RELIABILITY GROWTH OF REPAIRABLE SYSTEMS

Stephen A. Smith and Shmuel S. Oren*

Analysis Research Group
Xerox Palo Alto Research Center
Palo Alto, California

ABSTRACT

This paper considers the problem of modeling the reliability of a repairable system or device that is experiencing reliability improvement. Such a situation arises when system failure modes are gradually being corrected by a test-fix-test-fix procedure, which may include design changes. A dynamic reliability model for this process is discussed and statistical techniques are derived for estimating the model parameters and for testing the goodness-of-fit to observed data. The reliability model analyzed was first proposed as a graphical technique known as Duane plots, but can also be viewed as a nonhomogeneous Poisson process with a particular mean value function.

1. INTRODUCTION

Predicting the reliability of a system or piece of equipment during its development process is an important practical problem. Reliability standards are often a major issue in the development of transportation facilities, military systems, and communication networks. For commercial products that are to be leased and maintained in a competitive marketplace, system reliability estimates strongly influence predicted profitability and customer acceptance. When considering a system that is modified in response to observed failures, most classical statistical estimation techniques are not applicable. This is because the system reliability is improving with time, while most statistical techniques require repeated samples under identical conditions.

A frequently used graphical model of reliability growth of repairable systems is known as "Duane Plots," proposed by J. T. Duane [9]. This model is based on the empirical observation that, for many large systems undergoing a reliability improvement program, a plot of cumulative failure rate versus cumulative test time closely follows a straight line on log-log paper. Several recent papers present applications of Duane plots, e.g., [4], [9] and [10]. Estimating the parameters of the Duane model, i.e., the slope and intercept of the straight line fit, is somewhat difficult to do directly on the graph [5]. Weighted least squares and regression techniques are sometimes used ([9], [10]) to obtain parameter values.

An underlying probabilistic failure model that is consistent with the Duane reliability model is the nonhomogeneous Poisson process (NHPP) whose intensity is total test time raised to some power. (See [7] and [8]). Assuming the sample data consists of all the individual failure times, Crow [7] derived maximum likelihood estimates for the Duane model parameters and a goodness-of-fit test based on the Cramer-von Mises statistic (Parzen [12, p. 143]). A more general NHPP model was proposed by Ascher and Feingold [1], which also used

*Now with Dept. of Engineering-Economic Systems, Stanford University, Stanford, CA.
the Cramer-von Mises statistic for goodness-of-fit testing. Critical values of this statistic, however, must be obtained by Monte Carlo simulation for each sample size. Crow [7, p. 403] calculated and tabulated values for sample sizes up to sixty. These parameter estimates and goodness-of-fit test deal effectively with Duane model applications having small sample sizes. The facts that all failure times must be stored and the goodness-of-fit measure must be evaluated by simulation make this approach difficult for larger sample sizes. A recent paper by Singpurwalla [13] proposes a time series model for reliability dynamics. This model can, of course, be applied to any type of reliability trend data, but requires data tabulation at a larger number of time stages and does not have the intuitive appeal of the Poisson process for modeling failure occurrences in certain systems.

Our paper develops statistical estimators for the Duane model parameters based on tabulating the number of failures between fixed points in time. This approach has the advantage of using "sufficient statistics" for the data collection, i.e., the dimension of the data does not increase with sample size. Parameter estimates are obtained by maximum likelihood and a goodness-of-fit test based on the Fisher chi-square statistic is derived. This test has the advantage that chi-square tables are readily available for all sample sizes and significance levels. The accuracy of the chi-square test decreases, however, as the sample size gets small. Sample sizes for which the techniques of this paper apply are found in developmental systems that experience frequent, minor failures such as paper jams in photocopiers, voltage fluctuations in power supply systems, faults in semiconductor manufacturing processes, etc. The last section of this paper illustrates the application of the estimation and goodness-of-fit techniques to a representative set of simulated failure data.

Regardless of how the parameters of the Duane model are obtained, considerable caution is required when extrapolating reliability trends beyond the observed data to future time points. Major breakthroughs or setbacks in the reliability improvement program may cause significant deviations from the straight line projections. Some users recommend reinitializing the model and shifting to a new straight line fit when major changes in the program occur. Even if one is uneasy about extrapolating the reliability growth model to estimate future reliability, it remains a valuable tool for obtaining a "smoothed" estimate of current system reliability. While reliability is changing, sample sizes at any point in time are not sufficient for conventional statistical estimation techniques. With a dynamic reliability model, past and current failure data can be combined to obtain estimates of current reliability based on fitting all observed data.

2. THE DUANE MODEL

The Duane model states that cumulative failure rate versus cumulative test time, when plotted on log-log paper, follows approximately a straight line. More precisely, if we let \( N(0,t) \) represent the total number of failures observed up to time \( t \), we have that

\[
\log[N(0,t)/t] \approx -b \log t + a,
\]

where the fitted parameters are \( a, b > 0 \). The relationship is meaningless at \( t = 0 \) but, as most users point out ([5],[9]), a certain amount of early data is generally excluded from the fit because it is influenced by factors such as training of personnel, changes in test procedures, etc. Equation (2.1) therefore implies that

\[
N(0,t)/t = \alpha t^{-b}, \quad \text{where } \alpha = \log a,
\]

for \( t \) beyond a certain point. It should be emphasized that, in all applications, time \( t \) corresponds to cumulative operating time or test time. For the results of this paper it is most convenient to write the Duane model as:

\[
N(0,t) \approx \alpha t^{\beta}, \quad \text{where } \beta = 1 - b.
\]
For a fairly diverse set of observed systems, Codier [5, p. 460] has found $b$ to be generally between 0.3 and 0.5, corresponding to $\beta$ between 0.5 and 0.7.

3. AN UNDERLYING STATISTICAL MODEL

In this section we describe a statistical model for the failure process that is consistent with assuming that the observed failure data fits the Duane model. Suppose the probability that the system fails at time $t$ (strictly speaking in a small interval $[t, t + dt]$), regardless of the past, is determined by a hazard function $h(t)$. That is,

$$ h(t)dt \approx P \{ \text{the system fails in the interval} [t, t + dt] \}, $$

independent of its previous failure and repair history. The expected number of failures in any time interval $[t_1, t_2]$ of operating time is then given by the mean value function

$$ M(t_1, t_2) = \int_{t_1}^{t_2} h(t)dt. $$

Furthermore, it can be shown (See Parzen [12, Sect. 4.2]) that $N(t_1, t_2)$, the number of failures observed in some future time interval $[t_1, t_2]$, has probability distribution

$$ P[N(t_1, t_2) = k] = \left( M(t_1, t_2) \right)^k/k! \exp[-M(t_1, t_2)] \quad k = 0, 1, 2, \ldots . $$

In addition to its mathematical convenience, this model has considerable intuitive appeal. The simple Poisson process has been used successfully to model the failure occurrences of many devices, or collections of devices operating in series. One may think of a system having a nonhomogeneous Poisson failure process as a large collection of simpler devices in series, with individual device failure modes being gradually removed with time.

The mean value function

$$ M(t_1, t_2) = \alpha (t_2^\beta - t_1^\beta), \text{ where } \alpha, \beta > 0; $$

corresponding to $h(t) = \alpha t^{\beta-1}$, is of particular interest. Crow [7, p. 405] pointed out that the number of failures from a process with this mean value function will approximate the Duane Model by observing that

$$ \log[M(t)/t] = \log \alpha + (\beta - 1) \log t, \text{ where } M(t) = M(0, t). $$

This means that system failure data from a NHPP with mean value function $M(t)$ will approach the Duane model with probability one. Conversely, this process with mean value function $M(t)$ is the only model with independent increments that approximates the Duane model in a probabilistic sense for sufficiently large sample sizes. We will not give a proof of these statements but refer the reader to Parzen [12, ch. 4] or Donelson [8] for a complete discussion.

4. SELECTING A STARTING TIME

The Duane reliability model and the expected number of failures in Equation (3.3) are both nonlinearly dependent on the choice of the time origin. That is, if we begin observing failures at time $t = t_0 > 0$ and ignore the first $N(0, t_0)$ failures and the time interval $[0, t_0]$, we do not obtain the same parameters $\alpha$ and $\beta$ by fitting the subsequent data. Since the logarithm is a strictly concave function, there is only one choice of $t_0$ that can give a straight line fit to the data on log-log paper. Specifying the operating time $t_0$ that is assumed to have elapsed before the beginning of the modeling process is therefore an important step.

Some users of the Duane Model ([5],[10]) suggest reducing the cumulative failures and observation time by removing early data to obtain a straight line fit. This is done graphically by
successively shifting the origin to the right and replottting the data until a straight line fit is obtained. With each shift to the right, the shape of the graph of cumulative failures versus cumulative observation time becomes more downward bending (concave), so it is not hard to tell when the best point has been located.

Sometimes a terminal straight line trend on log-log paper is observed before the noisy early data is dropped. If the origin is shifted further to the right, the straight line shape will become concave. Therefore, the most that can be said in this case is that, for $t$ greater than some $t_1$, the data fits the Duane model. The statistical model \((3.3)\) can still be applied, however, by testing to see if the number of failures \(N(t_1, t)\) after the first \(N(0, t_1)\) fit the NHPP with mean value function \(M(t)\) for \(t > t_1\).

5. ESTIMATING THE MODEL PARAMETERS

If the Duane model is applied graphically, the user can attempt to estimate the parameters $\alpha$ and $\beta$ by drawing the best straight line through the plotted points. This is somewhat tricky because, with cumulative failure data, the later points should be weighted more heavily in determining the fit. This section describes a statistical estimation procedure based on the NHPP model of the failure process. We consider two possibilities for collecting and recording system failure times. The first is to record the occurrence time of each failure, which yields a sequence of observed times $T_1, T_2, \ldots, T_N$. This case has been analyzed by Crow in [7] and the maximum likelihood estimates are given by

\[
\alpha^* = \frac{N}{T_N^*}
\]

and

\[
\beta^* = -N \sum_{i=1}^{N} \log(T_i/T_N).
\]

A goodness-of-fit test corresponding to these estimators is derived in [7] and critical values of the error statistic are tabulated for sample sizes 2-60.

If large numbers of failures are observed, it is often convenient to record only the aggregate number of failures between each pair in a sequence of fixed time points $t_0, t_1, \ldots, t_n$. In this case the data is in the form $N_1, N_2, \ldots, N_n$, where $N_i$ = number of failures observed in the interval $[t_{i-1}, t_i)$. Maximum likelihood estimates and a goodness-of-fit criterion for observations in this form are developed in the next few paragraphs.

Maximum Likelihood Estimates for the Aggregated Case

We first calculate the likelihood function for the data $N_1, N_2, \ldots, N_n$, given the time points $t_0, t_1, \ldots, t_n$ and the assumed form of the mean value function in Equation \((3.3)\). The probability of $N_i$ system failures in the interval $[t_{i-1}, t_i)$ is obtained from Equation \((3.2)\). Since the underlying model assumes that each of the time segments is independent, the likelihood function can be written as a product of these probabilities,

\[
L(\alpha, \beta) = \prod_{i=1}^{n} P[N(t_{i-1}, t_i) = N_i] = \exp(-M(t_0, t_n)) \prod_{i=1}^{n} (M(t_{i-1}, t_i)^{N_i}/N_i!).
\]

To simplify the calculation of the estimators, we take the log of $L(\alpha, \beta)$, noting that maximizing the log will yield the same maximum likelihood estimates. From \((5.3)\) we have

\[
\log L(\alpha, \beta) = -\alpha(t_n^\beta - t_0^\beta) + \sum_{i=1}^{n} N_i \log \alpha + \log(t^\beta - t_{i-1}^\beta)] - \sum_{i=1}^{n} \log N_i!.
\]
Taking the partial derivatives \((\partial \log L) / \partial \alpha = 0\) and \((\partial \log L) / \partial \beta = 0\), we obtain the equations for the maximum likelihood estimates,

\[
\alpha^* = N / (t_n^\beta - t_0^\beta^*), \quad \text{where } N = \sum_{i=1}^{n} N_i
\]

\[
0 = \sum_{i=1}^{n} N_i \left[ \log t_i - \rho_i \log t_i - \frac{\log t_i - \rho_0 \log t_0}{1 - \rho_i} \right],
\]

where

\[
\rho_i = \left( \frac{t_{i-1}}{t_i} \right)^{\beta^*}, \quad i = 1, 2, \ldots, n, \quad \text{and } \rho_0 = \left( \frac{t_0}{t_n} \right)^{\beta^*}.
\]

Equation (5.6) is an implicit function of \(\beta^*\), but can be solved iteratively by a computer algorithm or programmable calculator, because the right hand side is strictly decreasing in \(\beta^*\). To verify this fact, consider any two times \(t, t'\) and compute the derivative

\[
(\partial / \partial \beta) \left( \log t - T^\beta \log t' / (1 - T^\beta) \right) = -T^\beta (\log T) / (1 - T^\beta)^2, \quad \text{where } T = t / t'.
\]

This derivative is negative and decreasing in \(T\) for \(0 < T, \beta < 1\). The derivative of the sum in (5.6) is a sum of terms involving the difference of the derivative (5.7) evaluated at \(T = t_i / t_{i-1}\) and \(t_0 / t_n\). The fact that (5.6) is decreasing in \(\beta^*\) follows from the fact that (5.7) is decreasing in \(T\), i.e., its largest or least negative value occurs at \(T = t_0 / t_n\). Therefore, (5.6) has a unique solution.

6. GOODNESS-OF-FIT CRITERION

This section describes a procedure for testing the goodness-of-fit of the observed failure data to the NHPP. We assume that the parameters \(\alpha^*\) and \(\beta^*\) are obtained from the maximum likelihood estimates (5.5) and (5.6). From the form of (5.5), it is clear that the estimate \(\alpha^*\) is defined in such a way that the total number of observed failures \(N\) always equals the expected number of failures for the time period \((t_0, t_n)\). That is, \(\alpha^*\) is defined so that

\[
N = E[N] = \alpha^* (t_n^\beta - t_0^\beta^*).
\]

Therefore, there is no difference between the observed versus predicted total number of failures. The goodness-of-fit measure must therefore be based on the differences between the observed incremental failures \(N_1, N_2, \ldots, N_n\), and the predicted values

\[
E[N_i] = \alpha^* (t_i^\beta - t_{i-1}^\beta^*), \quad i = 1, 2, \ldots, n.
\]

Assuming the estimate (5.5) is used for \(\alpha^*\), the likelihood function for a goodness-of-fit statistic will be expressed only in terms of \(\beta^*\). Since the NHPP has independent increments, the probability that a given failure occurs in the interval \([t_{i-1}, t_i]\) is the expected number of failures for that interval, divided by the total number of failures. This is written as

\[
p_i = \rho_i (\beta^*) = [\alpha^* (t_i^\beta - t_{i-1}^\beta^*)] / [\alpha^* (t_n^\beta - t_0^\beta^*)], \quad i = 1, 2, \ldots, n.
\]

where the \(\alpha^*\) parameter obviously cancels out. The likelihood function for a set of observed failures \(N_1, N_2, \ldots, N_n\), given \(N\), is therefore the multinomial

\[
\left( \begin{array}{c} N \\ N_1, N_2, \ldots, N_n \end{array} \right) p_1^{N_1} p_2^{N_2} \cdots p_n^{N_n}, \quad \text{where } N_1 + N_2 + \ldots + N_n = N,
\]

which depends only on \(\beta^*\). The parameter \(\alpha^*\) can be regarded as a scale parameter that guarantees the model will fit the total number observed of failures \(N\).
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We now show how the goodness-of-fit of the incremental failure statistic data can be measured by the Fisher chi-square statistic

\[ \chi^2 = \sum_{i=1}^{n} \frac{(N_i - N_\beta)^2}{N_\beta}. \]

The use of this statistic as a goodness-of-fit measure is based on the following theorem, which has been restated in the context of this discussion.

**THEOREM 6.1:** Let the parameters \( p_1, p_2, \ldots, p_n \), with \( \Sigma p_i = 1 \), be functions of a parameter \( \beta \) and let a particular value \( \beta' \) be determined from

\[ 0 = \sum_{i=1}^{n} \left( \frac{N_i}{p_i} \right) \left( \frac{\partial p_i}{\partial \beta} \right) \bigg|_{\beta = \beta'}. \]

Then the statistic (6.4) with \( p_i = p_i(\beta') \), \( i = 1, 2, \ldots, n \), has approximately a chi-square distribution with \( n - 1 \) degrees of freedom \( (\chi^2(n - 1)) \) for large \( N \). The proof of this result is quite lengthy and can be found in [6, pp. 424-434].

To apply this result to our particular problem, we must show that \( \beta' \) equals the estimator \( \beta^* \) defined by Equation (5.6). Using \( p_i(\beta) \) as defined in Equation (6.2), and differentiating with respect to \( \beta \), one can verify that Equation (6.5) reduces to Equation (5.6). Thus, \( \beta' = \beta^* \) and, since (5.6) has only one solution, the value is unique.

The chi-square error statistic (6.4) has an additional intuitive interpretation for this application. Suppose \( \alpha \) and \( \beta \) are the "true" parameters of the underlying inhomogeneous Poisson process, i.e., the values to which the estimators \( \alpha^* \) and \( \beta^* \) must eventually converge for very large sample sizes. Then the "true" variance of the number of observed failures in \([t_{i-1}, t_i)\), i.e., the limiting value for the sample variance of a large number of observations, is given by

\[ \text{Var}[N_i|\alpha, \beta] = \alpha (t_{i-1}^\beta - t_i^\beta) \quad i = 1, 2, \ldots, n. \]

Consider

\[ W(\alpha^*, \beta^*) = \sum_{i=1}^{n} \frac{(N_i - E[N_i|\alpha^*, \beta^*])^2}{\text{Var}[N_i|\alpha, \beta]}, \]

which is the sum of square errors between the observed and estimated failures, weighted by the true variance for each of the time intervals. Suppose we minimize this with respect to \( \alpha^* \) and \( \beta^* \) by solving \( \frac{\partial W}{\partial \alpha^*} = 0 \) and \( \frac{\partial W}{\partial \beta^*} = 0 \). If then substitute our "best estimates", \( \alpha^* \) for \( \alpha \) and \( \beta^* \) for \( \beta \), these two equations reduce to the maximum likelihood equations, (5.5) and (5.6), respectively. Birnbaum [2, p. 251-2] also points that if we minimize the chi-square statistic (6.4) with respect to \( \beta \), the estimate obtained must approach the estimate \( \beta' \) that satisfies (6.5) as the sample size approaches infinity.

This goodness-of-fit criterion measures, in effect, how well the observed data fits a NHPP with mean value function \( M(t) \), where \( \beta^* \) is the "best" growth parameter for the observed data. If the \( \chi^2(n - 1) \) statistic (6.4) exceeds the critical value at a reasonable significance level, such as 0.05 or 0.1, the model should be rejected. Since Theorem 6.1 gives only an asymptotic result, it is important to discuss the sample size requirements for applying it. Given the popularity of this test, there has been considerable experience with various types of data. A common criterion is that \( N \) and, in this case the time points \( t_0, t_1, \ldots, t_n, \) must be such that \( N p_i \geq 10 \) for all \( i \). (See Birnbaum [2, p. 248]).
7. APPLICATION EXAMPLE

As an illustration, we will determine the estimators $\alpha^*$ and $\beta^*$ and apply the goodness-of-fit test to the sample data in Table 1. We assume that the failures of the system were only monitored at fixed points of time so that the observed data consists of the first two columns of the table. These data points were generated by computer simulation with failures sampled from a NHPP with mean value function $M(t)$, having parameters $\alpha = 10.0$, $\beta = 0.5$. Failure times $T_1, T_2, \ldots$ from this distribution can be generated sequentially from a set of random samples $U_1, U_2, \ldots$ from the uniform distribution by means of the transformation

$$T_{i+1} = [T_i^\beta - (1/\alpha) \log U_{i+1}]^{1/\beta}, \quad T_0 = 0, \quad i = 0, 1, 2, \ldots$$

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>Observed Failures</th>
<th>Predicted Failures</th>
<th>Standard Deviation</th>
<th>Normalized Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 400 - 800</td>
<td>63</td>
<td>78</td>
<td>8.8</td>
<td>2.88</td>
</tr>
<tr>
<td>2 800 - 1200</td>
<td>63</td>
<td>61</td>
<td>7.8</td>
<td>0.07</td>
</tr>
<tr>
<td>3 1200 - 1600</td>
<td>54</td>
<td>51</td>
<td>7.1</td>
<td>0.18</td>
</tr>
<tr>
<td>4 1600 - 2000</td>
<td>51</td>
<td>46</td>
<td>6.8</td>
<td>0.54</td>
</tr>
<tr>
<td>5 2000 - 2500</td>
<td>68</td>
<td>51</td>
<td>7.1</td>
<td>5.67</td>
</tr>
<tr>
<td>6 2500 - 3000</td>
<td>49</td>
<td>46</td>
<td>6.8</td>
<td>0.20</td>
</tr>
<tr>
<td>7 3000 - 3500</td>
<td>34</td>
<td>43</td>
<td>6.6</td>
<td>1.88</td>
</tr>
<tr>
<td>8 3500 - 4000</td>
<td>39</td>
<td>40</td>
<td>6.3</td>
<td>0.03</td>
</tr>
<tr>
<td>9 4000 - 4500</td>
<td>39</td>
<td>38</td>
<td>6.2</td>
<td>0.02</td>
</tr>
<tr>
<td>10 4500 - 5000</td>
<td>43</td>
<td>36</td>
<td>6.0</td>
<td>1.36</td>
</tr>
<tr>
<td>11 5000 - 5500</td>
<td>39</td>
<td>34</td>
<td>5.8</td>
<td>0.74</td>
</tr>
<tr>
<td>12 5500 - 6000</td>
<td>39</td>
<td>33</td>
<td>5.7</td>
<td>0.27</td>
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<tr>
<td>13 6000 - 6500</td>
<td>28</td>
<td>31</td>
<td>5.6</td>
<td>0.29</td>
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<tr>
<td>14 6500 - 7000</td>
<td>22</td>
<td>30</td>
<td>5.5</td>
<td>2.13</td>
</tr>
<tr>
<td>15 7000 - 7500</td>
<td>35</td>
<td>29</td>
<td>5.4</td>
<td>1.24</td>
</tr>
<tr>
<td>16 7500 - 8000</td>
<td>32</td>
<td>28</td>
<td>5.3</td>
<td>0.57</td>
</tr>
<tr>
<td>17 8000 - 8500</td>
<td>22</td>
<td>27</td>
<td>5.2</td>
<td>0.93</td>
</tr>
<tr>
<td>18 8500 - 9000</td>
<td>19</td>
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<td>5.2</td>
<td>2.37</td>
</tr>
<tr>
<td>19 9000 - 9500</td>
<td>19</td>
<td>26</td>
<td>5.1</td>
<td>1.88</td>
</tr>
</tbody>
</table>

The accuracy of $\beta^*$ is reasonably close to the correct value $\beta = 0.5$, but the estimate of $\alpha^*$ is off by more than 20%. Other calculations with different sets of random numbers produced errors in both directions but generally resulted in an $\alpha^*$ error several times larger than the $\beta^*$ error, on a percentage basis. This seems to indicate that one is more likely to estimate slopes of the Duane Plot lines accurately than to estimate the intercepts accurately with the maximum likelihood estimates. Naturally, as the number of observation points in Table 1 is increased, the estimates become more accurate. Accuracy was not improved much by increasing the number of time points from 20, as shown in the table, to 100 and the sign of the error for a given example generally did not change as the number of observation points was increased,
while holding the underlying failure points fixed. Bringing the estimate $\alpha^*$ to within 5% of the correct value typically required 300 to 500 observation time points for the computed examples.

To illustrate the use of the goodness-of-fit test we calculate the chi-square statistic (6.4) for this table. The "Predicted Failures" between the various time points are given by

$$Np_i = \alpha^*(t_{i+1}^\alpha - t_i^\alpha), \quad i = 1, 2, \ldots, 19.$$  

The normalized error terms as in (6.4) are given by

$$(N_i - Np_i)^2/(Np_i).$$

The sum of these errors, when compared with a $\chi^2(18)$ error table, is less than the critical values 25.99 and 28.87, associated with significance levels 0.1 and 0.05, respectively.

For many applications of the model it is more important to predict the number of failures that will occur in the next time period than to obtain accurate estimates for $\alpha$ and $\beta$. In such cases the estimators obtained from 10-20 time points appear to be sufficiently accurate. This is because there is a range of $\alpha, \beta$ pairs that provide almost as good a fit to the observed data as the optimal ones and any parameters in this range provide a satisfactory predictive model.

To illustrate the prediction accuracy of the estimates $\beta^* = 0.52$, $\alpha^* = 7.97$ obtained from Table 1, we generated simulated failures out to 40,000 time units. The number of failures predicted by extrapolating with the estimated parameters and with the true parameters are compared in Table 2. The errors in predicting failures caused by inaccuracy in estimating the parameters is much less than the random errors that occur due to stochastic variations of the failure process. This was found to be the case in several similar experiments.

<table>
<thead>
<tr>
<th>Time Interval</th>
<th>Simulated Failures</th>
<th>Estimated Extrapolation</th>
<th>True Extrapolation</th>
<th>Standard Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>9500 - 10,000</td>
<td>24</td>
<td>25</td>
<td>25</td>
<td>5.0</td>
</tr>
<tr>
<td>9500 - 15,000</td>
<td>235</td>
<td>251</td>
<td>250</td>
<td>15.8</td>
</tr>
<tr>
<td>9500 - 20,000</td>
<td>412</td>
<td>443</td>
<td>439</td>
<td>21.0</td>
</tr>
<tr>
<td>9500 - 30,000</td>
<td>715</td>
<td>766</td>
<td>757</td>
<td>27.5</td>
</tr>
<tr>
<td>9500 - 40,000</td>
<td>999</td>
<td>1041</td>
<td>1025</td>
<td>32.0</td>
</tr>
</tbody>
</table>

8. CONCLUSION

Choosing the fixed time points between which to tabulate failures is mainly a question of engineering judgement. The time points might be selected, for example, to correspond to milestones in the reliability development program. The parameter estimates and goodness-of-fit tests obtained in this paper and those obtained by Crow are essentially complementary with respect to various applications of the Duane model. It is not possible to determine the precise sample size at which one approach becomes more advantageous than the other. Based on experience, the chi-square goodness-of-fit test tends to reject most sample data, including data that fits the model, when sample sizes are too small. Therefore, rejection of the model by the chi-square test, based on data with a questionable total number of samples, might be viewed as inconclusive and the more accurate test developed by Crow could then be applied. For large sample sizes that have at least 10 failures between time points, the chi-square test should be accurate and is computationally easier. Data that fails to fit the NHPP model with mean value
function $M(r)$ based on these tests requires a more general approach. A NHPP model with a different intensity such as discussed in [1], or a less constrained model such as [13] might then be tested.

BIBLIOGRAPHY