcontracting is more expensive than production in every time period, that is, $f_t \leq g_t$ and $p_t \leq s_t$ for all t .

Since the cost structure studied in this section differs from the one in §3.2 for only subcontracting, computing block costs v_{bk}^p under the assumption that there is no subcontracting period in a block is done as in §3.2 by Algorithm 2.

In the remaining of this section we establish that v_{bk}^s , the cost of block $[b, k]$ under the assumption that there is a subcontracting period in block $[b, k]$, can be computed in $O(n)$. Let

$$\rho(t) = \min_{1 \leq j \leq n} \{j : f_j + c(p_j + H_t - H_j) \leq cs_t\},$$

where $\rho(t) = \infty$ if there exists no $1 \leq j \leq n$ that satisfies $f_j + c(p_j + H_t - H_j) \leq cs_t$. Thus, if $\rho(t) \leq t$, then $\rho(t)$ is the earliest time period that one would rather produce at full-capacity c and carry c units of inventory from period $\rho(t)$ to period t than increase subcontracting by c units in period t . And, if $\rho(t) > t$, then $\rho(t)$ is the earliest time period that one would rather produce at full-capacity c than increase subcontracting by c units in period t and carry c units of inventory from period t to period $\rho(t)$. We also define

$$\sigma(t) = \max_{1 \leq j \leq t} \{j : d_{jt} \geq c\},$$

where $\sigma(t) = 0$ if there exists no $1 \leq j \leq t$ that satisfies $d_{jt} \geq c$.

Recall that due to Proposition 10, we only need to consider solutions with no fractional production period and a single subcontracting period. We now present properties of an optimal solution of CC that will be useful for computing v_{bk}^s efficiently.

PROPOSITION 22. *There exists an optimal solution (i, x, y) of CC that satisfies the following properties for a block $[b, k]$ with $y_t > 0$, where $b \leq t \leq k$.*

- (a) If $0 < y_t \leq c$, then $x_t = c$.
- (b) If $t < \rho(t)$, then $x_j = 0$ for $j = t, t+1, \dots, \rho(t)-1$.
- (c) If $\rho(t) \leq \sigma(k)$, then $x_{\sigma(k)} = c$.
- (d) If $\rho(t) \leq t$ and $y_t \geq c$, then $x_j = c$ for $j = \rho(t), \rho(t)+1, \dots, t$.

PROOF. The validity of these properties is established by simple contrapositive arguments.

(a) If $x_t < c$, then we can construct a cheaper solution by moving $\min\{y_t, c - x_t\}$ units from subcontracting to production in period t since $p_t \leq s_t$ and $f_t \leq g_t$. Note that if $f_t + cp_t > cs_t$, then we may have $y_t > c$ and $x_t = 0$ in an optimal solution.

(b) Suppose not. From Proposition 6 we can assume that $x_j = c$. Since $t < \rho(t)$, $cp_j + f_j > c(s_t + H_j - H_t)$ for all $t \leq j \leq \rho(t)-1$. Thus subcontracting c units more in period t and not producing in period j is feasible and cheaper.

(c) Suppose not. From Proposition 6 we can assume that $x_{\sigma(k)} = 0$. Let l be the last time period before $\sigma(k)$ with production. If $l > t$, then $i_{l-1} + c = d_{l(\sigma(k)-1)} + d_{\sigma(k)k} \geq d_{l(\sigma(k)-1)} + c$. Thus we can move the production in period l to period $\sigma(k)$ without violating feasibility, and doing so reduces the cost as production cost is nonspeculative. Otherwise, $i_{l-1} + y_t + x_t = d_{l(\sigma(k)-1)} + d_{\sigma(k)k} \geq d_{l(\sigma(k)-1)} + c$. If $y_t < c$ (by assumption $y_t > 0$), from (a) $x_t = c$ and we can move the production from t to $\sigma(k)$; else we can reduce the subcontract by c units and produce c units in time period $\sigma(k)$ without violating feasibility, and doing so reduces the total cost since $\rho(t) \leq \sigma(k)$.

(d) Suppose not. Since $\rho(t) \leq j$, it is cheaper to produce c units in period j and to subcontract c units less in period t . $\square$

Assume that the subcontracting period is t for some $b \leq t \leq k$. We now construct an optimal solution for

block $[b, k]$ in linear time under this assumption. Notice that it is enough to consider $t \leq \max_{b \leq j \leq k} \{j : d_{jk} > c\}$ for subcontracting, because otherwise $y_t \leq c$ and by Proposition 22 (a) $x_t + y_t > c \geq d_{tk}$, which is infeasible since $i_k = 0$. Therefore we may assume that $t \leq \sigma(k)$. Set $k' \leftarrow k$. The solution will be constructed depending on where $\rho(t)$ lies in relation to t and $\sigma(k')$. Consider the following three cases:

Case 1. $\sigma(k') < \rho(t)$. All feasible solutions for this block have $x_j = 0$ for $j = \sigma(k') + 1, \sigma(k') + 2, \dots, k'$. By Proposition 22 (b), we have $x_j = 0$ for all $j = t, t + 1, \dots, \sigma(k')$. Thus we construct an inventory minimal solution from b up to $t - 1$ and let $y_t = d_{tk'} - i_{t-1}$. If $d_{tk'} - i_{t-1} \leq c$, then there is no optimal solution with $y_t > 0$ for this block.

Case 2. $t \leq \rho(t) \leq \sigma(k')$. By Proposition 22 (c) we let $x_{\sigma(k')} = c$. If $t = \sigma(k')$, then we construct an inventory minimal solution from b up to $t - 1$ and let $y_t = d_{tk'} - i_{t-1} - c$. However, if $t < \sigma(k')$, then set $d_{\sigma(k')-1} \leftarrow d_{(\sigma(k')-1)k'} - c$ and $k' \leftarrow \sigma(k') - 1$. So we are either in Case 1 or Case 2 again.

Case 3. $\rho(t) \leq t$. Since $t \leq \sigma(k')$, as in Case 2, we let $x_{\sigma(k')} = c$. If $t = \sigma(k')$, then we construct an inventory minimal solution from b up to $t - 1$ and let $y_t = d_{tk'} - i_{t-1} - c$. By Proposition 22 (d), if $x_j = 0$ for $\rho(t) \leq j \leq t$ and $y_t \geq c$, then produce in period j and reduce subcontract in period t by c units. However, if $t < \sigma(k')$, set $d_{\sigma(k')-1} \leftarrow d_{(\sigma(k')-1)k'} - c$ and $k' \leftarrow \sigma(k') - 1$. We are in Case 3 again.

It is easy to see that the construction described above of an optimal solution for block $[b, k]$ under the assumption that subcontracting period is t requires $O(n)$. Since there are at most n periods to consider for subcontracting in a block, v_{bk}^s can be computed in $O(n^2)$. However, we can improve this complexity to $O(n)$ by using the nonspeculative structure of the subcontracting costs. Since $s_t + h_t \geq s_{t+1}$, we have $\rho(t-1) \leq \rho(t)$, which implies that the solution constructed with $y_t > 0$ can be used to construct a solution with $y_{t-1} > 0$ rather than starting from scratch. Based on this observation, Algorithm 4 computes v_{bk}^s by updating only the necessary production and subcontracting decisions in a backward pass along the periods of a block in $O(n)$. In this algorithm, stack S is used to keep the time periods with no production, whereas stack T is used to keep the time periods with

full production. $\text{Top}(T)$ refers to the last inserted element in stack T . In one forward pass, Algorithm 4 first finds an inventory minimal solution with no subcontracting. Then in a single backward pass, it considers subcontracting at each time period, while updating only the necessary production periods at each iteration in the backward pass. Observe that all $\rho(t)$ and $\sigma(t)$ can be computed in $O(n)$. Since $v_{bk} = \min\{v_{bk}^p, v_{bk}^s\}$, the block cost is computed in $O(n)$.

Algorithm 4 Computing v_{bk}^s in one forward and one backward pass.

```

1:  $S \leftarrow \emptyset$ .  $\backslash\backslash$  Stack for periods with no production in
   an inventory minimal solution.
2:  $T \leftarrow \emptyset$ .  $\backslash\backslash$  Stack for periods with full production in
   an inventory minimal solution.
3:  $i_{b-1} \leftarrow 0$ ;  $\text{cost} \leftarrow 0$ ;  $v_{bk}^s \leftarrow \infty$ .
4: for  $t = b$  to  $k$  do
5:    $\text{Push}(S, t)$ .
6:    $x_t \leftarrow 0$ ;  $i_t \leftarrow i_{t-1} - d_t$ .
7:   while  $(t < k \text{ and } i_t \leq 0)$  or
      $(t = k \text{ and } S \neq \emptyset \text{ and } i_t \leq -c)$  do
8:      $j \leftarrow \text{Pop}(S)$ ;  $\text{Push}(T, j)$ .
9:      $x_j \leftarrow c$ ;  $i_t \leftarrow i_t + c$ .
10:     $\text{cost} \leftarrow \text{cost} + f_j + (p_j + H_t - H_j)c$ .
11:  end while
12: end for
13:  $k' \leftarrow k$ .
14: for  $t = k$  to  $b$  do
15:   while  $i_{t-1} - d_t < i_t$  and  $\text{Top}(T) < \rho(t)$  do
16:      $t \leftarrow \text{Pop}(T)$ .
17:      $x_j \leftarrow 0$ ,  $i_t \leftarrow i_t - c$ .
18:      $\text{cost} \leftarrow \text{cost} - f_j - (p_j + H_t - H_j)c$ .
19:   end while
20:    $y_t \leftarrow -i_t$ .  $\backslash\backslash$  Subcontract to cover part of  $d_t$ .
21:    $v_t \leftarrow \text{cost}$ .
22:   while  $j < \sigma(k')$  and  $\rho(t) \leq \sigma(k')$  do
23:      $x_{\sigma(k')} \leftarrow c$ .  $d_{\sigma(k')-1} \leftarrow d_{(\sigma(k')-1)k'} - c$ .
24:      $v_t \leftarrow v_t + (p_{\sigma(k')} + H_{\sigma(k')} - H_t)c$ .
25:      $k' \leftarrow \sigma(k') - 1$ .
26:   end while
27:    $y_t \leftarrow y_t + d_{tk'}$ ;  $v_t \leftarrow v_t + g_t + s_t y_t$ .  $\backslash\backslash$  If  $y_t = 0$  then  $v_t \leftarrow \infty$ .
28:   If  $v_t < v_{bk}^p$  then
29:      $v_{bk}^p \leftarrow v_t$ .
30:   end if
31: end for

```

THEOREM 23. *The dynamic programming algorithm DP, which uses Algorithms 2 and 4 for computing the block costs, solves the constant capacity lot-sizing and subcontracting problem (CC) with nonspeculative fixed-charge production and subcontracting costs in $O(n^3)$.*

Figure 4 Layered Shortest Path Network for Computing v_{bk}^s

COROLLARY 24. *There exists an $O(n^6)$ algorithm to solve the capacity acquisition and subcontracting problem (CSI) with nonspeculative fixed-charge production and subcontracting costs.*

PROOF. Follows from Theorem 23 and Corollary 7. $\square$

3.5. Concave Production and Subcontracting Costs

In this section, we discuss how to compute the optimal block cost v_{bk} when the costs associated with subcontracting and production are general concave functions. First suppose that there is no subcontracting period in block $[b, k]$. Then, from Proposition 10, there is at most one fractional production in the block with production level equal to $\epsilon = d_{bk} - qc$, where $0 \leq \epsilon < c$ and q integer. So q is the number of time periods that one must have production at full capacity in an extreme-point solution of CC for this block when there is no subcontracting in the block. Clearly such a solution is feasible if and only if $q \leq k - b$ for $\epsilon > 0$ and $q \leq k - b + 1$ for $\epsilon = 0$. Therefore, the cumulative amount produced up to any period in the block equals one of the, at most $2n + 1$, quantities $0, \epsilon, c, c + \epsilon, 2c, \dots, qc, qc + \epsilon$. Then v_{bk}^p , the cost of block $[b, k]$ under the assumption that there is no subcontracting period, can be computed in $O(n^2)$ by solving a shortest path on a directed network with nodes representing possible cumulative production quantities as shown by Florian and Klein (1971).

Now suppose there is a period with subcontracting in block $[b, k]$. Let $\epsilon = d_{bk} - qc$, where $0 \leq \epsilon < c$ and q integer. When subcontracting is considered to satisfy the demand, $q \leq k - b + 1$ does not necessarily hold for a feasible solution. From Proposition 10, extreme points of CC have no fractional production in this block. So the subcontracted amount equals $\epsilon + (q - r)c$ in an extreme-point solution, where $r (\leq \min\{q, k - b + 1\})$ is the number of periods with production equal to full capacity. Therefore, the cumulative amount produced and subcontracted up to any period in the block equals one of the, at most $2n + 1$, quantities $0, c, 2c, 3c, \dots, lc, (q - l)c + \epsilon, (q - l + 1)c + \epsilon, \dots, qc + \epsilon$, where $l = \min\{q, k - b + 1\}$.

To find an optimal solution for block $[b, k]$ among all possible extreme-point solutions with subcontracting in $[b, k]$ efficiently, we construct a layered network with one layer for each time period as in Figure 4. Nodes in a layer of the network represent possible cumulative production and subcontracted quantities up to the corresponding time period. From layer t to layer $t + 1$ arcs $(ic, (i + 1)c)$ and $(ic + \epsilon, (i + 1)c + \epsilon)$ represent production at full capacity, and arcs (ic, ic) and $(ic + \epsilon, ic + \epsilon)$ represent no activity in period t . Arcs $(ic, jc + \epsilon)$ with $i \leq j \leq q$ represent either subcontracting $(j - i)c + \epsilon$ units in period t or producing at full capacity and subcontracting $(j - i - 1)c + \epsilon$ units in period t if $i < j$, whichever is cheaper. Then v_{bk}^s is

computed by solving a shortest path problem on this network.

THEOREM 25. *The dynamic programming algorithm DP solves the constant capacity lot-sizing and subcontracting problem (CC) with concave production and subcontracting costs in $O(n^5)$.*

PROOF. For a block $[b, k]$, v_{bk}^s is computed by solving a shortest path problem on an acyclic network with $O(n^2)$ nodes and $O(n^3)$ arcs, which can be done in $O(n^3)$. Since v_{bk}^p is computed in $O(n^2)$ as shown by Florian and Klein (1971), computing v_{bk}^s is the bottleneck for finding the block cost $v_{bk} = \min\{v_{bk}^p, v_{bk}^s\}$. There are $O(n^2)$ blocks and since given the block costs, the optimal value v_n can be computed in $O(n^2)$ by the dynamic programming algorithm DP, the constant capacity lot-sizing and subcontracting problem (CC) with concave costs is solved $O(n^5)$. $\square$

COROLLARY 26. *There exists an $O(n^8)$ algorithm to solve the capacity acquisition and subcontracting problem (CSI) with concave production and concave subcontracting cost functions.*

PROOF. Follows from Theorem 25 and Corollary 7. $\square$

4. Conclusions

This paper provides a framework for comprehensively balancing the trade-offs between acquiring capacity and subcontracting, production, and inventory decisions. In all prior work these decisions are analyzed in isolation without considering the cross-impacts on other affected decisions. The classes of cost functions we study, in relation to capacity acquisition and subcontracting models, are appropriate for a large variety of practical situations. For all these cost functions the paper presents polynomial time algorithms.

An important aspect of our study is the feature of “forecast robustness” of the optimal solutions. The implications of forecast robustness are discussed in some detail for the model trading off capacity and subcontracting. We plan to expand the study of the nature of forecast-robustness and its implications for other scenarios as well.

Numerous other practical and important problems related to capacity acquisition and subcontracting

remain to be investigated. In particular, we plan to focus future research on problems involving dynamic adjustment of capacity—expansion or contraction of current capacity in relation to subcontracting, production, and inventory decisions. Another important area is the effect of having multiple products competing for the same capacity resources on the optimal solutions. Here an open problem is how to best allocate capacity among different products with varying subcontracting, production, and inventory costs.

Acknowledgments

The authors are grateful to Prof. Candace A. Yano for several insightful discussions. Alper Atamtürk was supported, in part, by NSF Grants DMI-9908705 and DMI-0070127. Dorit Hochbaum was supported, in part, by NSF Grants DMI-9713482 and DMI-9908705.

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Accepted by Thomas M. Liebling; received June 2000. This paper was with the authors 1 month for 1 revision.