Journal of Economic Dynamics and Control 9 (1985) 25–39. North-Holland

A 'FUNNEL' TURNPIKE THEOREM FOR OPTIMAL GROWTH PROBLEMS WITH DISCOUNTING

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Received May 1983, final version received April 1985

A turnpike theorem for the optimal control problem with discounting is given. The optimal trajectory is shown to lie in an exponentially bounded region of the optimal steady-state. This region, referred to as a funnel, is determined by the discount rate of the problem. The funnel theorem reduces to the classical turnpike theorem when the discount rate is zero.

1. Introduction

A particularly interesting question in mathematical economics is the 'turnpike' problem. There have been many papers written on the subject, bearing out Nikaido's (1964) prediction that 'it will continue to bewitch many people'. In this paper we present an extension to the classical turnpike theory for the discounted optimal control problem, developed in the context of Implicit Programming theory.

The analytic background and development of turnpike theory may be found in the papers of Morishima (1961), Radner (1961), Atsumi (1963), Nikaido (1964), Inada (1964), Samuelson (1965), Cass (1966), Gale (1967), McKenzie (1968, 1976), Liviatan and Samuelson (1969), Rockafellar (1973, 1976), and Haurie (1976). The original description of the phenomenon was given by Dorfman, Samuelson and Solow (1958, p. 331). This is only a partial list of publications, but these particular papers form an interesting sequence that share a similar set of assumptions and analyze a similar model. The earlier papers are in discrete time, with the exception of Cass' and Samuelson's. Haurie extends McKenzie's (1968) results to continuous time. It is also important to note that only Cass, Liviatan–Samuelson, and Rockafellar (1976) deal with discounted problems.

The problem structure that we will consider in this paper is given as follows:

$$\min \int_0^{t_f} e^{-\rho t} l(x, u)\, dt, \tag{1a}$$

subject to

$$\dot{x}(t) = f(x, u), \tag{1b}$$

$$x(0) = x_0, \tag{1c}$$

$$\big(x(t_f)\big) = x_f, \tag{1d}$$

$$(x(t), u(t)) \in X \times U \subseteq \mathbb{R}^n \times \mathbb{R}^m \quad \text{for each} \quad t \in [0, \infty). \tag{1e}$$

Problem (1) defines the optimal control problem with discounting on a fixed, finite-time horizon, t_f. The variable $x \in \mathbb{R}^n$ is the state variable and the variable $u \in \mathbb{R}^m$ is the control variable. The set $U \subseteq \mathbb{R}^m$ may depend on x explicitly. In that case, we shall write $U = U(x)$. l is a real-valued function, $\rho > 0$ is the discount rate, and f takes values in $\mathbb{R}^n$.

The essential feature of turnpike theory is that, under certain conditions, the optimal trajectory of problem (1) will spend most of the planning period in a neighborhood of a particular steady-state trajectory, called the 'turnpike'. The turnpike will be a pair (x^*, u^*) where (x^*, u^*) satisfies $f(x^*, u') = \theta$.

There are both finite-time and infinite-time horizon statements of the turnpike property. The finite-time statement, loosely, is that given an initial state x_0 and a final state x_f, specified in constraints (1c) and (1d), there exists a time T such that the optimal trajectory $(x^*(t), u^*(t))$ that transfers the system from x_0 to x_f in time $t_f \geq T$ will be arbitrarily near the turnpike (x^*, u^*) for all but an arbitrarily small fraction of the total time t_f. The infinite-time statement is that there exists an optimal control $u^*(t)$ such that the optimal trajectory converges to the turnpike (x^*, u^*) as $t_f \to +\infty$, where constraint (1d) is suppressed.

A statement of the infinite-time turnpike property was given by Rockafellar (1976). The result was based on the application of convexity theory to problem (1), and the convergence of the optimal trajectories of the infinite horizon problem was shown for sufficiently small values of the discount rate, ρ. For the non-discounted problem Haurie (1976) gave both finite- and infinite-time horizon results, and Rockafellar (1973) gave an infinite-horizon result. We will give a finite-horizon result for the discounted problem in this paper.

In particular, we are interested in establishing a result that describes the behavior of the optimal trajectory for the (finite-horizon) discounted problem over most of the planning period. We know of only one such result, that of

Cass (1966), and that result was given for the one-dimensional problem. The method of proof is not extendable to multi-dimensional systems, since Cass used arguments based on the phase-plane portraits of the trajectories of problem (1). Haurie (1976) gave a multi-dimensional result, but did not consider the problem with discounting. McKenzie (1976, p. 853 ff.), considering the discrete-time problem, gave results that were valid only for the undiscounted problem. Our results generalize the turnpike property for the discounted problem and are aimed at completing the missing element of the theory.

Our analysis begins with a static characterization of the optimal steady-states of problem (1). It has been shown [Feinstein and Luenberger (1981)] that the solution to the Implicit Programming problem determines an optimal steady-state of problem (1). We will briefly discuss this result.

The Implicit Programming problem associated with the optimal control problem (1) is

$$\min l(x,u), \tag{2a}$$

subject to

$$f(x,u)-\rho(x-x^*)=\theta, \tag{2b}$$

$$x\in X, \qquad u\in U(x). \tag{2c}$$

This problem is a well-defined mathematical programming problem with the unique structural feature that the state-component of the solution, x^*, appears in the constraint (2b); in other words, the constraint is defined implicitly by the solution to the problem itself.

It is natural to embed problem (2) in the family of mathematical programming problems:

$$\min l(x,u), \tag{3a}$$

subject to

$$f(x,u)-\rho(x-c)=\theta, \tag{3b}$$

$$x\in X, \qquad u\in U(x), \tag{3c}$$

where $c\in \mathbb{R}^n$ is a parameter. Hence, the solution to any member of the embedding family may be written $(x^*(c),u^*(c))$, and the fixed points of the mapping $c\to x^*(c)$ are the feasible points of (2). This embedding motivates the definition of the Lagrangian

$$\begin{aligned} L_\rho(x,u,\lambda;c) &= l(x,u)-\langle\lambda, f(x,u)-\rho(x-c)\rangle, \\ &= +\infty \quad \text{if} \quad x\notin X \quad \text{or} \quad u\notin U(x). \end{aligned} \tag{4}$$

It has been shown that, under sufficient convexity assumptions [essentially strict convexity-concavity of the Hamiltonian of problem (1), $H^*(x,q) = \sup_u\{\langle q, f(x,u)\rangle - l(x,u)\colon\ u \in U(x)\}$], the solution of (2), (x^*, u^*), is an optimal steady-state in the class of trajectories initiated at $x(0) = x^*$ such that $e^{-\rho t}x(t)$ remains bounded as t approaches infinity [Feinstein and Luenberger (1981, theorem 6.4)]. This result motivates us to characterize the turnpike properties of the solutions of problem (1) using the Implicit Programming formulation. Further, we identify the Lagrange multiplier, λ^*, of the Implicit Programming problem with the steady-state costate variable, q^*, of the optimal control problem. Therefore, we will refer to the turnpike as a triple, (x^*, u^*, λ^*), where λ^* is the Lagrange multiplier of the constraints (2b) at the solution (x^*, u^*).

2. The funnel turnpike theorem

In the context of optimal control theory, the turnpike property of the optimal trajectory results from assumptions that are *sufficient* for dynamic optimality. The statement of a sufficient maximum principle was given by Peterson (1971) and extended by Feinstein and Luenberger (1981). For our present purposes, we recall an essential aspect of this sufficiency theory, the support property.

Definition 1. A trajectory $(x^*(t), u^*(t))$ of problem (1) defined on an infinite-time horizon is said to be supported if there exists a continuous, piecewise continuously differentiable function $p^*\colon [0,\infty) \to \mathbb{R}^n$ such that

$$\begin{aligned} &-e^{-\rho t} l(x^*(t), u^*(t)) + \mathrm{d}/\mathrm{d}t\,[\langle p^*(t), x^*(t)\rangle] \\ &\geq -e^{-\rho t} l(x,u) + \langle p^*(t), f(x,u)\rangle + \langle \dot{p}^*(t), x\rangle, \qquad (5) \\ &\forall (x,u) \in X \times U, \quad \text{for almost every} \quad t \in [0,\infty). \end{aligned}$$

The support property was defined by Gale (1967) (he referred to a supported trajectory as 'competitive'). More recently, Haurie (1980) generalized the support property to enable analysis of a broad class of problems of the form (1). It can be shown [Feinstein and Luenberger (1981, theorem 4.1)] that the supporting function $p^*(t)$ can be identified with the costate trajectory given by the maximum principle. Thus, for problem (1), we may write $p^*(t) = e^{-\rho t}q^*(t)$, and for a supported steady-state, $(x^*(t), u^*(t), q^*(t)) = (x^*, u^*, q^*)$, $\forall t$, the support property becomes

$$l(x^*, u^*) \leq l(x,u) - \langle q^*, f(x,u) - \rho(x - x^*)\rangle, \quad \forall (x,u) \in D. \quad (6)$$

The fundamental property of the turnpike triple (x^*, u^*, λ^*) is that it satisfies a strong version of the support property. This strong support property is generally referred to as Atsumi's Lemma (although a similar result was first given by Radner). There are various ways to derive the result (for the undiscounted problem), which rely essentially on a compactness assumption on the so-called *transformation set*, $X \times F(x) = \{(x, l, v)\colon x \in X,\ l = l(x, u),\ v = f(x, u),\ u \in U(x)\}$. [See especially Radner (1961, p. 102), Atsumi (1963, p. 132), Haurie (1976, p. 90), and Brock and Haurie (1976, p. 344).] However, we prefer to derive the result by strengthening the convexity assumptions. We impose the following assumption:

Assumption 1. Let (x^*, u^*, λ^*) be a solution of the Implicit Programming problem (2). We assume that the Lagrangian, evaluated at λ^*, with $c = x^*$, $L_\rho(x, u, \lambda^*; x^*)$ (4), is convex on the convex set $D = \{(x, u)\colon x \in X, u \in U(x)\}$. In particular, we assume that L_ρ is a $\mathbf{C}^2$ function with positive-definite Hessian matrix for any $(x, u) \in D$. In addition, we assume that

$$\inf_{(x,u)} \{m(x, u)\colon (x, u) \in D\} = m > 0, \tag{7}$$

where $m(x, u)$ is the minimum eigenvalue of the Hessian of the Lagrangian evaluated at $(x, u) \in D$.

This is a strong assumption. Assumption 1 extends the basic assumption of Local Duality Theory [Luenberger (1973)]. We may now prove 'Atsumi's Lemma'.

Lemma 1. Strong Support Property. *Let (x^*, u^*, λ^*), $(x^*, u^*) \in D^\circ$, be a solution of the Implicit Programming problem (2) that satisfies the support property (6). Suppose that Assumption 1 holds. Then*

$$\forall \varepsilon > 0 \quad \textit{and} \quad \forall (x, u) \in D, \quad \exists \delta_\varepsilon > 0, \quad \textit{such that}$$

$$\|(x, u) - (x^*, u^*)\|^2 > \varepsilon$$

$$\Rightarrow l(x^*, u^*) < l(x, u) - \langle \lambda^*, f(x, u) - \rho(x - x^*) \rangle - \delta_\varepsilon. \tag{8}$$

Proof. Expand the Lagrangian (4) about (x^*, u^*), letting $z^T = [(x - x^*)^T, (u - u^*)^T]$:

$$l(x, u) - \langle \lambda^*, f(x, u) - \rho(x - x^*) \rangle$$

$$= l(x^*, u^*) + \left[\nabla_{(x,u)} L_\rho(x^*, u^*, \lambda^*; x^*)\right] z$$

$$+ \tfrac{1}{2} z^T \left[\nabla^2_{(x,u)} L_\rho(\xi, \omega, \lambda^*; x^*)\right] z \quad \text{where} \quad (\xi, \omega) \in D.$$

The gradient of the Lagrangian vanishes by the first-order necessary conditions for the Implicit Programming problem. By Assumption 1, the Hessian of the Lagrangian is positive-definite on D, with minimum eigenvalue $m(\xi, \omega) > 0$. Hence,

$$
\begin{aligned}
& l(x, u) - \langle \lambda^*, f(x, u) - \rho(x - x^*) \rangle \\
& \geq l(x^*, u^*) + \tfrac{1}{2} m(\xi, \omega) \|(x, u) - (x^*, u^*)\|^2 \\
& > l(x^*, u^*) + \tfrac{1}{2} m(\xi, \omega) \varepsilon.
\end{aligned}
$$

Since, by Assumption 1,

$$\inf_{(\xi, \omega)} \{ m(\xi, \omega) \colon (\xi, \omega) \in D \} = m > 0,$$

we may set $\delta_\varepsilon = \frac{1}{2} m \varepsilon$, independently of (x, u). □

For the undiscounted optimal control problem the strong support property (8) permits the finite-time horizon turnpike property to be established [see Haurie (1976, theorem 3.4)]. However, for the problem with discounting, we require a variant of (8).

Corollary 1. Let (x, u) be a trajectory of the optimal control problem with discounting. Under the conditions of Lemma 1, if

$$\|(x(t), u(t)) - (x^*, u^*)\|^2 > \varepsilon \mathrm{e}^{\rho t}, \qquad t \in [0, \infty),$$

then

$$
\begin{aligned}
l(x^*, u^*) < {} & l(x(t), u(t)) - \langle \lambda^*, f(x(t), u(t)) - \rho(x(t) - x^*) \rangle \\
& - \delta_\varepsilon \mathrm{e}^{\rho t}.
\end{aligned}
\tag{9}
$$

We interpret the condition

$$\|(x(t), u(t)) - (x^*, u^*)\|^2 > \varepsilon \mathrm{e}^{\rho t} \tag{10a}$$

as defining all points $(x(t), u(t))$ of the trajectory (x, u) that are outside an exponential 'funnel' centered at the point $(x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m$ (see fig. 1). Equivalently, this condition may be expressed as

$$\|(\xi^*(t), \omega^*(t))\|^2 > \varepsilon, \tag{10b}$$

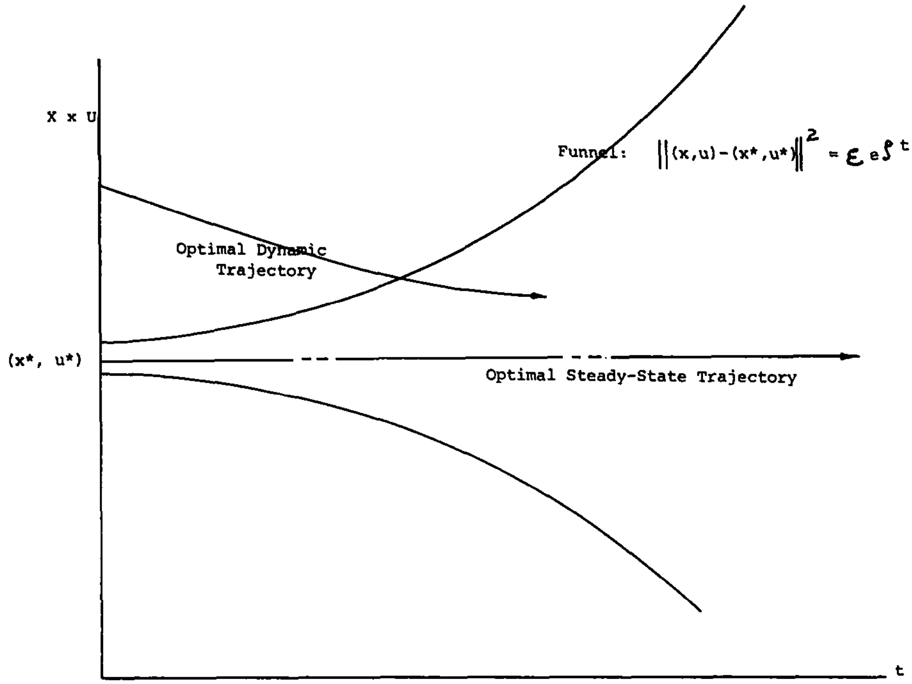

Fig. 1. The 'funnel' turnpike theorem.

which defines all points $(\xi^*(t), \omega^*(t))$ that are outside a 'tube' centered at $(\theta, \theta) \in \mathbb{R}^n \times \mathbb{R}^m$, where

$$\xi^*(t) = e^{-\frac{1}{2}\rho t}\left(x(t) - x^*\right), \tag{11a}$$

$$\omega^*(t) = e^{-\frac{1}{2}\rho t}\left(u(t) - u^*\right). \tag{11b}$$

The variables $(\xi^*(t), \omega^*(t))$ are called *mirage* variables by Magill (1977). As we shall show, it is the mirage variables that satisfy the classical turnpike theorem, for the stationary trajectory (θ, θ). The actual variables $(x(t), u(t))$ satisfy a 'funnel' turnpike property for the stationary trajectory (x^*, u^*). The terminology *mirage* suggests the fact that those variables always are closer to the optimal steady-state trajectory than the actual variables.

In order to establish the result, further assumptions are required to ensure controllability and the existence of an optimal control over a finite-time horizon. We require:

Assumption 2. Let x_o and x_f be points in X. We assume that for some (finite) time T, the finite-time horizon optimal control problem with discounting (1) has an optimal solution (x°, u°), for all t_f, $T \le t_f < \infty$.

Sufficient conditions for the existence of an optimal control for problem (1) are given by Lee and Markus (1967). An essential issue in establishing the existence of an optimal control is the question of reachability: does there exist an admissible controller such that the response of the dynamic system (1b) satisfies the boundary conditions $x(0) = x_o$ and $x(t_f) = x_f$? We shall make specific reachability assumptions in the next theorem.

We will now state and prove the funnel turnpike theorem:

Theorem 1. Finite-Time Horizon Turnpike Theorem. *Let* (x^*, u^*, λ^*), $(x^*, u^*) \in D^\circ$, *be a solution to the Implicit Programming problem, such that the support property holds. Let Assumptions 1 and 2 hold. Suppose further that both* x_o *and* x_f *are reachable from* x^* *by admissible controllers in finite times* T_1 *and* T_2, *respectively.*

Then, for any $\varepsilon > 0$, *there exists a number* $n(\varepsilon)$ *such that for all* $t_f \ge \max\{T, T_1 + T_2\}$ *there holds*

$$\mu\left[\left\{t\colon \left\|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\right\|^2 > \varepsilon \mathrm{e}^{\rho t}, t \in \left[0, t_f\right]\right\}\right] < n(\varepsilon), \quad (12)$$

where $\mu[\cdot]$ *is Lebesgue measure and* $n(\varepsilon)$ *is independent of* t_f.

Before proceeding to prove the theorem, we offer the following geometric interpretation. Let

$$\Sigma(\varepsilon, t_f, \rho) = \left\{t\colon \left\|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\right\|^2 > \varepsilon \mathrm{e}^{\rho t}, t \in \left[0, t_f\right]\right\}. \quad (13)$$

Then $\Sigma(\varepsilon, t_f, \rho)$ is the set of all time (instants) within the planning period such that the optimal state-control trajectory is outside an exponential envelope of the stationary point (x^*, u^*). We refer to this envelope as the 'funnel', as shown in fig. 1.

The theorem states that the optimal trajectory will be within the funnel for all but a set of times of bounded measure. The fact that the bound is independent of t_f makes the following corollary immediate. The corollary is analogous to the finite-horizon classical turnpike theorem.

Corollary 2. For any $\varepsilon > 0$ and any γ, $0 < \gamma < 1$, there exists a finite time $T(\varepsilon, \gamma)$, such that for all horizons $t_f \geq T(\varepsilon, \gamma)$, there holds

$$\mu\left[\Sigma(\varepsilon, t_f, \rho)\right] < \gamma t_f. \tag{14}$$

Proof of Theorem 1. The method of proof compares a particular feasible trajectory to the optimal trajectory [similar methods are employed in McKenzie (1976) and Haurie (1976)]. By the reachability assumptions, we can construct the following controller that transfers x_o to x_f for any $t_f \geq T_1 + T_2$. Let

$$\begin{aligned} u(t) &= u(t; x_0, x^*, T_1) && \text{for} \quad 0 \leq t \leq T_1, \\ &= u^* && \text{for} \quad T_1 < t < T_1 + T_{ss}, \\ &= u(t; x^*, x_f, T_2) && \text{for} \quad T_1 + T_{ss} \leq t \leq T_1 + T_{ss} + T_2, \end{aligned}$$

where $u(t; x_o, x^*, T_1)$ transfers x_o to x^* in T_1 and $u(t; x^*, x_f, T_2)$ transfers x^* to x_f in T_2. The feature of this controller is that the state remains at x^* for a period T_{ss} where T_{ss} is chosen such that $T_1 + T_2 + T_{ss} = t_f \geq T$.

By Assumption 2, for $t_f \geq \max\{T, T_1 + T_2\}$ there exists an optimal control $u^\circ(t)$ that transfers x_o to x_f in t_f. By optimality of $u^\circ(t)$, it follows that

$$\int_0^{t_f} e^{-\rho t} l(x^\circ(t), u^\circ(t))\, dt \leq \int_0^{t_f} e^{-\rho t} l(x(t), u(t))\, dt,$$

for $u(t)$ given above. Thus,

$$\begin{aligned} \int_0^{t_f} e^{-\rho t} l(x^\circ(t), u^\circ(t))\, dt \leq & \int_0^{T_1} e^{-\rho t} l(x_1(t), u_1(t))\, dt \\ & + e^{-\rho T_1} \int_0^{T_{ss}} e^{-\rho t} l(x^*, u^*)\, dt \\ & + e^{-\rho(T_1 + T_{ss})} \int_0^{T_2} e^{-\rho t} l(x_2(t), u_2(t))\, dt, \end{aligned} \tag{$*$}$$

where $(x_1(t), u_1(t))$ is the state-control path corresponding to $u(t; x_o, x^*, T_1)$ and $(x_2(t), u_2(t))$ is the state-control path corresponding to $u(t; x^*, x_f, T_2)$.

By Corollary 1, when $t \in \Sigma(\varepsilon, t_f, \rho)$, we have

$$\begin{aligned} l(x^*, u^*) < & \; l(x^\circ(t), u^\circ(t)) \\ & - \langle \lambda^*, f(x^\circ(t), u^\circ(t)) - \rho(x^\circ(t) - x^*) \rangle - \delta_\varepsilon e^{\rho t}. \end{aligned}$$

Thus,

$$e^{-\rho t}l(x^*, u^*) < e^{-\rho t}l(x^\circ(t), u^\circ(t)) \\ -\langle \lambda^*, d/dt\left[e^{-\rho t}(x^\circ(t) - x^*(t))\right]\rangle - \delta_\varepsilon.$$

Now, upon integrating over $[0, t_f]$, we find

$$\int_0^{t_f} e^{-\rho t}l(x,^*, u^*)\,dt + \langle \lambda^*, e^{-\rho t_f}(x_f - x^*) - (x_0 - x^*)\rangle \\ + \delta_\varepsilon\mu\left[\Sigma(\varepsilon, t_f, \rho)\right] \\ < \int_0^{t_f} e^{-\rho t}l(x^\circ(t), u^\circ(t))\,dt.$$

Combining $(*)$ and $(**)$,

$$\delta_\varepsilon\mu\left[\Sigma(\varepsilon, t_f, \rho)\right] < \int_0^{T_1} e^{-\rho t}\left[l(x_1(t), u_1(t)) - l(x^*, u^*)\right]dt \\ + \langle \lambda^*, x_0 - x^*\rangle + e^{-\rho(T_1 + T_{s})} \\ \times\left(\int_0^{T_2} e^{-\rho t}\left[l(x_2(t), u_2(t)) - l(x^*, u^*)\right]dt \right. \\ \left. - e^{-\rho T_2}\langle \lambda^*, x_f - x^*\rangle\right).$$

Now observe that

$$\left(\int_0^{T_2} e^{-\rho t}\left[l(x_2(t), u_2(t)) - l(x^*, u^*)\right]dt - e^{-\rho T_2}\langle \lambda^*, x_f - x^*\rangle\right)$$

This follows from the support property,

$$l(x^*, u^*) \le l(x, u) - \langle \lambda^*, f(x, u) - \rho(x - x^*)\rangle.$$

Since $x_2(0) = x^*$ and $x_2(T_2) = x_f$, upon integrating the inequality over [(after multiplication by $e^{-\rho t}$,

$$\int_0^{T_2} e^{-\rho t}\left[l(x_2(t), u_2(t)) - l(x^*, u^*)\right]dt \\ - \left\langle \lambda^*, \int_0^{T_2} d/dt\left[e^{-\rho t}(x_2(t) - x^*)\right]dt\right\rangle \ge 0,$$

or

$$\int_0^{T_2} e^{-\rho t}\left[l(x_2(t), u_2(t)) - l(x^*, u^*)\right] dt - e^{-\rho T_2}\langle \lambda^*, x_f - x^* \rangle \geq 0.$$

Then, for $T_{ss} \geq 0$, we have $e^{-\rho(T_1 + T_{ss})} \leq e^{-\rho T_1}$, hence we can bound $\mu[\Sigma(\varepsilon, t_f, \rho)]$ by

$$\mu\left[\Sigma(\varepsilon, t_f, \rho)\right] < (1/\delta_\varepsilon)K = n(\varepsilon), \tag{15a}$$

with

$$\begin{aligned} K = &\int_0^{T_1} e^{-\rho t}\left[l(x_1(t), u_1(t)) - l(x^*, u^*)\right] dt + \langle \lambda^*, x_0 - x^* \rangle \\ &+ e^{-\rho T_1}\left(\int_0^{T_2} e^{-\rho t}\left[l(x_2(t), u_2(t)) - l(x^*, u^*)\right] dt \right. \\ &\left. - e^{-\rho T_2}\langle \lambda^*, x_f - x^* \rangle\right). \end{aligned} \tag{15b}$$

Hence, $n(\varepsilon)$ depends only on ε (through δ_ε) and $(x_o, x_f, T_1, T_2, \lambda^*, x^*, u^*, \rho)$, and is independent of t_f. □

Theorem 1 indicates the existence of a funnel in which all but an arbitrarily small portion of the optimal trajectory can be placed. The classical theory places the trajectory in a 'tube' about the turnpike (x^*, u^*). A natural question to ask is whether the funnel can be placed in an arbitrary tube. We will discuss the relationship of the funnel theorem to the classical turnpike theorem below.

Before turning to that discussion, we state and prove a theorem regarding the cost sum provided by the optimal trajectory. Although we cannot conclude that the actual states and controls along the optimal trajectory are necessarily close to the optimal steady-state, we can assert that for most of the planning period, the difference between the discounted cost sum provided by the optimal trajectory and that provided by the optimal steady-state is arbitrarily small. To prove that result, we require a further assumption.

Assumption 3. Let (x^*, u^*, λ^*) be a solution of the Implicit Programming problem (2). We assume that the Lagrangian, evaluated at λ^*, with $c = x^*$, $L_\rho(x, u, \lambda^*; x^*)$ (4), is a $\mathbf{C}^2$ function with positive-definite Hessian matrix for any $(x, u) \in D$. In addition, we assume that

$$\sup_{(x,u)} \{M(x, u) : (x, u) \in D\} = M < \infty, \tag{16}$$

where $M(x, u)$ is the maximum eigenvalue of the Hessian of the Lagrangian evaluated at $(x, u) \in D$.

Theorem 2. Let Assumptions 1, 2, and 3 hold. For any interval $[t_1, t_2]$, *such that the optimal trajectory* (x°, u°) *is within the funnel, the discounted cost sum provided by the optimal trajectory is arbitrarily close to the discounted cost sum provided by the steady-state* (x^*, u^*).

Proof. Let $[t_1, t_2] \not\subset \Sigma(\varepsilon, t_f, \rho)$, where the set $\Sigma(\varepsilon, t_f, \rho)$ is defined by (13). Then, since $\dot{x}^\circ(t) = f(x^\circ(t), u^\circ(t))$, a.e.,

$$\int_{t_1}^{t_2} \mathrm{e}^{-\rho t} l(x^\circ(t), u^\circ(t))\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big[l(x^\circ(t), u^\circ(t)) - \langle \lambda^*, f(x^\circ(t), u^\circ(t)) - \dot{x}^\circ(t)\rangle\big]\,\mathrm{d}t.$$

Upon integrating by parts, this yields

$$\int_{t_1}^{t_2} \mathrm{e}^{-\rho t} l(x^\circ(t), u^\circ(t))\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big[l(x^\circ(t), u^\circ(t))$$
$$- \langle \lambda^*, f(x^\circ(t), u^\circ(t)) - \rho(x^\circ(t) - x^*)\rangle\big]\,\mathrm{d}t$$
$$+ \mathrm{e}^{-\rho t}\langle \lambda^*, (x^\circ(t) - x^*)\rangle\big|_{t_1}^{t_2},$$

or, recalling the definition of the Lagrangian (4),

$$\int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big[l(x^\circ(t), u^\circ(t)) - l(x^*, u^*)\big]\,\mathrm{d}t$$
$$= \int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big[L_\rho(x^\circ(t), u^\circ(t), \lambda^*; x^*) - l(x^*, u^*)\big]\,\mathrm{d}t$$
$$+ \mathrm{e}^{-\rho t}\langle \lambda^*, (x^\circ(t) - x^*)\rangle\big|_{t_1}^{t_2}.$$

Expanding the Lagrangian to second order, and inserting absolute values,

$$\left|\int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big[l(x^\circ(t), u^\circ(t)) - l(x^*, u^*)\big]\,\mathrm{d}t\right|$$
$$\le \tfrac{1}{2} M \int_{t_1}^{t_2} \mathrm{e}^{-\rho t}\big\|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\big\|^2\,\mathrm{d}t$$
$$+ \left|\mathrm{e}^{-\rho t}\langle \lambda^*, (x\circ(t) - x^*)\rangle\big|_{t_1}^{t_2}\right|,$$

since, by Assumption 3, the maximum eigenvalue of the Hessian of the Lagrangian is bounded by M.

Observe now that since the interval $[t_1, t_2]$ is outside the set $\Sigma(\varepsilon, t_f, \rho)$, it follows that $e^{-\rho t}\|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\|^2 < \varepsilon$. Similarly, the inner product may be bounded:

$$\left| e^{-\rho t}\langle \lambda^*, (x^\circ(t) - x^*)\rangle\Big|_{t_1}^{t_2} \right| < \|\lambda^*\| \varepsilon^{\frac{1}{2}}.$$

Hence,

$$\left| \int_{t_1}^{t_2} e^{-\rho t}\left[l(x^\circ(t), u^\circ(t)) - l(x^*, u^*) \right] \mathrm{d}t \right| \leq o(\varepsilon). \quad \square$$

3. Discussion and conclusions

The first question one might ask is whether the 'tube' theorem can be derived from the 'funnel theorem'. Analytically, the classical turnpike theorem would follow from Theorem 1 if, given ε_1 and γ_1, $0 < \gamma_1 < 1$, there exists a $T < +\infty$ such that

$$\begin{aligned} &\mu\left[\Sigma(\varepsilon, t_f, \rho)\right] < n(\varepsilon) \\ &\Rightarrow \mu\left[\left\{ t\colon \|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\|^2 > \varepsilon_1, t \in [0, T] \right\}\right] < \gamma_1 T, \end{aligned}$$

where $(x^\circ(t), u^\circ(t))$ is the optimal trajectory of Theorem 1.

The critical aspect of this issue is the dependence of $n(\varepsilon)$ on ε. We observe from (15a) that $n(\varepsilon) = K/\delta_\varepsilon$ where K is a constant, determined by the parameters $(x_o, x_f, T_1, T_2, \lambda^*, x^*, u^*, \rho)$. Since $\delta_\varepsilon = \frac{1}{2} m\varepsilon$, it follows that $n(\varepsilon) = 2K/m \cdot 1/\varepsilon$.

Now

$$\begin{aligned} \mu\left[\Sigma(\varepsilon, t_f, \rho)\right] &= \mu\left[\left\{ t\colon \|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\|^2 > \varepsilon e^{\rho t}, \right.\right. \\ &\qquad \left.\left. t \in [0, t_f] \right\}\right] \\ &\geq \mu\left[\left\{ t\colon \|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\|^2 > \varepsilon e^{\rho t_f}, \right.\right. \\ &\qquad \left.\left. t \in [0, t_f] \right\}\right], \end{aligned} \tag{17}$$

since

$$e^{\rho t_f} \geq e^{\rho t}, \qquad \forall t \in [0, t_f], \quad \rho \geq 0.$$

Suppose we let $\varepsilon_1 = \varepsilon\, e^{\rho T}$. Then, from (16),

$$\mu\left[\left\{t: \|(x^\circ(t), u^\circ(t)) - (x^*, u^*)\|^2 > \varepsilon_1, t \in [0, T]\right\}\right]$$
$$\leq \mu[\Sigma(\varepsilon, T, \rho)] < n(\varepsilon), \tag{18}$$

and if $n(\varepsilon) = 2K/m \cdot 1/\varepsilon \leq \gamma_1 T$, the result follows. Thus we are seeking a solution to the system

$$\varepsilon_1 \doteq \varepsilon\, e^{\rho T}, \quad \gamma_1 T \geq 2K/m \cdot 1/\varepsilon,$$

which becomes the single inequality

$$\gamma_1 T \geq 2K/m \cdot e^{\rho T}/\varepsilon_1. \tag{19}$$

However, for arbitrary $\rho > 0$, there will be no solution $T = T(\varepsilon_1, \gamma_1)$ to the inequality.

Consider the equation

$$2K/m \cdot 1/\gamma_1\varepsilon_1 = T/e^{\rho T}.$$

The infimum of the product $\gamma_1\varepsilon_1$ is zero. The left-hand side of this equation can take values anywhere on the positive real line. However, the right-hand side, as a function of T, is bounded. In fact it attains its maximum at $T = 1/\rho$, where $T/e^{\rho T} = 1/\rho e$. As ρ approaches zero, the maximum increases, but for fixed ρ, we can always find $(\gamma_1, \varepsilon_1)$ such that

$$2K/m \cdot 1/\gamma_1\varepsilon_1 > T/e^{\rho T}, \quad \forall T. \tag{20}$$

Thus, the result of Theorem 1 is not equivalent to the classical turnpike theorem for the variables $(x^\circ(t) - x^*, u^\circ(t) - u^*)$. However, it is easy to see that the funnel theorem is equivalent to the classical turnpike theorem for the variables (11),

$$\left(\xi^*(t), \omega^*(t)\right) = \left(e^{-\frac{1}{2}\rho t}\left(x^\circ(t) - x^*\right), e^{-\frac{1}{2}\rho t}\left(u^\circ(t) - u^*\right)\right).$$

Observe that the infinite-time horizon statement of the classical turnpike theorem indicates that, as $t_f \to \infty$, the variables $(\xi^*(t), \omega^*(t))$ converge to (θ, θ). This would seem to permit the actual variables to behave asymptotically as $O(e^{\frac{1}{2}\rho t})$, hence divergent. However, the infinite horizon results [Rockafellar (1976)] give actual convergence of the variables $(x^\circ(t) - x^*, u^\circ(t) - u^*)$, if the discount rate is sufficiently small. We conclude from this that the 'funnel' theorem is a general result, valid for any discount rate, whereas the infinite-time theorem is valid for a restricted range of discount rates.

Essentially, the funnel theorem provides a statement about cost sums. Even though we cannot conclude that the actual states and controls along the optimal trajectory are necessarily close to the optimal steady-state, we can assert, as indicated in Theorem 2, that for most of the planning period, the difference between the discounted cost sum provided by the actual optimal trajectory and that provided by the optimal steady-state is arbitrarily small.

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