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A Mathematical Theory of Man–Machine Text Editing

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***Abstract*—A mathematical representation of text editing tasks and a model of the implementation of such tasks by man–machine systems is introduced. The model assumes an abstract system that is characterized by a set of parameters such as insert rate, delete rate, etc. This model is used to derive the optimal operation strategy for a system and the expected editing rate as a function of the document's length and the parameters describing the work load and the system. Based on these results, some criteria are suggested for evaluating alternative text editing systems and for determining tradeoffs involved in designing such systems.**

I. Introduction

COMPUTER assisted text editing has played an important role to date in the office environment and has recently been propagated into the news and publishing industries. This computer application evolved from the basic editing features that were originally introduced in time sharing systems to enable on line editing of computer programs and was adopted and extended to aid document creation. Systems representing some of the early efforts in this area are described by Callahan and Grace [2] and by Magnuson [5]. Englebart and English [3] describe a more general system for "augmenting human intellect" in which text manipulation capabilities form a substantial part. Presently, computer assisted text editing systems are available on a commercial basis as part of time sharing computer services, as special purpose shared-logic systems driven by in house minicomputers and in the form of numerous stand alone units (see, for example, [9], [10]).

In general such systems, often referred to as word processing systems, consist of a keyboard, a means of display such as a sheet of paper, a CRT, etc., storage for file and buffer, logic with a capability to search, retrieve and edit stored information, and a hard copy printer. The merits of these types of equipment and their relative effectiveness have been extensively discussed on a qualitative basis in the *Word Processing Report*, a biweekly publication of the Word Processing Institute, and in other professional magazines. A somewhat more quantitative cost benefit analysis has been published by Skolnik and Jenkins [8] comparing the IBM

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Administrative Terminal System (ATS) with their Magnetic Tape Selectric Typewriter (MTST). However, no general theory has been developed that would provide the analytical framework for evaluating the effectiveness of such text editing equipment. Queueing theory, which has been extensively used to model and analyze different aspects of computer time sharing (see, for example, [6]), can provide the framework for analyzing problems associated with sharing logic among text editing units or problems associated with integrating such units into word processing systems. This theory, however, does not enable one to evaluate the effectiveness of an individual unit as a function of its technical features. This paper continues the work done by this author [7], which was a first attempt to model some aspects of man–machine word processing and to develop criteria for evaluating the characteristics of text editing machines with respect to specific applications. While the earlier paper focused on the process of synthesizing a document using stored paragraphs, this one considers the process of editing a stored document.

In general terms, the objective of editing is to implement a revision that may be viewed as a transformation of a text entity represented in some memory medium into another text entity shown on the same or a different medium. It may always be assumed that there is a certain degree of similarity between the original and revised text entities, and one of the main purposes of modern text editing equipment is to aid the operator in exploiting this similarity to reduce the time of producing the revised text representation.

The objective of this paper is to construct a mathematical model of man–machine text editing, to be used in evaluating the potential performance of a given editing machine or determining the tradeoffs involved in designing such a machine. First, we develop a mathematical representation of text revision to be used in characterizing the work load of a man–machine text editing system. Next, we assume an abstract editing man–machine system, which is characterized by a set of parameters. For this machine we obtain an equation for deriving the expected editing time of a given length document and determine the optimal operation policy to minimize this time. This equation is then solved analytically under some simplifying assumption regarding the work load characteristic, and the results are used to derive criteria for evaluating text editing machines. An example is given in which it is demonstrated how such criteria may be used to answer certain questions regarding a specific configuration of a text editing machine.

II. Mathematical Representation of Text Revision

In an abstract sense, a document may be viewed as a string of characters and a revision as a transformation mapping a source string (S_0) onto a target string (S_T). Consequently, text editing may be viewed as the process of implementing a revision. In general, a revision will consist of a series of numerous editing functions which are applied to S_0. However, for simplicity we shall assume that only three editing functions are used: 1) reusing substrings of S_0 (skipping), 2) inserting substrings of S_T, and 3) deleting substrings of S_0. More complex editing functions such as moving a substring from one place to another or substituting one substring for another can be viewed as combinations of these three basic editing functions. For convenience of representation, it will be assumed that the three editing functions are applied periodically in the following order: reusing, inserting, and deleting. This assumption is valid since we allow reusing, inserting, and deleting of null strings, and it allows us to represent a revision as a sequence of 3-tuples $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n$, where Λ_i denotes reused substrings, Υ_i inserted substrings, and Δ_i deleted substrings.

It is clear that for $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n$ to represent a revision from S_0 to S_T the following must hold.

$$S_0 = \sum_{i=1}^{n} (\Lambda_i + \Delta_i)$$

$$S_T = \sum_{i=1}^{n} (\Lambda_i + \Upsilon_i) \tag{1}$$

where $\sum$ and $(+)$ denote concatenation. It follows from (1) that the representation $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n$ completely specifies S_0 and S_T, and thus it can be used to describe the target and source string as well as how the editing is implemented. This is actually done in word processing where an originator usually describes the desired version of a document by marking deletions and insertions on a hard copy of the original version.

It is clear at this point that any revision that satisfies relations (1) will transform S_0 onto S_T. Thus we may define the set of revisions transforming S_0 onto S_T as

$$\mathscr{R}(S_0,S_T) \equiv \left\{ \{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n \;\middle|\; \sum_{i=1}^{n} (\Lambda_i + \Delta_i) = S_0, \; \sum_{i=1}^{n} (\Lambda_i + \Upsilon_i) = S_T \right\}$$

where n is any integer. The set $\mathscr{R}(S_0,S_T)$ consists of a large number of revisions performing the same text editing. Thus one may expect that different revisions will be implemented by different operators for the same text editing task. The revision implemented by a typist, for instance, will be different from that used by the originator to describe S_T. An implemented revision usually depends on subjective considerations of the typist, the equipment, the storage medium containing the source string, and the required form of output. An extreme case is one where a clean hard copy (no striking over) has to be produced on a regular typewriter from a hard copy draft. In this case, the whole document must be retyped. This can be viewed as deletion of the entire source string and insertion of the entire target string, so that the implemented revision is $\{\text{null}, S_T,S_0\}_1^1$. Clearly, all the man–machine implemented revisions transforming S_0 into S_T form a subset of $\mathscr{R}(S_0,S_T)$. By postulating the structure of such revisions, we shall now characterize this subset. If we compared revisions used by originators to describe S_T and the ones implemented by the typists, we would find that they are based on the same "matching" of substrings in S_0 and S_T, while the differences

are in the amount of reused text. In other words, a typist may choose to delete and reinsert a substring, which in the originator's revision was reused, or reuse a substring of S_0, which the originator deleted and reinserted. We assume that this relation exists among all man–machine implemented revisions in $\mathscr{R}(S_0,S_T)$. Furthermore, we assume that there exists a revision in $\mathscr{R}(S_0,S_T)$ from which all man–machine implemented revisions may be constructed through replacement of reused substrings with deletions and reinsertions. To formalize these assumptions we introduce two concepts. The first to be defined is the "embedding" operation, which is a mapping from $\mathscr{R}(S_0,S_T)$ into itself. This operation consists of replacing reused substrings in a revision by deleted and inserted substrings, and it will be characterized by a Boolean sequence $\{k_i\}$ that determines which of the reused substrings in the original revision will be substituted. The following is a formal definition of "embedding".

Definition 1: Let $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n \in \mathscr{R}(S_0,S_T)$ and $\{k_i\}_1^n$ be a Boolean sequence. Then an $\{k_i\}_1^n$ embedding of $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n$ denoted by $[\{\Lambda_i,\Upsilon_i,\Delta_i\},k_i]_1^n$ is a revision such that

$$[\{\Lambda_i,\Upsilon_i,\Delta_i\},k_i]_1^n \triangleq \{\bar{\Lambda}_i,\bar{\Upsilon}_i,\bar{\Delta}_i\}_1^n \in \mathscr{R}(S_0,S_T)$$

where, for $i = 1,2,\cdots,n$

$$\left.\begin{aligned}\bar{\Lambda}_i &= \text{null}\\ \bar{\Upsilon}_i &= \Lambda_i + \Upsilon_i\\ \bar{\Delta}_i &= \Lambda_i + \Delta_i\end{aligned}\right\}\text{ if } k_i = 1 \qquad \left.\begin{aligned}\bar{\Lambda}_i &= \Lambda_i\\ \bar{\Upsilon}_i &= \Upsilon_i\\ \bar{\Delta}_i &= \Delta_i\end{aligned}\right\}\text{ if } k_i = 0.$$

It follows from Definition 1 that if a certain revision can be obtained by embedding another revision, then the first one can be viewed as a reduction of the second since it reduces the amount of deleted and inserted text. This leads to the definition of "irreducible revision".

Definition 2: A revision is said to be irreducible and is denoted by $\{\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i\}_1^n$ if it satisfies the following properties: 1) $\{\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i\}_1^n = [\{\Lambda_i,\Upsilon_i,\Delta_i\},k_i]_1^n$, only if $\{\Lambda_i,\Upsilon_i,\Delta_i\}_1^n = \{\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i\}_1^n$; 2) $\hat{\Upsilon}_i + \hat{\Delta}_i \neq$ null, for $i = 1,2,\cdots,n$; 3) $\hat{\Lambda}_i \neq$ null, for $i = 1,2,\cdots,n$.

The first property of an irreducible revision implies that it can be only an embedding of itself, while the last two properties imply that this revision is represented by the smallest possible number of 3-tuples. In terms of these definitions, a key assumption of our model is that man–machine implemented revisions are of the form $[\{\hat{\Lambda}_i,\Upsilon_i,\hat{\Delta}_i\},k_i]_1^n$. In other words, the subset of $\mathscr{R}(S_0,S_T)$ consisting of all man–machine implemented revisions consists of embeddings of some irreducible revision in $\mathscr{R}(S_0,S_T)$. Furthermore, we assume that the irreducible revision is an objective characteristic of the pair (S_0,S_T), while the Boolean sequence $\{k_i\}_1^n$ characterizes the particular embedding implemented by the operator.

These assumptions allow us to describe text editing as a multistage control process in which the operator perceives the editing task in terms of a sequence of 3-tuples $\{\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i\}_1^n$ forming an irreducible revision, and he controls the implementation of this task by selecting (consciously or subconsciously) the zero-one control variables k_i. Another key assumption, which allows us to represent the control sequence $\{k_i\}$ by a single scalar, is that of a threshold control policy. In particular, we assume that the criterion for selecting k_i at each stage is the length l_i of the reused substring $\hat{\Lambda}_i$. Whenever this length is below a threshold length r, the substring $\hat{\Lambda}_i$ is deleted and reinserted; while if it is longer than r, it is reused. In other words, $k_i = 1$, if $l_i < r$, $k_i = 0$, if $l_i \geq r$. The intuitive motivation for this assumption is that switching back and forth between the "delete/insert" and "skip" modes in an editing process involves some inconvenience, which the operator will be reluctant to undertake unless it can save him a significant amount of rekeyboarding.

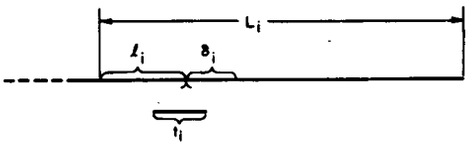

Fig. 1. Illustration of substring lengths l_i, t_i, δ_i, L_i.

So far we have considered the representation of individual editing tasks. For evaluating the performance of a text editing man–machine system, we have to consider a whole spectrum of such tasks forming a typical work load.

The textual content of the revisions is of less importance for such an analysis; hence, the work load nature may be characterized in terms of the statistics on the length of reused, inserted, and deleted substrings. In view of these considerations, we shall analyze the processing of a representative revision $[\{\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i\},k_i]_1^n$, in which the length l_i,t_i,δ_i of the substrings $\hat{\Lambda}_i,\hat{\Upsilon}_i,\hat{\Delta}_i$ are samples from a joint probability distribution on these lengths, which encodes the aforementioned statistics. Clearly, at any stage of the revision the length of deleted and reused substrings cannot exceed the remaining length in the source string. Thus the probability distribution will be conditional on L_i, which denotes the length from the start of the reused substring Λ_i to the end of the document (see Fig. 1). For simplicity we shall treat string length as a continuous variable so that l_i, t_i, δ_i will be sampled from a conditional joint density function $p(l,t,\delta|L)$. Given the total length of the document, this density function also implicitly determines the probability distribution on the total number of 3-tuples n. The sequence $\{k_i\}$ in the representative revision is determined by a threshold policy described earlier, which was defined in terms of a threshold length r selected by the operator. It is clear from Fig. 1 that $\delta_i < L_i - l_i$; hence, $p(l,t,\delta|L)$ has to satisfy $p(l,t,\delta|L) = 0$, for $\delta > L - l$. It is also obvious that $l_i < L_i$. However, instead of requiring that $p(l,t,\delta|L) = 0$, for $l > L$, it is convenient to let $p(l,t,\delta|L) \neq 0$ for this range and use the cumulative

$$P^{>}(L) = \int_L^\infty dl \int_0^\infty dt \int_0^{L-l} p(l,t,\delta \mid L)\, d\delta$$

to encode the finite probability of not having any changes in the remaining L length of the document. The monotonic decrease of $P^{>}(L)$ with L is consistent with the intuitive properties of this probability.

III. Recursive Equation for the Expected Editing Time of a Given Document

In this section we derive a recursive equation for the expected editing time of a document, given the threshold length r and the probability density function describing the editing load $p(l,t,\delta|L)$. For the sake of generality we shall not confine our analysis to any specific text editing machine, form of text representation, or storage medium. Instead, we assume a general text editing man–machine system, which can be viewed as a black box characterized by the following four parameters:

- c_δ deletion speed (time/unit length),
- c_t insertion speed (time/unit length),
- c_l reusing (skip or copy) speed (time/unit length),
- c edit control time per reused substring.

It is assumed that c_t includes the time for correcting typing errors during insertion and proofreading of the insertion. The expected editing time of a document may be expressed now in terms of the aforementioned parameters of the system. Let $T(L,r)$ denote the expected time of editing a document of length L using the threshold r. Assuming this is equivalent to the editing, the L remaining length of a longer document $T(L,r)$ satisfies the following recursive equations:

$$\begin{aligned} T(L,r) = &\int_0^r \int_0^\infty \int_0^{L-l} p(l,t,\delta \mid L)[(c_\delta + c_t)l + c_\delta\delta \\ &+ c_t t + T(L - l - \delta, r)]\, d\delta\, dt\, dl \\ &+ \int_r^L \int_0^\infty \int_0^{L-l} p(l,t,\delta \mid L)[c_l l + c_\delta\delta + c_t t \\ &+ c + T(L - l - \delta, r)]\, d\delta\, dt\, dl \\ &+ \int_L^\infty \int_0^\infty \int_0^{L-l} p(l,t,\delta \mid L) c_l L\, d\delta\, dt\, dl, \end{aligned} \quad \text{for } L > r. \tag{2.a}$$

$$\begin{aligned} T(L,r) = &\int_0^L \int_0^\infty \int_0^{L-l} p(l,t,\delta \mid L)[c_\delta + c_t)l + c_\delta\delta + c_t t \\ &+ T(L - l - \delta, r)]\, d\delta\, dt\, dl \\ &+ \int_L^\infty \int_0^\infty \int_0^{L-l} p(l,t,\delta \mid L) c_l L\, d\delta\, dt\, dl, \end{aligned} \quad \text{for } L \le r. \tag{2.b}$$

Equations (2.a) and (2.b) express $T(L,r)$ as the expected sum of the processing time of the current triplet (l,t,δ) and the expected editing time of what is left after that, i.e., $T(L - l - \delta, r)$. Equation (2.a) considers the case where $L > r$; the first term in this equation accounts for the case where l is below the threshold. In this case the reusable substring is deleted and reinserted, which allows augmenting the deletion and insertion to the preceding ones, thus saving the additional edit control. The second term in (2.a) accounts for the case where l is above the threshold, in which case the reusable segment is reused, and edit control is required to re-enter the delete/insert mode. The third term accounts for the event that no changes are required in the remaining text, in which case it is all reused. Equation (2.b) considers the case where $L \le r$. This equation may be viewed as a restriction of (2.a), where no reusable substring exceeds the threshold r; therefore, the second term is omitted. Furthermore, since the reusable substring is no longer than L, the integration over l in the first term goes only up to L.

The threshold length r in (2.a) and (2.b) is allowed to be a function of L. However, we shall consider only the case where r is fixed. For future reference, we denote the right side of (2.a) by $\tau(L,r)$. Thus (2.a) and (2.b) may be rewritten in the compact form

$$T(L,r) = \begin{cases} \tau(L,r), & \text{for } r < L \\ \tau(L,L), & \text{for } r \ge L. \end{cases} \tag{3}$$

IV. The Optimal Threshold Policy

The problem of determining the optimal threshold length $\hat{r}$, which will minimize the expected editing time of a document, can be formulated as a dynamic programming problem. Let $\hat{\tau}(L,r)$ denote the right side of (2.a) after $T(L - l - \delta, r)$ is replaced by $T(L - l - \delta, \hat{r})$. By Bellman's [1] principle of optimality

$$\begin{aligned} T(L,\hat{r}) &= \min \left[\min_{0 \le r \le L} \hat{\tau}(L,r), \hat{\tau}(L,L) \right] \\ &= \min_{0 \le r \le L} \hat{\tau}(L,r) \triangleq \tau(L,\hat{r}). \end{aligned} \tag{4}$$

For $\hat{r}$ to minimize $\hat{\tau}(L,r)$ subject to $0 \le r \le L$, it has to satisfy the necessary conditions

$$\left.\frac{d\hat{\tau}(L,r)}{dr}\right|_{r=\hat{r}} \begin{cases} \le 0, & \text{if } \hat{r} = L \\ = 0, & \text{if } 0 \le \hat{r} \le L \\ \ge 0, & \text{if } \hat{r} = 0 \end{cases} \tag{5}$$

$$\begin{aligned} \frac{d\hat{\tau}(L,r)}{dr} = &\int_0^\infty \int_0^{L-r} p(r,t,\delta \mid L)[(c_\delta + c_t)\, r + c_\delta\, \delta \\ &+ c_t t + T(L - r - \delta, \hat{r})]\, d\delta\, dt \\ &- \int_0^\infty \int_0^{L-r} p(r,t,\delta \mid L)[c_l r + c_\delta\, \delta + c_t t \\ &+ c + T(L - r - \delta, \hat{r})]\, d\delta\, dt \\ = &\int_0^\infty \int_0^{L-r} p(r,t,\delta \mid L)[(c_t + c_\delta - c_l)r - c]\, d\delta\, dt. \end{aligned} \tag{6}$$

Let

$$\tilde{r} = \frac{c}{c_t + c_\delta - c_l}. \tag{7}$$

Then by (5) and (6)

$$\hat{r} = \begin{cases} L, & \text{if } L \le \tilde{r} \\ \tilde{r}, & \text{if } 0 \le \tilde{r} \le L \\ L, & \tilde{r} \le 0. \end{cases} \tag{8}$$

If $\tilde{r}$ is negative, then $c_t + c_\delta < c_l$, which means that

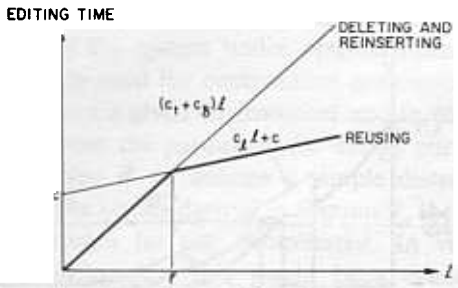

Fig. 2. Processing time of reusable substring via reusing versus deleting and reinserting as function of substring length.

deleting and reinserting is faster than reusing. It is obvious that in such a case the optimal strategy is to delete and retype the entire document, thus the optimal threshold is L. The case just given is unlikely in practical situations and will be excluded. This allows one to rewrite (8) as

$$\hat{r} = \min [\tilde{r}, L]. \tag{9}$$

Fig. 2 illustrates the cost as a function of length for processing a reusable substring by reusing versus deleting and reinserting. It is evident from this figure that $\tilde{r}$ is the breakpoint at which the first alternative becomes better. Furthermore, the optimal threshold $\hat{r}$ is the threshold length that yields the shortest processing time for each reusable substring (marked in Fig. 2 by a heavy line).

V. An Explicit Solution for $T(L,r)$

In this section we shall assume a special form of $p(l,t,\delta/L)$ which will allow us to analytically solve the integral equations (2.a) and (2.b). The solution will be given in terms of the threshold r, which is assumed at this point to be a fixed positive scalar.

It is true, in general, that $p(l,t,\delta/L) = p(\delta/t,l,L)p(t/l,L) \cdot p(l/L)$. We assume that t and l are independent of l or L, i.e., $p(t/l,L) = p(t)$ and $p(l/L) = p(l)$, and that δ depends only on $L - l$ so that $p(\delta/t,l,L) = p(\delta/L - l)$, where $p(\delta/L - l) = 0$, for $\delta > L - l$. Substituting this into the right side of (2.a) yields

$$\begin{aligned}\tau(L,r) = \int_0^r p(l) \Big[& (c_\delta + c_t)l + c_\delta \cdot \langle \delta \mid L - l \rangle + c_t \cdot \langle t \rangle \\ & + \int_0^{L-l} p(\delta \mid L - l) T(L - l - \delta, r)\, d\delta \Big]\, dl \\ & + \int_r^L p(l) \Big[(c_l l + c_\delta \cdot \langle \delta \mid L - l \rangle + c_t \cdot \langle t \rangle + c \\ & + \int_0^{L-l} p(\delta \mid L - l) T(L - l - \delta, r)\, d\delta \Big]\, dl \\ & + c_l L \cdot \int_L^\infty p(l)\, dl. \end{aligned} \tag{10}$$

The notation $\langle \cdot \rangle$ denotes expected value. We shall now assume $p(l)$ and $p(t)$ are exponential distributions, and $p(\delta/L - l)$ is a truncated exponential. Thus we set

$$p(l) = \eta e^{-\eta l} \tag{11}$$

$$p(t) = \lambda e^{-\lambda t} \tag{12}$$

and

$$p(\delta \mid L - l) = \begin{cases} \mu e^{-\mu\delta}/(1 - e^{-\mu(L-l)}), & \text{for } \delta \le L - l \\ 0, & \text{for } \delta > L - l. \end{cases} \tag{13}$$

The expected values of these distributions are, respectively

$$\langle l \rangle = 1/\eta \tag{14}$$

$$\langle t \rangle = 1/\lambda \tag{15}$$

$$\langle \delta \mid L - l \rangle = 1/\mu - (L - l)/(e^{\mu(L-l)} - 1) \tag{16}$$

and

$$P^{>}(L) = \int_L^\infty p(l)\, dl = e^{-\eta L}. \tag{17}$$

The length l of reused substrings may also be viewed as the distance between the start and end of subsequent changes in the source document. Thus assuming an exponential distribution for l is equivalent to the assumption that changes have an equally likely probability of starting at any distance (within the document) from the end of the previous change, while the probability of no changes decreases exponentially with L.

Similarly, assuming the exponential and truncated exponential distributions for t and δ is equivalent to assuming that an insertion has an equally likely probability to end at any distance from where it started, and a deletion has an equally likely probability to end at any place between its starting point and the end of the document. These distributions assess a higher probability for short reused, deleted, and inserted substrings in the irreducible revision, which is intuitively reasonable.

Substituting (11)–(17) into (10), along with some manipulation,[1] yields

$$\begin{aligned}\tau(L,r) = \alpha & + e^{-\eta r}\left(\tilde{r} - r - \frac{1}{\eta}\right)\frac{c}{\tilde{r}} \\ & + e^{-\eta L}\Big[-\alpha - \left(\tilde{r} - \frac{1}{\eta}\right)\frac{c}{\tilde{r}} - c_\delta \int_0^L \frac{\eta e^{\eta l} l}{(e^{\mu l} - 1)}\, dl \\ & + \int_0^L \int_0^l \frac{\eta\mu e^{\eta l}}{e^{\mu l} -} e^{\mu\delta} T(\delta,r)\, d\delta\, dl\Big] \end{aligned} \tag{18}$$

$$\alpha = c_t\left(\frac{1}{\eta} + \frac{1}{\lambda}\right) + c_\delta\left(\frac{1}{\eta} + \frac{1}{\mu}\right) \tag{19}$$

and $\tilde{r}$ is defined by (7). As shown earlier, $T(L,r)$ is defined by the integral equations

$$T(L,r) = \tau(L,r), \quad \text{for } L > r \tag{20.a}$$

$$T(L,r) = \tau(L,L), \quad \text{for } L \le r \tag{20.b}$$

and the boundary condition $T(0,r) = 0$. Differentiating (20.a) and (20.b) twice with respect to L, along with some

[1] The detailed derivation of the steps described in this section is given in the Appendix.

manipulation, yields a differential equation of the form

$$\frac{dR(L,r)}{dL} + \left(\eta + \frac{\mu}{1 - e^{-\mu L}}\right) R(L,r) = F(L,r) \quad (21)$$

where

$$R(L,r) = \frac{dT(L,r)}{dL}$$

and

$$F(L,r) \begin{cases} \dfrac{\mu\eta\alpha - \eta c_\delta e^{-\mu L} + (\mu\eta c/\tilde{r})(\tilde{r} - r - 1/\eta)e^{-\eta r}}{1 - e^{-\mu L}} & \text{for } L > r \\[2ex] \dfrac{\mu\eta\alpha - \eta c_\delta e^{-\mu L} + (\mu\eta c/\tilde{r})[(1 - e^{-\mu L})/\mu - 1/\eta]e^{-\eta L}}{1 - e^{-\mu L}}, & \text{for } L \le r. \end{cases} \quad (22)$$

Substituting $L = 0$ and $L = r$ in the first derivative of (20) yields the boundary conditions

$$R(0,r) = c_l + \frac{\eta}{\lambda} c_t \quad (23)$$

and

$$R(r^+,r) = R(r^-,r) + \eta c \left(1 - \frac{r}{\tilde{r}}\right) e^{-\eta r}. \quad (24)$$

The differential equation (21) has the solution

$$R(L,r) = \frac{1}{e^{\eta L}(e^{\mu L} - 1)} \left(K + \int_0^L e^{\eta l}(e^{\mu l} - 1)F(l,r)\, dl\right) \quad (25)$$

where K is a constant to be determined from the boundary conditions. Substituting $F(l,r)$ defined by (22) into (25) and using the conditions (23) and (24) to obtain K, for $L \le r$ and for $L > r$, results in the following expressions for $R(L,r)$:

$$R(L,r) = \frac{1}{e^{\eta L}(e^{\mu L} - 1)} \left\{ \frac{\mu\eta\alpha}{\eta + \mu}[e^{(\eta+\mu)L} - 1] - c_\delta[e^{\eta L} - 1] + \frac{\mu\eta c}{\tilde{r}(\eta + \mu)}\left[\tilde{r} - r - \frac{1}{\eta}\right][e^{(\mu+\eta)(L-r)} - 1]e^{\mu r} + \frac{c\eta}{\tilde{r}}\left[(e^{\mu r} - 1)\left(\tilde{r} - r - \frac{1}{\eta} + \frac{1}{\mu}\right) - r\right]\right\}, \quad \text{for } L > r. \quad (26.a)$$

$$R(L,r) = \frac{1}{e^{\eta L}(e^{\mu L} - 1)} \left\{ \frac{\mu\eta\alpha}{\eta + \mu}[e^{(\eta+\mu)L} - 1] - c_\delta[e^{\eta L} - 1] + \frac{c\eta}{\tilde{r}}\left[(e^{\mu L} - 1)\left(\frac{1}{\mu} - \frac{1}{\eta}\right) - L\right]\right\}, \quad \text{for } L \le r. \quad (26.b)$$

The expression (26), together with (7) and (19), allow us to evaluate $R(L,r)$, the marginal expected editing time per additional unit length of the source document, for a given man–machine system characterized by the parameter, c_l, c_t, c_δ, c, r, and a work load characterized by the parameters λ, μ, η. Fig. 3 illustrates qualitatively the dependency of $R(L,r)$ on L. We note that as L increases, $R(L,r)$ approaches an asymptotic value $R(\infty,r)$, which may be derived by letting $L \to \infty$ in (26.a)

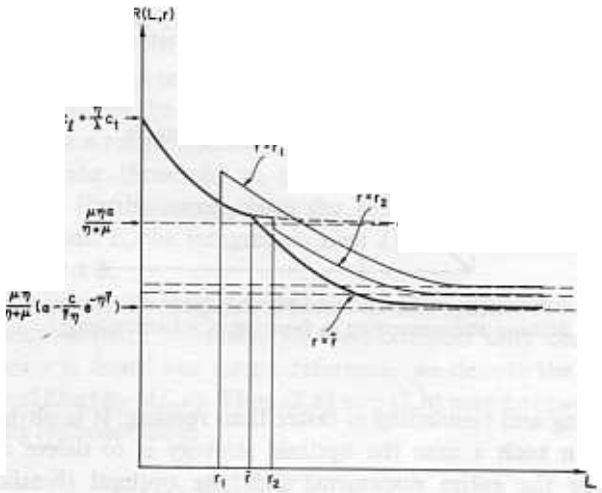

Fig. 3. Marginal expected editing time per unit length of source document as function of total length.

$$R(\infty,r) = \lim_{L\to\infty} R(L,r) = \frac{\mu\eta}{\eta + \mu}\left[\alpha + \frac{c}{\tilde{r}}\left(\tilde{r} - r - \frac{1}{\eta}\right) e^{-\eta\tilde{r}}\right]. \quad (27)$$

For the optimal threshold policy

$$R(\infty,\hat{r}) = \frac{\mu\eta}{\eta + \mu}\left(\alpha - \frac{c}{\tilde{r}\eta} e^{-\eta\tilde{r}}\right). \quad (28)$$

We also note from (24) that in this case $R(L,\hat{r})$ is continuous at $L = r = \tilde{r}$. The total expected editing time $T(L,r)$ can be obtained by integrating (26.a) and (26.b), and using the conditions that $T(0,r) = 0$ and $T(L,r)$ is continuous at $L = r$.

VI. Criteria for Evaluation of Text Editing Systems

Since the editing time is a major factor in determining the editing cost of a document, it seems attractive to measure the relative efficiency of alternative text editing systems in terms of the average editing time per unit document length. For a given length document the expected editing time per unit length is given by $T(L,r)/L$ and may be evaluated analytically under the assumptions of the previous section. Suppose now that $p(L)$ is a probability distribution representing document length statistics in the particular environment under consideration over a long period of time. Then the average editing time per unit length will be

$$E(r) = \int_0^\infty \frac{T(L,r)}{L} p(L)\, dL. \quad (29)$$

This criterion is a function of the systems parameters c_l, c_t, c_δ, c, the work load statistics, and the threshold r. Since r may be regarded as an input to the system, we would like to eliminate it from our criterion. This may be done by substituting $r = \hat{r}$ in (29). The resulting criterion $E(\hat{r})$, which depends only on the system parameters and work

load statistics, may be interpreted as the "ideal" average editing rate of the system under optimal operation. This criterion can be used for comparative evaluation of a text editing system in a given environment and for determining tradeoffs between the parameter for design purposes. Unfortunately, even if we assume a simple distribution for $p(L)$ and use the results derived in Section V, the expression for $E(\hat{r})$ becomes far too complicated. In view of the simplistic assumptions, this would be a crude measure anyhow, and the effort required to evaluate it is not worthwhile. A simpler criterion can be obtained by considering the asymptotic behavior of $T(L,r)/L$ under the assumptions of Section V. Since $R(L,r)$ is monotonically decreasing with respect to L, $T(L,r)$ is concave and

$$\frac{T(L,r)}{L} \geq R(L,r), \qquad \text{for all } L \in [0,\infty]). \tag{30}$$

Furthermore, $T(L,r)/L$ is monotonically decreasing and

$$\lim_{L\to\infty} \frac{T(L,r)}{L} = R(\infty,r) \tag{31}$$

One can easily show that

$$R(\infty,\hat{r}) = \inf_{r\in[0,\infty)} R(\infty,r).$$

Thus,

$$\frac{T(L,r)}{L} \geq R(\infty,\hat{r}), \qquad \text{for all } r, L \in [0,\infty) \tag{33}$$

with equality for $r = \hat{r}$ and $L \to \infty$. Consequently,

$$E(\hat{r}) \geq R(\infty,\hat{r}), \qquad \text{for any distribution } p(L). \tag{34}$$

In view of the these considerations and the simple form of (28), it seems attractive to use $R(\infty,\hat{r})$ as a crude criterion for evaluating the relative effectiveness of text editing systems and determining crude design tradeoffs for such systems.

Another criterion that one may want to consider particularly for design purposes is the average expected cost per unit length of document. One may assume that for a given configuration the man–machine system cost per unit time is a function of its parameters, say $\mathscr{C}(c,c_l,c_t,c_\delta)$. Thus, the average expected editing cost per unit length will be $E(r)\cdot\mathscr{C}(c,c_l,c_t,c_\delta)$. Following the previous argument, $E(r)$ may be replaced by $R(\infty,\hat{r})$, which by (28) will yield the criterion

$$C = \frac{\mu\eta}{\eta+\mu}\left(\alpha - \frac{c}{\tilde{r}\eta}e^{-\eta\tilde{r}}\right)\mathscr{C}(c,c_l,c_t,c_\delta) \tag{35}$$

where $\tilde{r}$ and α are defined by (7) and (19). This criterion might be used as an objective function for optimal design. Assuming that $\mathscr{C}(\cdot)$ is available, one may obtain the optimal parameter for a system by minimizing C with respect to these parameters and subject to the constraints imposed by feasibility considerations.

VII. A Special Case

To demonstrate the potential applications of the results derived in the previous section we present a simple example in which we illustrate how these results may be used to answer basic questions regarding a specific configuration of text editing systems.

We consider a text editing system consisting of a typical CRT terminal equipped with a pointing device and connected to CPU and storage. In such a configuration, which may exist as a stand-alone unit or as a part of a shared-logic system, the edited document and the operator's key strokes are stored in an electronic memory and displayed on the CRT. The editing is implemented on such a system by using the pointing device to bracket the deleted substring (or indicate the insertion point if there is no deletion involved) and then type the appropriate command. The control time c in this case will thus be twice the mean time for bringing the pointer to a desired spot on the screen plus the mean time for typing the control command. The deletion and skipping rates c_l and c_δ are zero, while the insertion rate c_t is the typing rate of the operator (including on line proofreading and error correction). An important question that may arise in designing a system is the marginal value of improving the control time in terms of the overall performance of the system. To answer this question, we use the criterion $R(\infty,\hat{r})$ as a measure of performance and substitute $c_l = c_\delta = 0$ in (28), (7), and (19). This yields

$$\tilde{r} = \frac{c}{c_t} \tag{36}$$

$$\alpha = c_t\left(\frac{1}{\lambda} + \frac{1}{\eta}\right) \tag{37}$$

and consequently

$$R(\infty,\hat{r}) = \frac{\mu c_t}{\eta+\mu}\left[\frac{\eta}{\lambda} + (1 - e^{-\eta c/c_t})\right]. \tag{38}$$

The marginal influence of the control time is then given by

$$\frac{dR(\infty,\hat{r})}{dc} = \frac{\eta\mu}{\eta+\mu}e^{-\eta c/c_t} < \frac{\eta\mu}{\eta+\mu} \tag{39}$$

Fig. 4 illustrates the dependency of $dR(\infty,\hat{r})/dc$ on the systems and work load parameters. This figure shows that the marginal benefit from reducing the control time decreases as the control gets larger. For a fixed control time this marginal benefit increases with the ratio c_t/η. To interpret this result, we note that $1/\eta$ is the mean length of reusable substrings and c_t is the typing rate. Thus c_t/η may be interpreted as the mean time it would take to retype a reusable substring. As this time gets larger, an operator will tend to reuse more substrings and, hence, use the control more often. This obviously increases the benefit from reducing the control time as indicated by the aforementioned results. We observe further from (39) and Fig. 4 that $dR(\infty,\hat{r})/dc$ is proportional to the term $\eta\mu/(\eta+\mu)$, which is also its upper bound. To interpret this term, we note that $1/\mu$ is the mean deletion length in the irreducible revision, and as mentioned previously, $1/\eta$ is the mean length of reusable substrings. Thus, this term, which may also be written as $1/(1/\eta + 1/\mu)$, can be interpreted as the mean number of times per unit length of the original document that one would have to use the control if he implemented

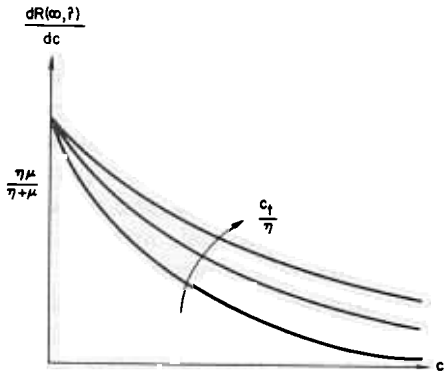

Fig. 4. Dependency of marginal value of control time on system and work load parameters for CRT text editing system.

the irreducible revisions of an infinitely long document. This clearly sets an upper bound on the marginal benefit from reducing the control time.

VIII. Conclusions

In this paper we introduced a mathematical representation of text editing tasks. This representation was used to characterize the work load in a model describing the implementation of text editing by an abstract man–machine system described by a set of parameters. For this model we obtained the optimal operating strategy of such a system. With some simplistic assumptions about the work load, we also derived expressions for the expected editing rate as a function of the document length and the parameters describing the work load and the system. These results, though based on somewhat simplistic assumptions, may be used to derive crude quantitative criteria for comparing alternative text editing systems and for determining the tradeoffs involved in designing such systems. These criteria depend on the work load characteristics that are encoded in terms of the three parameters λ, μ, η describing the theoretical probability density function of the length of reused, inserted, and deleted substrings in irreducible revisions. In a practical application, this theoretical pdf may be viewed as a first-order approximation of the actual pdf. Thus λ, μ, η can be inferred from the mean length of the reusable, inserted, and deleted substrings in the work load under consideration.

In summary, one should emphasize that the model and the analysis presented in this paper are not restricted by any means to the applications described. Furthermore, the methodology used here may be applied as well to editing systems that cannot be described in terms of the abstract configuration used in this paper.

Appendix

Detailed Derivation of (18) to (26)

Substituting (11)–(17) into (10) yields

$$\tau(L,r) = (c_\delta + c_t)\eta \int_0^r e^{-\eta l} l\, dl + c_l \eta \int_r^L e^{-\eta l} l\, dl + c\eta \int_r^L e^{-\eta l}\, dl + c_\delta \eta \int_0^L e^{-\eta l} \left[\frac{1}{\mu} - \frac{L-l}{e^{\mu(L-l)}-1}\right] dl$$
$$+ c_t \frac{\eta}{\lambda} \int_0^L e^{-\eta l}\, dl + \int_0^L \int_0^{L-l} \frac{\eta e^{-\eta l} \mu e^{-\mu\delta}}{1 - e^{-\mu(L-l)}} T(L - l - \delta, r)\, d\delta\, dl + c_l L e^{-\eta L}.$$

The sixth term in (A.1) can be rewritten and modified by proper substitution as follows:

$$\int_0^L \left[\frac{\eta\mu e^{-\eta l}}{1 - e^{-\mu(L-l)}} \int_0^{L-l} e^{-\mu\delta} T(L - l - \delta, r)\, d\delta\right] dl$$
$$= \int_0^L \eta\mu e^{-\eta l} \left[\frac{1}{1 - e^{-\mu(L-l)}} \int_0^{L-l} e^{-\mu(L-l-\delta)} T(\delta,r)\, d\delta\right] dl$$
$$= \int_0^L \eta\mu e^{-\eta(L-l)} \left[\frac{1}{1 - e^{-\mu l}} \int_0^{l} e^{-\mu(l-\delta)} T(\delta,r)\, d\delta\right] dl$$
$$= e^{-\eta L} \int_0^L \int_0^l \frac{\eta\mu e^{\mu l} e^{-\mu(l-\delta)}}{1 - e^{-\mu l}} T(\delta,r)\, d\delta\, dl. \tag{A.2}$$

Similarly, the second part of the fourth term in (A.1) can be modified, using the relation

$$\int_0^L e^{-\eta l} \frac{L - l}{e^{\mu(L-l)} - 1}\, dl = \int_0^L e^{-\eta(L-l)} \frac{l}{e^{\mu l} - 1}\, dl$$
$$= e^{-\eta L} \int_0^L \frac{e^{\eta l} l}{e^{\mu l} - 1}\, dl.$$

Substituting (A.2) and (A.3) into (A.1) and carrying the integrations in the remaining terms yields

$$\tau(L,r) = \left[c_t\left(\frac{1}{\eta} + \frac{1}{\lambda}\right) + c_\delta\left(\frac{1}{\eta} + \frac{1}{\mu}\right)\right]$$
$$+ \left[(c_l - c_\delta - c_t)\left(\frac{1}{\eta} + r\right) + c\right] e^{-\eta r}$$
$$+ e^{-\eta L}\left[-\left(\frac{c_\delta}{\mu} + \frac{c_t}{\lambda} + c + \frac{c_l}{\eta}\right)\right.$$
$$- \eta c_\delta \int_0^L \frac{e^{\eta l} l}{e^{\mu l} - 1}\, dl$$
$$\left. + \int_0^L \int_0^l \frac{\eta\mu e^{\eta l} e^{\mu\delta}}{e^{\mu l} - 1} T(\delta,r)\, d\delta\, dl\right]. \tag{A.4}$$

Substituting α and $\tilde{r}$, as defined in (7) and (19), into (A.4) yields (18). The integral equations (20.a) and (20.b) are both of the form

$$T(L,r) = a_i + e^{-\eta L} b_i(L,r) \tag{A.5}$$

where $i = 1$, for $L > r$, and $i = 2$, for $L \le r$. In both cases, $b_i(L,r)$ contains a double integral. To eliminate this integral, (A.5) must be differentiated twice with respect to L after rewriting it in the form

$$(T(L,r) - a_i)e^{\eta L} = b_i(L,r). \tag{A.6}$$

This yields

$$e^{\eta L}\left(\frac{dT}{dL} + \eta T - \eta a_i\right) = \frac{db_i}{dL}$$

and

$$e^{\eta L}\left(\frac{d^2T}{dL^2} + 2\eta\frac{dT}{dL} + \eta^2 T - \eta^2 a_i\right) = \frac{d^2 b_i}{dL^2}. \quad \text{(A.8)}$$

For $L > r$ we have $T(L,r) = \tau(L,r)$. Hence, by (18)

$$a_1 \triangleq \alpha + \left(\tilde{r} - r - \frac{1}{\eta}\right)\frac{c}{\tilde{r}}e^{-\eta r} \quad \text{(A.9)}$$

$$b_1(L,r) \triangleq -\alpha - \frac{c}{\tilde{r}}\left(\tilde{r} - \frac{1}{\eta}\right) - \eta c_\delta \int_0^L \frac{e^{\eta l} l}{e^{\mu l} - 1}\, dl + \int_0^L \int_0^l \frac{\eta\mu e^{\eta l} e^{\mu\delta}}{e^{\mu l} - 1} T(\delta,r)\, d\delta\, dl \quad \text{(A.10)}$$

$$\frac{db_1}{dL} = -c_\delta \eta \frac{e^{\eta L} L}{e^{\mu L} - 1} + \frac{\eta\mu e^{\eta L}}{e^{\mu L} - 1}\int_0^L e^{\mu\delta} T(\delta,r)\, d\delta. \quad \text{(A.11)}$$

This can be rewritten as

$$\left(\frac{e^{(\mu-\eta)L} - e^{-\eta L}}{\mu\eta}\right)\frac{db_1}{dL} = \int_0^L e^{\mu\delta} T(\delta,r)\, d\delta - \frac{c_\delta}{\mu} L. \quad \text{(A.12)}$$

Differentiating (A.12) with respect to L yields

$$\left[\frac{(\mu - \eta)e^{(\mu-\eta)L} + \eta e^{-\eta L}}{\mu\eta}\right]\frac{db_1}{dL} + \left[\frac{e^{(\mu-\eta)L} - e^{-\eta L}}{\mu\eta}\right]\frac{d^2 b_1}{dL^2} = e^{\mu L} T(L,r) - \frac{c_\delta}{\mu}. \quad \text{(A.13)}$$

Substituting (A.7) and (A.8) for $i =$ in (A.13) yields

$$\frac{1}{\mu\eta}\left\{[(\mu - \eta)e^{\mu L} + \eta]\left[\frac{dT}{dL} + \eta T - \eta a_1\right] + [e^{\mu L} - 1] \cdot \left[\frac{d^2T}{dL^2} + 2\eta\frac{dT}{dL} + \eta^2 T - \eta^2 a_1\right]\right\} = e^{\mu L} T - \frac{c_\delta}{\mu}. \quad \text{(A.14)}$$

Collecting terms reduces (A.14) to the form

$$\frac{d^2T}{dL^2} + \left(\eta + \frac{\mu}{1 - e^{-\mu L}}\right)\frac{dT}{dL} = \frac{a_1\eta\mu - \eta c_\delta e^{-\mu L}}{1 - e^{-\mu L}}. \quad \text{(A.15)}$$

Substituting (A.9) into (A.15) yields (21) and (22) for the case $L > r$.

For $L \leq r$ we have $T(L,r) = T(L,L)$. Hence, by (18) and (A.10),

$$a_2 \triangleq \alpha \quad \text{(A.16)}$$

$$b_2(L,r) \triangleq b_1(L,r) + \left(\tilde{r} - L - \frac{1}{\eta}\right)\frac{c}{\tilde{r}} \quad \text{(A.17)}$$

Consequently,

$$\frac{db_2}{dL} = -\frac{c}{\tilde{r}} + \frac{db_1}{dL} \quad \text{(A.18)}$$

and

$$\frac{d^2 b_2}{dL^2} = \frac{d^2 b_1}{dL^2} \quad \text{(A.19)}$$

Substituting db_1/dL and d^2b_1/dL^2 from (A.18) and (A.19) into (A.13) and then using (A.7) and (A.8) for $i = 2$ to eliminate db_2/dL and $db_2{}^2/dL^2$ yields

$$\frac{1}{\mu\eta}\left\{[(\mu - \eta)e^{\mu L} + \eta]\left[\frac{dT}{dL} + \eta T - \eta a_2 + \frac{c}{\tilde{r}}e^{-\eta L}\right] + [e^{\mu L} - 1]\left[\frac{d^2T}{dL^2} + 2\eta\frac{dT}{dL} + \eta^2 T - \eta^2 a_2\right]\right\} = e^{\mu L} T - \frac{c_\delta}{\mu}. \quad \text{(A.20)}$$

Collecting terms in (A.20) and substituting a_2 using (A.16) yields (21) and (22) for the case $L \leq r$.

To derive the boundary condition (23) we set $L = 0$ in (A.7) with $i = 2$ and use (A.16) and (A.18). This yields

$$\left.\frac{dT}{dL}\right|_{L=0} + \eta T(0,r) - \eta\alpha = -\frac{c}{\tilde{r}} + \lim_{L\to 0}\left(\frac{db_1}{dL}\right). \quad \text{(A.21)}$$

Applying L'Hopital's rule to (A.11) and using the condition $T(0,r) = 0$ yields

$$\left.\frac{dT}{dL}\right|_{L=0} - \eta\alpha = -\frac{c}{\tilde{r}} - \frac{c_\delta\eta}{\mu}. \quad \text{(A.22)}$$

Equation (23) is obtained from (A.22) if we substitute α and $\tilde{r}$ using (A.7) and (A.19).

To derive (24) we subtract (A.7) with $i = 1$, evaluated at $L = r^+$ from (A.7) with $i = 2$ evaluated at r^-. After using (A.9), (A.16), and (A.18), this yields

$$e^{\eta r}\left[\left(\left.\frac{dT}{dL}\right|_{L=r^+} - \left.\frac{dT}{dL}\right|_{L=r^-}\right) + \eta(T(r+,r) - T(r-,r))\right] - \eta\left(\tilde{r} - r - \frac{1}{\eta}\right)\frac{c}{\tilde{r}} = \left.\frac{db_1}{dL}\right|_{L=r^+} - \left.\frac{db_1}{dL}\right|_{L=r^-} + \frac{c}{\tilde{r}}. \quad \text{(A.23)}$$

Equation (24) follows from (A.23) and the fact that $T(L,r)$ and db_1/dL are continuous at $L = r$.

To solve the differential equation (21) we first consider its homogeneous part, which may be written in the form

$$\frac{R'}{R} = -\left(\eta + \frac{\mu e^{\mu L}}{e^{\mu L} - 1}\right). \quad \text{(A.24)}$$

Integrating both sides of (A.24) yields the homogeneous solution of (21)

$$R^H(L,r) = \frac{K}{e^{\eta L}(e^{\mu L} - 1)}. \quad \text{(A.25)}$$

Using the variation of constants technique, we assume a particular solution

$$R^P(L,r) = \frac{K(L)}{e^{\eta L}(e^{\mu L} - 1)}. \quad \text{(A.26)}$$

Substituting (A.26) into (21) yields

$$\frac{dK(L)}{dL} = e^{\eta L}(e^{\mu L} - 1)F(L,r). \quad \text{(A.27)}$$

Thus

$$R^P(L,r) = \frac{1}{e^{\eta L}(e^{\mu L} - 1)}\int_0^L e^{\eta l}(e^{\mu l} - 1)F(l,r)\, dl. \quad \text{(A.28)}$$

The total solution described by (25) is the sum of the homogeneous and particular solutions given by (A.25) and (A.28). Using (22), we obtain, for $L > r$,

$$\int_0^L e^{\eta l}(e^{\mu l} - 1)F(l,r)\,dl$$
$$= \int_0^L \left[\mu\eta\alpha e^{(\eta+\mu)l} - \eta c_\delta e^{\eta l} + \frac{\mu\eta c}{\tilde{r}} \left(\tilde{r} - r - \frac{1}{\eta}\right) e^{(\mu+\eta)l-\eta r}\right] dl$$
$$\frac{\mu\eta\alpha}{\eta+\mu}(e^{(\eta+\mu)L} - 1) - c_\delta(e^{\eta L} - 1) + \frac{\mu\eta c}{\tilde{r}(\eta+\mu)} \left(\tilde{r} - r - \frac{1}{\eta}\right) e^{-\eta r}(e^{(\eta+\mu)L} - 1). \quad \text{(A.29)}$$

Similarly, for $L \leq r$,

$$\int_0^L e^{\eta l}(e^{\mu l} - 1)F(l,r)\,dl$$
$$= \frac{\mu\eta\alpha}{\eta+\mu}(e^{(\eta+\mu)L} - 1) + c_\delta(e^{\eta L} - 1) + \frac{\eta c}{\tilde{r}} \cdot \left[(e^{\mu L} - 1)\left(\frac{1}{\mu} - \frac{1}{\eta}\right) - L\right]. \quad \text{(A.30)}$$

Substituting (A.30) into (25) and letting $L \to 0$ yields, after using L'Hopital's rule and replacing α and $\tilde{r}$ with the appropriate expressions,

$$R(0,r) = K \lim_{L\to 0}\left(\frac{1}{e^{\eta L}(e^{\mu L} - 1)}\right) + c_l + c_t\frac{\eta}{\lambda}. \quad \text{(A.31)}$$

Thus, by (23), $K = 0$ for $L \leq r$. Substituting $K = 0$ and (A.30) into (25) yields (26.b). To obtain the value of K for $L > r$, we use the condition (24). Substituting (A.29) into (25) and setting $L = r$ yields $R(r^+,r)$. Similarly, substituting $L = r$ in (26.b) yields $R(r^-,r)$. Using these expressions for $R(r^+,r)$ and $R(r^-,r)$ in (24) results in

$$\frac{1}{e^{\eta r}(e^{\mu r} - 1)}\left\{K + \frac{\mu\eta c}{\tilde{r}(\eta+\mu)}\left[\tilde{r} - r - \frac{1}{\eta}\right] e^{-\eta r}[e^{(\eta+\mu)r} - 1] - \frac{c}{\tilde{r}}\left[(e^{\mu r} - 1)\left(\frac{\eta}{\mu} - 1\right) - \eta r\right]\right\}$$
$$= \eta\frac{c}{\tilde{r}}(\tilde{r} - r)e^{-\eta r}. \quad \text{(A.32)}$$

By (A.32) we obtain

$$K = -\frac{\mu\eta c}{\tilde{r}(\eta+\mu)}\left(\tilde{r} - r - \frac{1}{\eta}\right) e^{-\eta r}(e^{(\mu+\eta)r} - 1) + \frac{c\eta}{\tilde{r}}\left[(e^{\mu r} - 1)\left(\tilde{r} - r - \frac{1}{\eta} + \frac{1}{\mu}\right) - r\right] \quad \text{(A.33)}$$

Equation (26.a) is obtained by substituting (A.33) and (A.29) in (25).

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