Math. Program., Ser. A (2010) 122:1–20
DOI 10.1007/s10107-008-0239-4

FULL LENGTH PAPER

Conic mixed-integer rounding cuts

Alper Atamtürk · Vishnu Narayanan

Received: 24 March 2007 / Accepted: 6 May 2008 / Published online: 21 August 2008

Abstract A conic integer program is an integer programming problem with conic constraints. Many problems in finance, engineering, statistical learning, and probabilistic optimization are modeled using conic constraints. Here we study mixed-integer sets defined by second-order conic constraints. We introduce general-purpose cuts for conic mixed-integer programming based on polyhedral conic substructures of second-order conic sets. These cuts can be readily incorporated in branch-and-bound algorithms that solve either second-order conic programming or linear programming relaxations of conic integer programs at the nodes of the branch-and-bound tree. Central to our approach is a reformulation of the second-order conic constraints with polyhedral second-order conic constraints in a higher dimensional space. In this representation the cuts we develop are linear, even though they are nonlinear in the original space of variables. This feature leads to a computationally efficient implementation of nonlinear cuts for conic mixed-integer programming. The reformulation also allows the use of polyhedral methods for conic integer programming. We report computational results on solving unstructured second-order conic mixed-integer problems as well as mean–variance capital budgeting problems and least-squares estimation problems with binary inputs. Our computational experiments show that conic mixed-integer rounding cuts are very effective in reducing the integrality gap of continuous relaxations of conic mixed-integer programs and, hence, improving their solvability.

This research has been supported, in part, by Grant # DMI0700203 from the National Science Foundation.

A. Atamtürk (✉) · V. Narayanan
Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720-1777, USA
e-mail: atamturk@berkeley.edu

V. Narayanan
e-mail: vishnu@ieor.berkeley.edu

Keywords Conic programming · Integer programming · Mixed-integer rounding · Branch-and-cut algorithms

Mathematics Subject Classification (2000) 90C11 · 90C25 · 90C57

1 Introduction

In the last two decades there have been major advances in our capability of solving linear integer programming problems. Strong cutting planes obtained through polyhedral analysis of problem structure contributed to this success substantially by strengthening linear programming relaxations of integer problems. Powerful cutting planes based on simpler substructures of problems have become standard features of leading optimization software packages. The employment of such structural cuts improve the performance of the linear integer programming solvers dramatically.

On another front, since the late 1980s we have experienced significant advances in convex optimization, particularly in conic optimization. Starting with Nesterov and Nemirovski [29–31], polynomial interior point algorithms that have earlier been developed for linear programming have been generalized to conic optimization problems such as convex quadratically constrained quadratic programs and semidefinite programs.

Development of efficient algorithms and publicly available software, e.g., CSDP [11], DSDP [9], SDPA [45], SDPT3 [42], SeDuMi [39], for conic optimization spurred many optimization and control applications in diverse areas ranging from medical imaging to signal processing, from robust portfolio optimization to truss design. Commercial software vendors, e.g., ILOG and MOSEK, have responded to the demand for solving (continuous) conic optimization problems by including solvers for second-order cone programming (SOCP) in their recent versions.

Unfortunately, the phenomenal advances in continuous conic programming and linear integer programming have so far not translated to improvements in conic integer programming, i.e., integer programs with conic constraints. Solution methods for conic integer programming are still limited to branch-and-bound algorithms that solve their continuous relaxations at the nodes of the search tree. In terms of development, conic integer programming today is where linear integer programming was before 1980s when solvers relied on pure branch-and-bound algorithms without the use of any cuts for improving the continuous relaxations at the nodes of the search tree.

Here we attempt to improve the solvability of conic integer programs. We develop general purpose cuts that can be incorporated into branch-and-bound solvers for conic integer programs. Toward this end, we describe valid cuts for the second-order conic mixed-integer constraints (defined in Sect. 2). The choice of second-order conic mixed-integer constraint for this study is based on

1. the existence of many important applications modeled with such constraints,
2. the availability of efficient and stable solvers for their continuous SOCP relaxations, and

3. the fact that one can form SOCP reformulations and/or relaxations for more general conic programs, which make the cuts given here widely applicable to conic integer programming.

Outline. In Sect. 2, we introduce conic mixed-integer programming, briefly review the relevant literature and explain our approach for generating valid cuts. In Sect. 3, we describe conic mixed-integer rounding cuts for second-order conic mixed-integer programming. In Sect. 4, we summarize our computational results with the cuts for solving unstructured second-order conic mixed-integer problems as well as mean–variance capital budgeting problems and least-squares estimation problems with binary inputs. Finally, we conclude in Sect. 5.

2 Conic integer programming

A cone $\mathcal{K}$ is a subset of $\mathbb{R}^m$ such that $x \in \mathcal{K}$ implies $\lambda x \in \mathcal{K}$ for all $\lambda \geq 0$. Let $\mathcal{K} \subseteq \mathbb{R}^m$ be a pointed, closed, convex cone with nonempty interior (for instance, the nonnegative orthant $\mathbb{R}^m_+$). These conditions on $\mathcal{K}$ imply that the binary relation $\preccurlyeq_{\mathcal{K}}$ on $\mathbb{R}^m$ defined as

$$x \preccurlyeq_{\mathcal{K}} y \;\Leftrightarrow\; y - x \in \mathcal{K}$$

is a partial order. For $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ consider the system of inequalities $Ax \preccurlyeq_{\mathcal{K}} b$. For instance, if $\mathcal{K} = \mathbb{R}^m_+$, then $Ax \preccurlyeq_{\mathcal{K}} b$ is the usual system of linear inequalities $Ax \leq b$, defining the feasible region of a linear program (LP).

A conic program (CP) is an optimization problem of a linear function over a subset of $\mathbb{R}^n$ defined with constraints $Ax \preccurlyeq_{\mathcal{K}} b$. Thus, by definition, it generalizes linear programming. Starting with Nesterov and Nemirovski [29–31], polynomial-time interior point algorithms for LP have been generalized to conic programming. The other generalization of linear programming is the linear integer programming (LIP) obtained by adding discrete variables. Even though LIP is $\mathcal{NP}$-hard [33], branch-and-cut algorithms that employ strong cuts are effectively used for finding provably optimal solutions to large-scale instances of many practical problems [27]. A conic integer program (CIP) is an integer program with conic constraints; thus, it is the natural generalization of CP and LIP.

A particularly interesting (nonlinear) cone with many applications in engineering and science is the *second-order* (or *Lorentz*) *cone*

$$\mathcal{Q}^{m+1} := \left\{(t, t_o) \in \mathbb{R}^m \times \mathbb{R} : \| \, t \, \| \leq t_o\right\},$$

where $\| \cdot \|$ denotes the Euclidean norm. In this paper, we focus on second-order conic integer programming. However, as one can reformulate or relax more general conic programs to second-order conic programs [17] our results are indeed applicable more generally.

Specifically, a *second-order conic mixed-integer program* is an optimization problem of the form

$$
\text{(SOCMIP)} \quad \begin{aligned} & \min\ cx + ry \\ & \text{s.t. } \| A_i x + G_i y - b_i \| \leq d_i x + e_i y - h_i, \quad i = 1, 2, \ldots, k \\ & \quad x \in \mathbb{Z}^n, \ y \in \mathbb{R}^p. \end{aligned} \tag{1}
$$

Here A_i, G_i, b are rational matrices with m_i rows, and c, r, d_i, e_i are rational row vectors of appropriate dimension, and h_i is a rational scalar. Each constraint of SOCMIP can be equivalently stated as $(A_i x + G_i y - b_i, d_i x + e_i y - h_i) \in \mathcal{Q}^{m_i+1}$. We assume that bounds on the variables are included in the constraints of SOCMIP.

For $n = 0$, SOCMIP reduces to a second-order cone program (SOCP), which is a generalization of linear programming as well as convex quadratically constrained quadratic programming. In addition to $n = 0$, if $G_i = 0$ for all i, then SOCP reduces to linear programming; if $e_i = 0$ for all i, then it reduces to (convex) quadratically constrained quadratic program (QCQP) after squaring the constraints. Convex optimization problems with more general norms, fractional quadratic functions, hyperbolic functions and others can be formulated as an SOCP. We refer the reader to [3,8,12,20,32] for a detailed exposure to conic optimization and many applications of SOCP.

2.1 Relevant literature

There has been significant work on deriving conic (in particular semidefinite) relaxations for (linear) combinatorial optimization problems [2,15,21] for obtaining stronger bounds than the ones given by their natural linear programming relaxations. We refer the reader to Goemans [14] for a survey on this topic. Our interest here, however, is not to find conic relaxations for linear integer problems, but for conic integer problems.

Clearly any method for general nonlinear integer programming applies to conic integer programming as well. Reformulation–Linearization Technique (RLT) of Sherali and Adams [34] initially developed for linear 0–1 programming has been extended to nonconvex optimization problems [36]. Stubbs and Mehrotra [37,38] generalize the lift-and-project method [6] of Balas et al. for 0–1 mixed convex programming. See also Balas [5] and Sherali and Shetti [35] on disjunctive programming methods. Kojima and Tunçel [18] give successive semidefinite relaxations converging to the convex hull of a nonconvex set defined by quadratic functions. Lasserre [19] describes a hierarchy of semidefinite relaxations of nonlinear 0–1 programs. Common to all of these general approaches is a hierarchy of convex relaxations in higher dimensional spaces whose size grows exponentially with the size of the original formulation. Therefore, using such convex relaxations in higher dimensions is impractical except for very small instances. On the other hand, projecting these formulations to the original space of variables is also difficult except for certain special cases. A partial convexification based on a small number of disjunctions is a practical tradeoff.

Another stream of research is the development of branch-and-bound algorithms for nonlinear integer programming based on linear outer approximations [1,10,40,41,43]. The advantage of linear approximations is that they can be solved fast; however, the bounds from linear approximations may not be strong. In the case of conic programming, and in particular second-order cone programming, existence of efficient algorithms permits the use of continuous conic relaxations at the nodes of the branch-and-bound tree, although the lack of effective warm-starts is a significant disadvantage.

Çezik and Iyengar [13] develop valid inequalities for conic integer sets. Given $S = \{(x, y) \in \mathbb{Z}^n \times \mathbb{R}^m : Ax + Gy \preccurlyeq_{\mathcal{K}} b\}$, their approach is to write a valid linear inequality

$$\lambda' Ax + \lambda' Gy \leq \lambda' b \quad \text{for some} \quad \lambda \succeq_{\mathcal{K}^*} 0, \tag{2}$$

where $\mathcal{K}^*$ is the dual cone of $\mathcal{K}$, for the continuous relaxation of S and then apply Chvátal–Gomory (CG) integer rounding cuts or mixed-integer rounding (MIR) cuts [27] to this linear inequality, as appropriate. However, as also noted by the authors, it is not clear how to pick λ for implementing such a cut generation approach. Çezik and Iyengar do not report an implementation of this approach. Note that the resulting MIR cuts for (2) would be linear in (x, y). For the mixed-integer case, the convex hull of feasible points is not polyhedral and has curved boundary (see Fig. 2 in Sect. 3). Therefore, nonlinear inequalities may be more effective for describing or approximating the convex hull of solutions. Recently, Atamtürk and Narayanan [4] describe lifting of conic inequalities for conic mixed-integer programming.

2.2 A new approach

Our approach for deriving valid inequalities for SOCMIP is to reformulate second-order conic constraints in a higher dimensional space that leads to a natural decomposition into simpler polyhedral sets and to analyze each of these sets. Specifically, given a second-order conic constraint

$$\| Ax + Gy - b \| \leq dx + ey - h \tag{3}$$

and the corresponding second-order conic mixed-integer set

$$C := \left\{ x \in \mathbb{Z}_+^n,\ y \in \mathbb{R}_+^p : (x, y) \text{ satisfies } (3) \right\},$$

by introducing auxiliary variables $(t, t_o) \in \mathbb{R}^{m+1}$, we reformulate (3) as

$$t_o \leq dx + ey - h \tag{4}$$
$$t_i \geq |a_i x + g_i y - b_i|, \quad i = 1, \ldots, m \tag{5}$$
$$t_o \geq \| t \|, \tag{6}$$

where a_i and g_i denote the ith rows of matrices A and G, respectively. Observe that each constraint (5) is indeed a second-order conic constraint as $(a_i x + g_i y - b_i, t_i) \in \mathcal{Q}^{1+1}$, yet polyhedral. Consequently, we refer to a constraint of the form (5) as a *polyhedral second-order conic constraint*.

Breaking (3) into polyhedral conic constraints allows us to exploit the implicit polyhedral set for each term in a second-order cone constraint. Cuts obtained for C in this way are linear in (x, y, t); however, they are nonlinear in the original space of (x, y). We will illustrate this point in the next section.

Our approach extends the successful polyhedral method for linear integer programming, in which one studies the facial structure of simpler substructures to second-order conic integer programming. To the best of our knowledge such an analysis for second-order conic mixed-integer sets has not been done previously.

3 Conic mixed-integer rounding

For a mixed integer set $X \subseteq \mathbb{Z}^n \times \mathbb{R}^p$, we use $\text{relax}(X)$ to denote its continuous relaxation in $\mathbb{R}^n \times \mathbb{R}^p$ obtained by dropping the integrality restrictions and $\text{conv}(X)$ to denote the convex hull of X. In this section we describe conic mixed-integer rounding cuts for conic mixed-integer programming. We will first present the inequalities on a simple case with a single integer variable; subsequently, we will derive the general inequalities.

3.1 The simple case

Let us first consider the mixed-integer set

$$S_0 := \left\{(x, y, w, t) \in \mathbb{Z} \times \mathbb{R}^3_+ \ : | \ x + y - w - b \ | \le t \right\}. \tag{7}$$

defined by a simple, yet non-trivial polyhedral second-order conic constraint with one integer variable. The continuous relaxation $\text{relax}(S_0)$ has four extreme rays: $(1, 0, 0, 1)$, $(-1, 0, 0, 1)$, $(1, 0, 1, 0)$, and $(-1, 1, 0, 0)$, and one extreme point: $(b, 0, 0, 0)$, which is infeasible for S_0 if $f := b - \lfloor b \rfloor > 0$. It is easy to see that if $f > 0$, $\text{conv}(S_0)$ has four extreme points: $(\lfloor b \rfloor, f, 0, 0)$, $(\lfloor b \rfloor, 0, 0, f)$, $(\lceil b \rceil, 0, 1 - f, 0)$ and $(\lceil b \rceil, 0, 0, 1 - f)$. Figure 1 illustrates S_0 for the restriction $y = w = 0$.

Proposition 1 *The simple conic mixed-integer rounding inequality*

$$(1 - 2f)(x - \lfloor b \rfloor) + f \le t + y + w \tag{8}$$

is valid for S_0 *and cuts off all points in* $\text{relax}(S_0) \setminus \text{conv}(S_0)$.

Proof We first show validity of (8) for S_0. Consider the *base inequality*

$$|x + y - w - b| \le t \tag{9}$$

of S_0. For $x = \lfloor b \rfloor - \alpha$ with $\alpha \ge 0$, (9) becomes $t \ge |y - w - f - \alpha|$ and (8) becomes $t \ge -y - w + f - \alpha(1 - 2f)$. Now since $|y - w - f - \alpha| - (-y - w + f - \alpha(1 - 2f)) = 2 \max\{y - f - \alpha f, w + \alpha(1 - f)\} \ge 0$, (8) is implied by (9) and $w \geq 0$ when $x \leq \lfloor b \rfloor$. On the other hand, for $x = \lceil b \rceil + \alpha$ with $\alpha \geq 0$, (9) becomes $t \geq |(1-f) + \alpha + y - w|$ and (8) becomes $t \geq -w - y + (1-f) + \alpha(1-2f)$. Then, since $|(1-f)+\alpha+y-w| - (-w-y+(1-f)+\alpha(1-2f)) = 2\max\{\alpha f + y, w - (1-f) - \alpha(1-f)\} \geq 0$, (8) is implied by (9) and $y \geq 0$ when $x \geq \lceil b \rceil$. Hence, inequality (8) is valid for S_0.

To see that (8) is sufficient to cut off all points in relax$(S_0)\setminus$ conv(S_0), consider the polyhedron S_0' defined by the inequalities:

$$x + y - w - b \leq t \tag{10}$$
$$-x - y + w + b \leq t \tag{11}$$
$$-y \leq 0 \tag{12}$$
$$-w \leq 0 \tag{13}$$
$$(1-2f)x - y - w - t \leq (1-2f)\lfloor b \rfloor - f. \tag{14}$$

Since S_0' has four variables, any basic solution of it is defined by four of the inequalities among (10)–(14) at equality. We enumerate all five basic solutions below:

1. Inequalities (10), (11), (12), (13): $(x, y, w, t) = (b, 0, 0, 0)$ (infeasible if $b \notin \mathbb{Z}$).
2. Inequalities (10), (11), (12), (14): $(x, y, w, t) = (\lceil b \rceil, 0, 1-f, 0)$.
3. Inequalities (10), (11), (13), (14): $(x, y, w, t) = (\lfloor b \rfloor, f, 0, 0)$.
4. Inequalities (10), (12), (13), (14): $(x, y, w, t) = (\lceil b \rceil, 0, 0, 1-f)$.
5. Inequalities (11), (12), (13), (14): $(x, y, w, t) = (\lfloor b \rfloor, 0, 0, f)$.

Hence the extreme points of S_0' are precisely the extreme points of conv(S_0). □

The simple conic mixed-integer rounding inequality (8) can be used to derive nonlinear conic mixed-integer inequalities for nonlinear conic mixed-integer sets. The first observation useful in this direction is that the piecewise-linear conic inequality

$$|(1-2f)(x - \lfloor b \rfloor) + f| \leq t + y + w \tag{15}$$

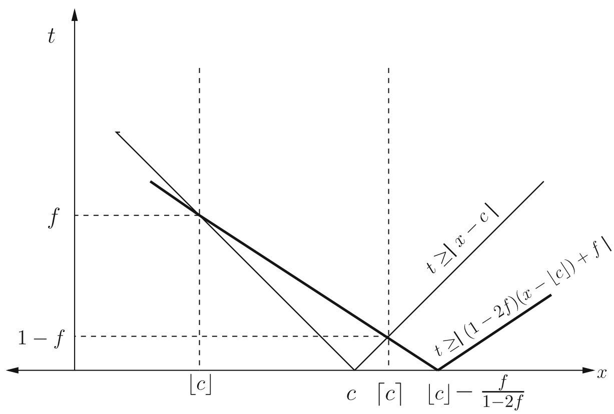

Fig. 1 Simple conic mixed-integer rounding cut

is valid for S_0. See Fig. 1 for the restriction $y = w = 0$.

In order to illustrate the nonlinear cuts, based on cuts for the polyhedral second-order conic constraints (5), let us now consider the simplest nonlinear second-order conic mixed-integer set

$$T_0 := \left\{ (x, y, t) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \ : \ \sqrt{(x-b)^2 + y^2} \le t \right\}. \tag{16}$$

The continuous relaxation $\operatorname{relax}(T_0)$ has exactly one extreme point $(x, y, t) = (b, 0, 0)$, which is infeasible for T_0 if $b \notin \mathbb{Z}$. Formulating T_0 as

$$t_1 \ge |x - b| \tag{17}$$

$$t_2 \ge |y| \tag{18}$$

$$t \ge \sqrt{t_1^2 + t_2^2} \tag{19}$$

we write the piecewise-linear conic inequality (15) for (17). Substituting out the auxiliary variables t_1, t_2, we obtain the *simple nonlinear conic mixed-integer rounding inequality*

$$\sqrt{((1-2f)(x - \lfloor b \rfloor) + f)^2 + y^2} \le t, \tag{20}$$

which is valid for T_0.

Proposition 2 *The simple nonlinear conic mixed-integer rounding inequality* (20) *cuts off all points in* $\operatorname{relax}(T_0) \setminus \operatorname{conv}(T_0)$.

Proof First, observe that for $x = \lfloor b \rfloor - \delta$, the constraint in (16) becomes $t \ge \sqrt{(\delta + f)^2 + y^2}$, and (20) becomes $t \ge \sqrt{(f - (1-2f)\delta)^2 + y^2}$. Since $(\delta + f)^2 - (f - (1-2f)\delta)^2 = 4f\delta(1+\delta)(1-f) \ge 0$ for $\delta \ge 0$ and for $\delta \le -1$, we see that (20) is dominated by $\operatorname{relax}(T_0)$ unless $\lfloor b \rfloor < x < \lceil b \rceil$. When $-1 < \delta < 0$ (i.e., $x \in (\lfloor b \rfloor, \lceil b \rceil)$), $4f\delta(1+\delta)(1-f) < 0$, implying that (20) dominates the constraint in (16).

We now show that if $(x_1, y_1, t_1) \in \operatorname{relax}(T_0)$ and satisfies (20), then $(x_1, y_1, t_1) \in \operatorname{conv}(T_0)$. If $x_1 \notin (\lfloor b \rfloor, \lceil b \rceil)$, it is sufficient to consider $(x_1, y_1, t_1) \in \operatorname{relax}(T_0)$ as (20) is dominated by $\operatorname{relax}(T_0)$ in this case. Now, the ray $R_1 := \{(b, 0, 0) + \alpha(x_1 - b, y_1, t_1) : \alpha \in \mathbb{R}_+\} \subseteq \operatorname{relax}(T_0)$. Let the intersections of R_1 with the hyperplanes $x = \lfloor x_1 \rfloor$ and $x = \lceil x_1 \rceil$ be $(\lfloor x_1 \rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil x_1 \rceil, \hat{y}_1, \hat{t}_1)$, which belong to T_0. Then (x_1, y_1, t_1) can be written as a convex combination of points $(\lfloor x_1 \rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil x_1 \rceil, \hat{y}_1, \hat{t}_1)$; hence $(x_1, y_1, t_1) \in \operatorname{conv}(T_0)$.

On the other hand, if $x_1 \in (\lfloor b \rfloor, \lceil b \rceil)$, it is sufficient to consider (x_1, y_1, t_1) that satisfies (20), since (20) dominates the constraint in (16) for $x \in [\lfloor b \rfloor, \lceil b \rceil]$. If $f = 1/2$, (x_1, y_1, t_1) is a convex combination of $(\lfloor b \rfloor, y_1, t_1)$ and $(\lceil b \rceil, y_1, t_1)$. Otherwise, all points on the ray $R_2 := \{(x_0, 0, 0) + \alpha(x_1 - x_0, y_1, t_1) \ : \ \alpha \in \mathbb{R}_+\}$, where $x_0 = \lfloor b \rfloor - \frac{f}{1-2f}$, satisfy (20). Let the intersections of R_2 with the hyperplanes $x = \lfloor b \rfloor$ and $x = \lceil b \rceil$ be $(\lfloor b \rfloor, \bar{y}_1, \bar{t}_1)$, $(\lceil b \rceil, \hat{y}_1, \hat{t}_1)$, which belong to T_0. Note that the intersections are nonempty because $x_0 \notin [\lfloor b \rfloor, \lceil b \rceil]$. Then we see that (x_1, y_1, t_1) can be written as

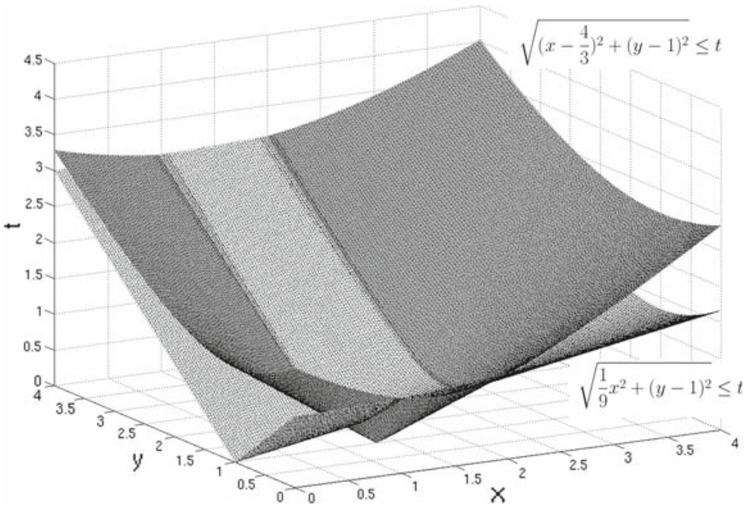

Fig. 2 Nonlinear conic mixed-integer rounding cut

a convex combination of $(\lfloor b \rfloor, \bar{y}, \bar{t})$ and $(\lceil b \rceil, \hat{y}, \hat{t})$. Hence, $(x_1, y_1, t_1) \in \text{conv}(T_0)$ in this case as well. □

Proposition 2 shows that the curved convex hull of T_0 can be described using only two second-order conic constraints. The following example illustrates Proposition 2.

Example 1 Consider the second-order conic set given as

$$T_0 = \left\{ (x, y, t) \in \mathbb{Z} \times \mathbb{R} \times \mathbb{R} \ : \ \sqrt{\left(x - \frac{4}{3}\right)^2 + (y-1)^2} \le t \right\}.$$

The unique extreme point of $\text{relax}(T_0)$ $(\frac{4}{3}, 1, 0)$ is fractional. Here $\lfloor b \rfloor = 1$ and $f = \frac{1}{3}$; therefore,

$$\text{conv}(T_0) = \left\{ (x, y, t) \in \mathbb{R}^3 : \sqrt{\left(x - \frac{4}{3}\right)^2 + (y-1)^2} \le t, \sqrt{\frac{1}{9}x^2 + (y-1)^2} \le t \right\}.$$

We show the inequality $\sqrt{\frac{1}{9}x^2 + (y-1)^2} \le t$ and the region it cuts off in Figure 2. Observe that the function values are equal at $x = 1$ and $x = 2$ and the cut eliminates the points between them.

3.2 The general case

In this section we present valid inequalities for the mixed-integer sets defined by general polyhedral second-order conic constraints (5). Toward this end, let

$$S := \{x \in \mathbb{Z}^n_+, \, y \in \mathbb{R}^p_+, \; t \in \mathbb{R} : \; t \geq |ax + gy - b|\}.$$

We refer to the inequalities $x \geq \mathbf{0}$, $y \geq \mathbf{0}$, and the base inequality $t \geq |ax + gy - b|$ as the *trivial inequalities*. We assume that the coefficients a_i and g_i are nonzero, since otherwise the corresponding variables can be dropped from S without loss of generality. The following result simplifies the presentation.

Proposition 3 *For any nontrivial facet-defining inequality $\gamma x + \pi y + \delta t \leq \pi_0$ of* $\mathrm{conv}(S)$ *the following statements are true*:

1. $\delta = -1$*;*
2. $\pi_i < 0$ *for all* $i = 1, \ldots, p$*;*
3. $\frac{\pi_i}{\pi_j} = \left|\frac{g_i}{g_j}\right|$ *for all* $i, j = 1, \ldots, p$.

Proof 1. Let $\gamma x + \pi y + \delta t \leq \pi_0$ be a nontrivial facet-defining inequality of $\mathrm{conv}(S)$. Since for any point (x, y, t) on the facet, the point $(x, y, t + \epsilon)$ with $\epsilon > 0$ is feasible, validity of the inequality implies $\delta \leq 0$. To see $\delta \neq 0$, let $(x, \mathbf{0}, t)$ be a feasible point such that $\gamma x > \pi_0$. As t can be chosen arbitrarily large such a point exists. Unless $\delta < 0$, inequality $\gamma x + \pi y + \delta t \leq \pi_0$ is not valid for this point. Therefore, by scaling we may assume $\delta = -1$.
2. If $g_i > 0$ let (x, y, t) be a point on the facet such that $ax + gy - b < t$. Such a point exists because the facet is nontrivial. Since the point $(x, y + \epsilon e_i, t - \epsilon g_i)$ is feasible for some small $\epsilon > 0$, we have $\pi_i < 0$. On the other hand, if $g_i < 0$ let (x, y, t) be a point on the facet such that $-ax - gy + b < t$. Such a point exists because the facet is nontrivial. Since the point $(x, y + \epsilon e_i, t - \epsilon g_i)$ is feasible for some small $\epsilon > 0$, we have $\pi_i < 0$ as well.
3. Suppose $g_i, \; g_j > 0$. Let (x, y, t) be a point on the facet such that $y_i > 0$. Such a point exists because the facet is distinct from $y_i = 0$. Then, the point $(x, y - \epsilon e_i + \epsilon \frac{g_i}{g_j} e_j, t)$ is also feasible. Evaluating the inequality for these points shows $\frac{\pi_i}{\pi_j} \leq \frac{g_i}{g_j}$. By symmetry, $\frac{\pi_i}{\pi_j} \geq \frac{g_i}{g_j}$ as well. The result for other pairs follows from this observation as S can be equivalently stated using $|-ax - gy + b| \leq t$ instead of $|ax + gy - b| \leq t$ and this rewriting of the base constraint has no impact on the facial structure.

□

Due to Proposition 3 it is sufficient to consider the polyhedral second-order conic constraint

$$\left|ax + y^+ - y^- - b\right| \leq t, \tag{21}$$

where all continuous variables with positive coefficients are aggregated into $y^+ \in \mathbb{R}_+$ and those with negative coefficients are aggregated into $y^- \in \mathbb{R}_+$ to represent a general polyhedral conic constraint of the form (5).

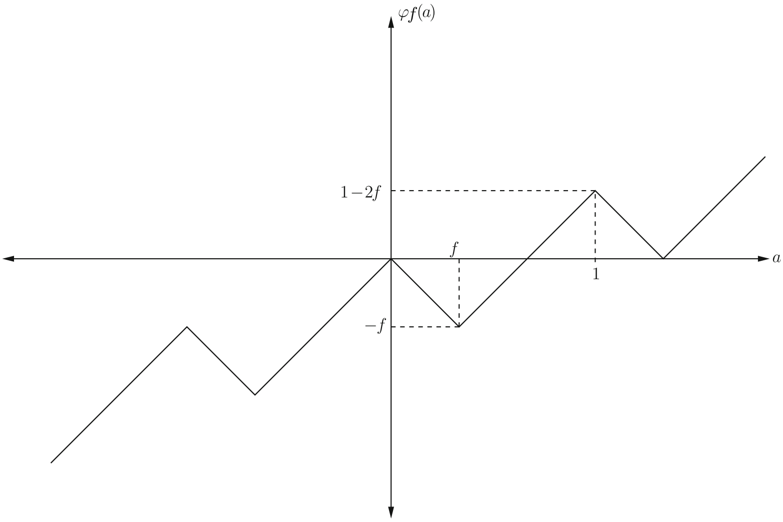

Fig. 3 Conic mixed-integer rounding function

Definition 1 For $0 \leq f < 1$ let the *conic mixed-integer rounding* (*MIR*) *function* $\varphi_f : \mathbb{R} \to \mathbb{R}$ be

$$\varphi_f(a) = \begin{cases} (1-2f)n - (a-n), & \text{if } n \leq a < n+f, \\ (1-2f)n + (a-n) - 2f, & \text{if } n+f \leq a < n+1. \end{cases} \quad n \in \mathbb{Z} \tag{22}$$

The conic mixed-integer rounding function φ_f is piecewise linear and continuous. Figure 3 illustrates φ_f. A function $\theta : \mathbb{R} \to \mathbb{R}$ is *superadditive* on $\mathbb{R}$ if $\theta(a) + \theta(b) \leq \theta(a+b)$ for all $a, b \in \mathbb{R}$.

Lemma 1 *The conic MIR function φ_f is superadditive on $\mathbb{R}$.*

Proof The proof follows from writing φ_f as a nonnegative combination of two superadditive functions. Observe that $\varphi_f(a) = \theta(a) + 2(1-f)\psi_f(a)$, where $\theta(a) = -a$ and $\psi_f(a) = \lfloor a \rfloor + (\frac{a - \lfloor a \rfloor - f}{1-f})^+$. The linear function θ is clearly superadditive and ψ_f is the superadditive MIR function for linear MIP [27]. $\square$

Theorem 1 *For any $\alpha \neq 0$ the conic mixed-integer rounding inequality*

$$\sum_{j=1}^{n} \varphi_{f_\alpha}(a_j/\alpha)x_j - \varphi_{f_\alpha}(b/\alpha) \leq (t + y^+ + y^-)/|\alpha| \tag{23}$$

where $f_\alpha = b/\alpha - \lfloor b/\alpha \rfloor$, is valid for S. Moreover, if α is chosen such that $\alpha = a_j$ and $b/a_j > 0$ for some $j \in \{1, \ldots, n\}$ and $a_i \leq b$ for all $i \in \{1, \ldots, n\} \setminus \{j\}$, then (23) *is facet-defining for* $\text{conv}(S)$.

Proof First consider the case $\alpha = 1$. Writing the conic inequality (21) as

$$\left| \left[\sum_{f_j \le f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j \right] + \left[\sum_{f_j \le f} f_j x_j + y^+ \right] - \left[\sum_{f_j > f} (1 - f_j) x_j + y^- \right] - b \right| \le t$$

where $f_j := a_j - \lfloor a_j \rfloor$, we see that $x' = \sum_{f_j \le f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j \in \mathbb{Z}$, $y' = \sum_{f_j \le f} f_j x_j + y^+ \in \mathbb{R}_+$, and $w' = \sum_{f_j > f} (1 - f_j) x_j + y^- \in \mathbb{R}_+$. Then the corresponding simple conic MIR inequality on variables (x', y', w', t) is

$$\begin{aligned} (1 - 2f) \left[\sum_{f_j \le f} \lfloor a_j \rfloor x_j + \sum_{f_j > f} \lceil a_j \rceil x_j - \lfloor b \rfloor \right] + f \\ \le t + \sum_{f_j \le f} f_j x_j + y^+ + \sum_{f_j > f} (1 - f_j) x_j + y^-. \end{aligned}$$

Rewriting this inequality using φ_f, we obtain

$$\sum_{j=1}^{n} \varphi_f(a_j) x_j - \varphi_f(b) \le t + y^+ + y^-.$$

To see that the inequality is facet-defining for $\operatorname{conv}(S)$, suppose conditions of the theorem are satisfied for $\alpha = 1$. Then, consider the four affinely points (x, y^+, y^-, t) on the face: $(\lfloor b \rfloor e_j, f, 0, 0)$, $(\lfloor b \rfloor e_j, 0, 0, f)$, $(\lceil b \rceil e_j, 0, 1 - f, 0)$ and $(\lceil b \rceil e_j, 0, 0, 1 - f)$. The remaining $n - 1$ affinely independent points on the face defined by (23) are as follows: For $i \in \{1, \ldots, n\} \setminus \{j\}$,

(1) if $a_i - \lfloor a_i \rfloor < f$: $(e_i + \lfloor b - a_i \rfloor e_j, \{b - a_i\}, 0, 0)$
(2) if $a_i - \lfloor a_i \rfloor \ge f$: $(e_i + \lceil b - a_i \rceil e_j, 0, 0, 1 - \{b - a_i\})$

where $\{b - a_i\} = b - a_i - \lfloor b - a_i \rfloor$. Finally, scaling the base inequality as

$$\left| \frac{1}{\alpha} (ax + gy - b) \right| \le t / |\alpha|$$

the result holds for $\alpha \ne 0$. □

Proposition 4 *Conic mixed-integer rounding inequalities with* $\alpha = a_j$, $j = 1, \ldots, n$ *are sufficient to cut off all fractional extreme points of* $\operatorname{relax}(S)$.

Proof After aggregating continuous variables as in (21), $\operatorname{relax}(S)$ has $n + 3$ variables and $n + 4$ constraints. Therefore, if $x_j > 0$ in an extreme point solution, then the remaining $n + 3$ constraints must be active. Thus, the continuous relaxation $\operatorname{relax}(S)$ has at most n fractional extreme points $(x^j, 0, 0, 0)$ of the form $x^j_j = b/a_j > 0$, and

$x_i^j = 0$ for all $i \neq j$, which are infeasible if $b/a_j \notin \mathbb{Z}$. But for such a fractional extreme point $(x^j, 0, 0, 0)$ inequality (23) reduces to

$$\varphi_{f_{a_j}}(1)x_j - \varphi_{f_{a_j}}(b/a_j) \leq (t + y^+ + y^-)/\left|a_j\right|,$$

or equivalently,

$$(1 - 2f_{a_j})x_j - (1 - 2f_{a_j})\lfloor b/a_j \rfloor - f_{a_j} \leq (t + y^+ + y^-)/\left|a_j\right|,$$

which by Proposition 1 cuts off the fractional point with $x_j^j = b/a_j \notin \mathbb{Z}$. □

Application to linear MIPs. Next we show that mixed-integer rounding (MIR) inequalities [16,27,28] for linear mixed-integer programming can be obtained as conic MIR inequalities. First, observe that any two linear inequalities

$$\alpha_1 x + \beta_1 y \leq \gamma_1 \quad \text{and} \quad \alpha_2 x + \beta_2 y \leq \gamma_2$$

can be written equivalently in the *conic form* as

$$\left|\left(\frac{\alpha_1 - \alpha_2}{2}\right)x + \left(\frac{\beta_1 - \beta_2}{2}\right)y - \frac{\gamma_1 - \gamma_2}{2}\right| \leq \frac{\gamma_1 + \gamma_2}{2} - \left(\frac{\alpha_1 + \alpha_2}{2}\right)x - \left(\frac{\beta_1 + \beta_2}{2}\right)y.$$

We will use this observation in the sequel.

Consider a linear mixed-integer set

$$ax - y \leq b, \; x \geq 0, \; y \geq 0, \quad x \in \mathbb{Z}^n, \; y \in \mathbb{R} \tag{24}$$

and the corresponding valid MIR inequality

$$\sum_{j=1}^{n} \left(\lfloor a_j \rfloor + \frac{(f_j - f)^+}{1 - f}\right)x_j - \frac{1}{1 - f}y \leq \lfloor b \rfloor, \tag{25}$$

where $f_j := a_j - \lfloor a_j \rfloor$ for $j = 1, \ldots, n$ and $f := b - \lfloor b \rfloor$.

Proposition 5 *Every MIR inequality* (25) *is a conic MIR* (23) *inequality.*

Proof After writing the inequalities $ax - y \leq b$ and $y \geq 0$, in the conic form

$$-ax + 2y + b \geq |ax - b|$$

we apply the conic mixed-integer rounding inequality to obtain get

$$-ax + 2y + b \geq \sum_{j=1}^{n} \varphi_f(a_j)x_j - \varphi_f(b).$$

After rearranging this inequality as

$$2y + 2(1-f)\lfloor b \rfloor \ge \sum_{f_j \le f} \big((1-2f)\lfloor a_j \rfloor - f_j + a_j\big)\, x_j + \sum_{f_j > f} \big((1-2f)\lceil a_j \rceil - (1-f_j) + a_j\big)\, x_j$$

and dividing it by $2(1-f)$ we obtain the MIR inequality (25). □

Example 2 In this example we illustrate that conic mixed-integer rounding cuts can be used to generate valid inequalities that are difficult to obtain by Chvátal–Gomory (CG) integer rounding in the case of pure integer programming. It is well-known that the CG rank of the polytope P given by inequalities

$$2kx_1 + x_2 \le 2k, \quad -2kx_1 + x_2 \le 0, \quad x_2 \ge 0$$

for a positive integer k equals exactly k [see 33, pg. 344]. Below we show that the non-trivial facet $x_2 \le 0$ of the convex hull of integer points in P can be obtained by a single application of the conic MIR cut.

Writing constraints $2kx_1 + x_2 \le 2k$ and $-2kx_1 + x_2 \le 0$ in conic form, we obtain

$$|2kx_1 - k| \le k - x_2. \tag{26}$$

Dividing the conic constraint (26) by $2k$ and treating $\frac{1}{2} - \frac{x_2}{2k}$ as a continuous variable, we obtain the conic MIR cut

$$\frac{1}{2} \le \frac{1}{2} - \frac{x_2}{2k},$$

which is equivalent to $x_2 \le 0$.

Writing two linear inequalities in conic form suggests a way of generating cuts that are based on two constraints of an MIP. Because in the conic form two constraints are used together, this approach may have a practical advantage for generating stronger cuts compared to the ones obtained from aggregating the two constraints. Exploring the potential effectiveness of cuts from a conic reformulation rather than an aggregation of constraints for linear MIP is worthy of an extensive computational study, but it is beyond the scope of the current paper, which is focused on conic MIP.

Conic Aggregation. It is possible to generate other cuts for the second order conic mixed integer set C by aggregating constraints (5) in conic form: for $\lambda, \mu \in \mathbb{R}^m_+$, we have $\lambda' t \ge \lambda'(Ax + Gy - b)$ and $\mu' t \ge \mu'(-Ax - Gy + b)$. Writing these two inequalities in conic form, we obtain

$$\left(\frac{\lambda+\mu}{2}\right)' t + \left(\frac{\mu-\lambda}{2}\right)' (Ax + Gy) + \left(\frac{\lambda-\mu}{2}\right)' b \ge \left| \left(\frac{\mu-\lambda}{2}\right)' t + \left(\frac{\lambda+\mu}{2}\right)' (Ax + Gy) - \left(\frac{\lambda+\mu}{2}\right)' b \right|. \tag{27}$$

Then we can write the corresponding conic MIR inequalities for (27) by treating the left-hand side of inequality (27) as a single continuous variable. Constraint (27) allows us to utilize multiple polyhedral conic constraints (5) in deriving a cut. We use this conic aggregation in the computational study presented next.

4 Computational experiments

In this section we report our computational results with the conic mixed-integer rounding inequalities. We tested the effectiveness of the cuts on three different problem sets with and without structure. The first set consists of randomly generated SOCMIP instances with no structure. The second set is from a mean–variance capital budgeting problem, whereas the third set is from a binary least-squares estimation problem. The data set used in the experiments can be downloaded from http://ieor.berkeley.edu/~atamturk/data.

All experiments were performed on a 3.2 GHz Pentium 4 Linux workstation with 1GB main memory using CPLEX[1] (Version 11.0) second-order conic MIP solver. We used CPLEX's barrier algorithm to solve SOCPs at the nodes of a branch-and-bound algorithm. CPLEX solver was run with default options, with the exception of primal heuristic options, which were turned off as they often increased the solution time. CPLEX restricts conic constraints that can be input into its solver to only $\sum_{i=1}^{n} t_i^2 \leq t_0^2$. Therefore, the auxiliary variables $t_0, t_1, \ldots, t_n$ had to be used explicitly for entering the problems to the solver. This allows, though, the cuts to be added as linear cuts in the (x, y, t) space.

Conic MIR cuts (23) were added only at the root node using a simple separation heuristic. We performed a simple version of conic aggregation (27) on pairs of constraints using only 0–1 valued multipliers λ and μ, and checked for violation of conic MIR cut (23) for each integer variable x_j with fractional value for the continuous relaxation using $\alpha \in \{1a_j, 2a_j, 4a_j, 6a_j, 8a_j, 10a_j\}$. For problems with binary variables (Sects. 4.2, 4.3), if the continuous relaxation value of a variable is greater than 0.7, we complement the variable before applying cut (23). Thus, the heuristic implementation of scaling and complementing variables follows Marchand and Wolsey [24] for linear MIP.

4.1 Random SOCMIP instances

In Table 1, we report our computations for randomly generated SOCMIP instances of the form (1) with cones $\mathcal{Q}^2$, $\mathcal{Q}^{25}$, and $\mathcal{Q}^{50}$. The coefficients of A, G, b, d, e, and h are generated from Uniforms(0,3). In the table we show the size of the cone (m), the number of integer variables in the formulation (n), the number of cuts, the integrality gap (the percentage gap between the optimal solution and the continuous relaxation), the number of nodes explored in the search tree, and CPU time (in seconds) with

[1] CPLEX is a registered trademark of ILOG, Inc.

Table 1 Computations with random SOCMIP instances

m	n	Without cuts (23)			With cuts (23)			
		`% gap`	`nodes`	`time`	`% gap`	`nodes`	`time`	`cuts`
2	200	90.84	27	0.4	0.51	1	0.2	21
	400	84.31	56	0.4	0.00	0	0.2	31
	600	79.31	82	0.5	0.00	0	0.3	51
	800	87.16	168	0.9	0.63	3	0.5	76
	1000	91.74	208	1.6	1.28	3	1.2	117
25	200	38.52	73	1.9	2.58	9	1.5	68
	400	61.78	5638	241.1	6.82	39	7.3	39
	600	54.93	328	19.3	1.73	6	3.1	61
	800	73.45	549	50.3	0.81	3	3.3	71
	1000	84.81	598	79.5	2.01	8	6.2	63
50	200	50.69	61	5.3	0.00	0	0.6	41
	400	53.71	126	16.8	2.94	5	4.6	39
	600	62.19	269	51.4	1.63	3	5.8	71
	800	79.25	392	97.4	1.37	4	8.0	104
	1000	83.64	568	188.3	1.43	4	8.1	127
Average		71.8	609.5	50.3	1.6	5.9	3.4	65.3

and without adding the conic mixed-integer rounding cuts (23). Each row of the table shows the averages for five instances.

We see that conic MIR cuts are very effective in closing the integrality gap. Most of the instances has 0% gap at the root node after adding the cuts and were solved without branching, and the remaining ones are solved within only a few nodes. Average integrality gap at the root node is reduced from 71.8% to only 1.6%. This improvement reduces the average number of nodes explored in the search from 609.5 to only 5.9. The solution time of continuous SOCP relaxation did not increase much with the addition of the cuts. This is probably due to the fact that the added cuts are linear in (x, y, t) space. A comparison of the overall computation time shows that with the addition of the conic MIR cuts computational effort is reduced by more than an order of magnitude.

4.2 Mean–variance capital budgeting

The second data set consists of capital budgeting problems with a mean–variance objective [7,25,44]

$$\max \left\{ rx - \gamma x' V x : cx \leq d,\ x \in \{0, 1\}^n \right\}, \tag{28}$$

where r is the expectation and V is the covariance matrix of uncertain return for n projects that must satisfy a budget constraint $cx \leq d$, and $\gamma > 0$ is the investor's risk-averseness parameter. As the purpose of the application is illustration only, for

Table 2 Mean–variance capital budgeting instances

γ	n	Without cuts (23)			With cuts (23)			
		`% gap`	`nodes`	`time`	`% gap`	`nodes`	`time`	`cuts`
1	20	27.92	228	2.6	7.99	88	1.1	13
	40	14.27	2932	99.0	5.63	1162	34.2	21
	60	12.21	5423	1233.3	3.93	3018	525.9	31
	80	11.32	11149	2066.9	2.81	4410	793.0	31
	100	9.61	20635	7728.3	2.68	8129	2271.5	35
2	20	29.19	267	2.8	7.62	79	0.96	11
	40	17.83	3118	106.8	4.92	1048	30.6	24
	60	15.37	5688	1482.5	4.09	3142	603.7	28
	80	13.24	12581	2270.1	2.96	4591	872.2	29
	100	10.63	22074	8062.3	2.59	8005	2547.7	32
5	20	34.11	376	4.2	8.02	103	1.26	15
	40	21.02	3682	130.3	5.31	1247	36.8	20
	60	17.84	5849	1639.0	4.51	3305	641.3	32
	80	14.39	14719	2581.6	3.40	4742	819.4	30
	100	11.42	24338	7849.2	3.76	7491	2841.3	36
Average		17.4	8870.6	2350.6	4.7	3370.7	801.4	25.9

simplicity of the model we ignore the financing structure and assume that a fixed budget d is available for the projects. Because the objective is concave quadratic, it can be written in conic quadratic form. In particular, for $a \in \mathbb{R}^n$ such that $a'V = \frac{1}{2\gamma}r$, problem (28) is equivalent to

$$a'a - \min\left\{x'Vx - 2a'Vx + a'a : cx \leq d,\ x \in \{0,1\}^n\right\}, \tag{29}$$

which can be stated in conic quadratic form as

$$a'a - \left(\min\left\{t : ||V^{1/2}x - a|| \leq t,\ cx \leq d,\ x \in \{0,1\}^n,\ t \in \mathbb{R}_+\right\}\right)^2. \tag{30}$$

Note that this transformation of a convex quadratic function into the conic quadratic form is different from the one suggested by Ben-Tal and Nemirovski [see 8, pg. 104] and fits our approach for deriving cutting planes.

For these experiments the coefficients of r, a, and $V^{1/2}$ are generated from Uniform(0,5) and b is set as $b = \sum_{i=1}^n a_i$. We observe in Table 2 that the capital budgeting instances are much harder to solve compared to the unstructured randomly generated SOCMIPs even though the integrality gaps are smaller. The integrality gaps decrease as the number of variables increases, but increase with γ as the quadratic term gains more weight. With the addition of cuts, the integrality gap at root node reduces from an average of 17.4 to 4.7% and we see close to threefold reduction in the number of nodes and computation time.

Table 3 Binary least squares estimation problem

n/m	n	Without cuts (23)			With cuts (23)			
		% gap	nodes	time	% gap	nodes	time	cuts
1	20	19.10	104	1.9	5.82	47	0.9	14
	40	11.81	2104	70.0	4.88	590	26.1	17
	60	9.42	4431	900.4	3.68	974	515.9	20
	80	6.05	8053	1498.5	2.04	3061	581.2	22
	100	4.34	22301	7731.4	1.83	5736	2018.9	28
5	20	16.03	109	0.4	7.01	49	0.1	5
	40	14.90	13085	57.4	6.14	1017	5.8	9
	60	17.51	17815	261.5	6.18	3242	68.3	11
	80	9.81	31783	1049.3	3.20	3928	117.5	16
	100	11.06	38509	1112.8	3.29	7804	316.4	20
10	20	12.25	49	0.2	4.96	10	0.2	2
	40	15.08	3620	25.9	5.20	918	4.0	9
	60	16.33	15691	210.5	5.31	2104	32.1	8
	80	9.40	39362	458.8	3.52	3489	38.2	10
	100	11.18	50579	788.6	3.63	8120	109.9	10
Average		12.3	16506.3	944.5	4.5	2739.3	255.7	13.4

4.3 Least squares estimation with binary inputs

The third data set comes from a least squares problem with binary inputs

$$\min\left\{||Qx - y|| : \ x \in \{-1, +1\}^n\right\}. \tag{31}$$

Least squares with binary inputs is a fundamental problem in signal processing in digital communication networks. Given an observed noisy output vector $y \in \mathbb{R}^m$, problem (31) must be solved for decoding the binary input vector $x \in \{-1, +1\}^n$ for a matrix $Q \in \mathbb{R}^{m\times n}$ [22,23,26]. Letting $z = \frac{1}{2}(\mathbf{1}+x)$ and $b = \frac{1}{2}Q\mathbf{1} + y$, we can state (31) in the conic quadratic form

$$\min\left\{t : ||Qz - b|| \le t, \ z \in \{0, 1\}^n, \ t \in \mathbb{R}_+\right\}.$$

For the experiments the coefficients of Q are generated from Uniform(0,5) and y from Uniform$(0, n/2)$. The results are reported in Table 3. We observe that, similar to the capital budgeting instances, integrality gaps decrease as the number of variables grows. There does not seem to be a consistent pattern in the gaps as a function of the output size m. Nevertheless, the problems become significantly easier as the output size decreases compared to the input size. With the addition of the cuts, the integrality gap at the root node reduces from an average of 12.3 to 4.5% and we see about 83% reduction in the number of nodes and 73% reduction in the computation time.

5 Conclusion

In this paper, we introduced conic mixed-integer rounding cuts for conic integer programming. Crucial to our approach is a reformulation of the second-order conic constraints with polyhedral second-order constraints in a higher dimensional space. In this representation the cuts are linear, even though they are nonlinear in the original space of variables. This feature leads to a computationally efficient implementation of nonlinear cuts for conic mixed-integer programming. The reformulation also allows the use of polyhedral methods for conic integer programming. Computations with unstructured SOCMIP problems as well as instances for mean–variance capital budgeting problems and least-squares estimation problems with binary inputs show the effectiveness of the conic MIR approach.

References

1. Abhishek, K., Leyffer, S., Linderoth, J.T.: FilMINT: An outer-approximation-based solver for nonlinear mixed integer programs. Preprint ANL/MCS-P1374-0906, Argonne National Laboratory, Mathematics and Computer Science Division, September 2006
2. Alizadeh, F.: Interior point methods in semidefinite programming and applications to combinatorial optimization. SIAM J. Optim. **5**, 13–51 (1995)
3. Alizadeh, F., Goldfarb, D.: Second-order cone programming. Math. Program. **95**, 3–51 (2003)
4. Atamtürk, A., Narayanan, V.: Lifting for conic mixed-integer programming. Technical Report BCOL.07.04, IEOR, University of California-Berkeley, October 2007
5. Balas, E.: Disjunctive Programming. Ann. Discrete Math. **5**, 3–51 (1979)
6. Balas, E., Ceria, S., Cornuéjols, G.: A lift-and-project cutting plane algorithm for mixed 0–1 programs. Math. Program. **58**, 295–324 (1993)
7. Baum, S., Carlson, R.C., Jucker, J.V.: Some problems in applying the continuous portfolio selection model to the discrete capital budgeting problem. J. Financ. Quant. Anal. **13**, 333–344 (1978)
8. Ben-Tal, A., Nemirovski, A.: Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications. MPS-SIAM Series on Optimization. SIAM, Philadelphia, 2001
9. Benson, S.J., Ye, Y.: DSDP5: Software for semidefinite programming. Technical Report ANL/MCS-P1289-0905, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, September 2005. ACM Transactions on Mathematical Software (submitted)
10. Bonami, P., Biegler, L.T., Conn, A.R., Cornuéjols, G., Grossmann, I.E., Laird, C.D., Lee, J., Lodi, A., Margot, F., Sawaya, N., Wächter, A.: An algorithmic framework for convex mixed integer nonlinear programs. Discrete Optim. **5**, 186–204 (2008)
11. Borchers, B.: CSDP, a C library for semidefinite programing. Optim. Methods Softw. **11**, 613–623 (1999)
12. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004)
13. Çezik, M.T., Iyengar, G.: Cuts for mixed 0–1 conic programming. Math. Program. **104**, 179–202 (2005)
14. Goemans, M.X.: Semidefinite programming in combinatorial optimization. Math. Program. **79**, 143–161 (1997)
15. Goemans, M.X., Williamson, D.P.: Improved approximation algorithms for maximum cut and satisfyibility problems using semidefinite programming. J. ACM **42**, 1115–1145 (1995)
16. Gomory, R.E.: An algorithm for the mixed integer problem. Technical Report RM-2597, The Rand Corporation (1960)
17. Kim, S., Kojima, M., Yamashita, M.: Second order cone programming relaxation of a positive semidefinite constraint. Optim. Methods Softw. **18**, 535–451 (2003)
18. Kojima, M., Tuncel, L.: Cones of matrices and successive convex relaxations of nonconvex sets. SIAM J. Optim. **10**, 750–778 (2000)
19. Lasserre, J.B.: An explicit exact SDP relaxation for nonlinear 0–1 programs. In: Aardal, K., Gerards, A.M.H. (eds.) Lecture Notes in Computer Science, vol. 2081, pp. 293–303 (2001)

20. Lobo, M., Vandenberghe, L., Boyd, S., Lebret, H.: Applications of second-order cone programming. Linear Algebra Appl. **284**, 193–228 (1998)
21. Lovász, L., Schrijver, A.: Cones of matrices and set-functions and 0–1 optimization. SIAM J. Optim. **1**, 166–190 (1991)
22. Luo, J., Pattipati, K.R., Willett, P., Levchuk, G.M.: Optimal and suboptimal any-time algorithms for cdma multiuser detection based on branch and bound. IEEE Trans. Commun. **52**, 632–642 (2004)
23. Luo, Z.-Q.: Applications of convex optimization in signal processing and digital communication. Math. Program. **97**, 117–207 (2003)
24. Marchand, H., Wolsey, L.A.: Aggregation and mixed integer rounding to solve MIPs. Oper. Res. **49**, 363–371 (2001)
25. McBride, R.D.: Finding the integer efficient frontier for quadratic capital budgeting problems. J. Financ. Quant. Anal. **16**, 247–253 (1981)
26. Mobasher, A., Taherzadeh, M., Khandani, A.K.: A near-maximum likelihood decoding algorithm for MIMO systems based on semi-definite programming. IEEE Trans. Inform. Theory **53**, 3869–3886 (2007)
27. Nemhauser, G.L., Wolsey, L.A.: Integer and Combinatorial Optimization. John Wiley and Sons, New York (1988)
28. Nemhauser, G.L., Wolsey, L.A.: A recursive procedure for generating all cuts for 0–1 mixed integer programs. Math. Program. **46**, 379–390 (1990)
29. Nesterov, Y., Nemirovski, A.: A general approach to polynomial-time algorithm design for convex programming. Technical report, Center. Econ. Math. Inst, USSR Acad. Sci., Moskow, USSR (1988)
30. Nesterov, Y., Nemirovski, A.: Self-concordant functions and polynomial time methods in convex programming. Technical report, Center. Econ. Math. Inst, USSR Acad. Sci., Moskow, USSR (1990)
31. Nesterov, Y., Nemirovski, A.: Conic formulation of a convex programming problem and duality. Technical report, Center. Econ. Math. Inst, USSR Acad. Sci., Moskow, USSR (1991)
32. Nesterov, Y., Nemirovski, A.: Interior-Point Polynomial Algorithms for Convex Programming. SIAM, Philedelphia (1993)
33. Schrijver, A.: Theory of Linear and Integer Programming. John Wiley and Sons, Chichester (1987)
34. Sherali, H.D., Adams, W.P.: Hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems. SIAM J. Discrete Math. **3**, 411–430 (1990)
35. Sherali, H.D., Shetti, C.: Optimization with disjunctive constraints, vol. 181 of Lectures on Econ. Math. Systems. Springer Verlag, Berlin, Heidelberg, New York (1980)
36. Sherali, H.D., Tunçbilek, C.H.: A reformulation-convexification approach for solving nonconvex quadratic programming problems. J. Global Optim. **7**, 1–31 (1995)
37. Stubbs, R., Mehrotra, S.: A branch-and-cut methods for 0–1 mixed convex programming. Math. Program. **86**, 515–532 (1999)
38. Stubbs, R., Mehrotra, S.: Generating convex polynomial inequalities for mixed 0-–1 programs. J. Global Optim. **24**, 311–332 (2002)
39. Sturm, J.: Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optim. Methods Softw. **11**, 625–653 (1999)
40. Tawarmalani, M., Sahinidis, N.V.: Global optimization of mixed-integer nonlinear programs: a theoretical and computational study. Math. Program. **99**, 563–591 (2004)
41. Tawarmalani, M., Sahinidis, N.V.: A polyhedral branch-and-cut approach to global optimization. Math. Program. **103**, 225–249 (2005)
42. Toh, K.C., Todd, M.J., Tütüncü, R.H.: SDPT3—a Matlab software package for semidefinite programming. Optimization Methods and Software, vol. 11/12, pp. 545–581 (1999)
43. Vielma, J.P., Ahmed, S., Nemhauser, G.L.: A lifted linear programming branch-and-bound algorithm for mixed integer conic quadratic programs. Manuscript, Georgia Institute of Technology (2007)
44. Weingartner, H.M.: Capital budgeting of interrelated projects: survey and synthesis. Manage. Sci. **12**, 485–516 (1966)
45. Yamashita, M., Fujisawa, K., Kojima, M.: Implementation and evaluation of SDPA 6.0 (SemiDefinite Programming Algorithm 6.0). Optim. Methods Softw. **18**, 491–505 (2003)