Multi-product pricing for electric power

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This paper describes methods for determining the prices for a discrete set of interrelated products offered by a single supplier. Each product has a fixed marginal rate per unit, the demand for each product depends upon the price levels of all the products and the supplier's cost function depends upon the demands for all the products. For specific demand function forms, methods are described for determining the prices that maximize a weighted sum of net revenue and consumer surplus.

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It has long been recognized that electric power is not a single product, but many products, whose distinguishing characteristics are the conditions of delivery of the electric power service. Common product attributes include maximum capacity, energy usage per time period, time-of-use variations and the contracted interruptibility level of the service. The product options that a customer is offered depend upon the type of customer (ie residential, industrial, etc). However, any rate schedule that offers different rates to the customer as a function of his usage pattern is in effect offering a choice of products.

This note considers the problem of determining the price levels for a set of distinct interrelated products. It is assumed that the demands for the products are interrelated, ie there will be varying amounts of cross-substitution among the products depending upon their attributes and relative prices. The total cost of supplying the products is assumed to be a general function of all the individual product demands. The primary restriction imposed on the model in this note is that the prices for each product must be linear in the quantity purchased.

Since an arbitrary supply cost function is permitted, only the general first-order necessary conditions for the optimal prices have been obtained in closed form. However, for any specific example, the determination of the prices is a mathematical optimization problem, which can be solved by a computer algorithm. An optimization algorithm has been prepared and programmed and is applied to several illustrative examples.

The pricing objective function can be varied, depending upon the revenue requirements that are to be met. In general, the objective function is a weighted sum of the supplier's net revenue and the consumer's surplus. By adjusting the weights, the resulting net revenue can be increased or decreased as necessary to meet externally imposed constraints on revenue. If revenue is not externally constrained it is observed that the resulting prices must be directly proportional to the marginal supply costs across the various products. The details of the model are described in the following sections.

Suppliers of products and services in many markets offer families of products that are partial substitutes for each other. This feature has long been an important consideration in pricing durable goods such as automobiles and appliances, as well as in a wide variety of consumer products. Recently, suppliers in certain major service industries such as telecommunications, electric power and transportation are moving toward

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increased product differentiation for a variety of reasons, which include technological advances, relaxed regulatory restrictions, and consumer demands for new services. As a supplier's family of products grows, interproduct competition becomes increasingly important, as does the prediction of the cross impacts on demand that result from new product offerings. Selecting prices for the various product offerings and the choice of other controllable product attributes becomes a complex planning task.

This paper proposes a modelling methodology for analysing pricing and cross-substitution effects among products offered by a single supplier. It is assumed that all products have a fixed price per unit (eg no quantity discounts), and that income effects can be neglected. Demand functions and a supplier cost function are specified for obtaining specific necessary conditions for the optimal prices. The objective function is defined as a weighted sum of supplier and consumer surplus. This includes as special cases various objectives, ranging from profit maximization to total surplus maximization subject to a revenue constraint. A specific demand function form is proposed so that explicit price determinations can be made. It is illustrated how the required demand function parameters can be estimated from observed consumption patterns. For n competing products, this particular functional form requires the estimation of only $n+1$ parameters, but is able to reflect cross-elasticities in a fairly robust manner. For given parameter values, the optimal prices must be determined by an iterative computational procedure. A Newton algorithm has been developed and programmed for searching for the optimum of this objective function and some illustrative examples are solved and analysed.

Product line pricing has been studied widely, in both the economics and marketing literature. Overviews of product pricing research in these areas are contained in two survey papers, Nagle [10] and Rao [14]. It is useful to summarize some aspects of this literature here, but this paper will not attempt a survey.

Theoretical foundations for multiple product pricing date back to Hotelling [4] and are based on 'spacial competition' models in which each product is represented by a geometric point that defines its characteristics relative to other products. Product characteristics are most commonly represented along a single attribute dimension (eg product quality) in addition to price. (See, for example, Stokey [18], Lancaster [6, 7], Schmalensee [15], Oren *et al* [11] and Smith [16].) Differentiating products along a single dimension allows closed form analytic solutions to be obtained for optimal product pricing and in some cases for product attribute positioning, if the product attribute is one for which all customers have the same preference ordering, and other regularity conditions are met. That is, for a product attribute such as quality, all customers must prefer more quality to less, but may differ in their willingness to pay for it. In some pricing research, price structure itself is used as the means of product differentiation. (See, for example, Mirman and Sibley [9], Spence [17] and Oren *et al* [12].) The theory of spacial competition has been extended to two dimensions, again using attributes for which consumers hold uniformly ordered preferences. In this case, geometric models can be used to obtain each product's market share, by assuming that products are infinitely divisible, in that the amount of each attribute obtained per dollar captures the relevant information for the consumer. This structure excludes non-linear relationships between products' demands and their prices. (See, for example, Hauser and Shugan [3], Huber and Puto [5] and Hauser and Gaskin [1]. Marketing and econometric researchers have developed mappings from customers' perceptions of products' attributes to their selections from the set of available products in the market. (See, for example, Hauser and Simmie [2] and McFadden [8].)

This paper's approach differs from much of the previous research discussed above in that the goal of obtaining closed-form analytic solutions for the products' optimal prices is abandoned. This allows removal of the restriction that customers' preference for the offered products be monotone in any uniform manner, which poses an unrealistic limitation in the case of certain product attributes such as time-of-use differentiation. A second restriction which is present in many deterministic product positioning models, namely that cross-substitution occurs only between spacially 'adjacent' products, is also not required in this paper's framework.

Pricing models that characterize products by one uniformly preferred attribute are able to incorporate more general price structures than the fixed marginal price per unit considered in this paper. This paper models consumers' demand for the offered products directly, as opposed to modelling preferences for product attributes and then mapping them onto the set of offered products. This approach is used by most of the models in the economics literature, while the marketing literature has focused on generic product attributes and consumers' perceptions of them. Modelling product demand directly has the advantage of simplicity and a reduction in the number of relationships that must be developed, but does not provide any sensitivity information concerning the choice of products' attributes (other than price).

Supplier competition can be analysed in a variety of ways. A simple competitive scenario is to define the objective function in terms of the products offered by a particular supplier and to assume that the prices of the products offered by other suppliers are fixed. The cross-substitution effects of the other suppliers' products are then reflected in the matrix of cross-elasticities, ignoring the possibility of competitors' reactions to price changes. This approach could easily be applied to this paper's model. If competitive reactions are considered explicitly, the pricing problem becomes much more complex, and the suppliers' prices must be obtained as solutions of an equilibrium model. This extension of the results in this paper has not been attempted.

Model specifications

Let us consider the situation in which there are n distinct products offered by a single supplier. Pricing is at a fixed marginal rate for each product, so that

m_i = marginal price per unit of product i, $i = 1, 2, \ldots, n$

A price plan is defined by the vector $\mathbf{m} = m_1, \ldots, m_m$. To formulate the problem of selecting an optimal price plan, we introduce the demand functions for each of the n products as a function of the price vector $\mathbf{m}$,

$q_i(\mathbf{m})$ = quantity of demand for the ith product when the prices are $\mathbf{m} = m_1, \ldots, m_n$

We assume the supplier of the good or service has a general production cost function of the form

$C(\mathbf{q})$ = cost of supplying q_i units of product i

where $\mathbf{q} = q_1, \ldots, q_n$.

In industries such as electric power, for example, $C(\mathbf{q})$ is typically convex over most quantity regions, since successively more costly production facilities are called into service as demand increases.

The supplier's net revenue with marginal price schedule $\mathbf{m}$ is given by

$$NR = \sum_{i=1}^{n} m_i q_i - C(\mathbf{q}) \tag{1}$$

where $q_i = q_i(\mathbf{m})$.

The partial derivatives of NR with respect to price are given by

$$\partial NR/\partial m_j = q_j + \sum_{i=1}^{n} [m_i - c_i(\mathbf{q})]\partial q_i/\partial m_j = 0 \qquad j = 1, 2, \ldots, n \tag{2}$$

where $c_i(\mathbf{q}) = \partial C(\mathbf{q})/\partial q_i$.

These equations correspond to the standard monopoly pricing conditions in vector form. To illustrate, let us define the matrix E of elasticities and cross-elasticities, where the elements are

$$E_{ij} = -(\partial q_i/\partial m_j)m_j/q_i \qquad i, j = 1, 2, \ldots, n \tag{3}$$

Then multiplying through by m_j, Equation (2) can be written as

$$[\mathbf{m}\circ\mathbf{q}](I - E) = [\mathbf{c}\circ\mathbf{q}](-E)$$

where $[\mathbf{m}\circ\mathbf{q}] = (m_1 q_1, m_2 q_2, \ldots, m_n q_n)$
$[\mathbf{c}\circ\mathbf{q}] = (c_1 q_1, c_2 q_2, \ldots, c_n q_n)$.

Multiplying from the left by E^{-1}, we obtain the vector equation for the standard monopoly pricing necessary condition

$$[\mathbf{m}\circ\mathbf{q}](I - E^{-1}) = [\mathbf{c}\circ\mathbf{q}] \tag{4}$$

This is a general multi-product optimal pricing condition for a monopoly supplier, as long as prices are required to be linear in the quantity purchased. The various products $i = 1, 2, \ldots, n$ can also be used to distinguish multiple attributes. For example, in electric power consumption, we might distinguish products both by time of day and by interruptibility level. In this case we would have a matrix of prices

m_{ij} = price per unit consumed at time period i at interruptibility level j

Then the $q_{ij}(\mathbf{m})$ would be the corresponding consumption quantities.

Cost function structure

It might be reasonable, in some cases, to assume that the production cost is additively separable, ie

$$C(\mathbf{q}) = \sum_i C_i(q_i) \tag{5}$$

This would be appropriate for time of use products, for example, if the cost of transition between time periods can be neglected. If, as demand increases for each product, more costly production methods are successively brought into use, each $C_i(q_i)$ would be a convex function of q_i.

When multi-product prices are offered in matrix form, more general cost function forms may be appropriate. For example, let i correspond to the time period and j correspond to the interruptibility level of the contract, with prices m_{ij} and quantities q_{ij}. The cost function in this case might have the alternate form

$$C(\mathbf{q}) = \sum_{i,j} C_{ij}\left(\sum_{k<j} q_{ik}\right) \tag{6}$$

which implies that the production cost of the jth level of interruptibility depends upon the quantities produced at all priority levels greater than or equal to the jth. More complex interdependencies can be included in Equation (4) as well. All of these formulations fall within the framework of the optimal pricing conditions in Equations (2) or (4).

Weighted sum objective function

The classical product pricing objective in regulated industries is to maximize total welfare, subject to a constraint on net revenue that guarantees an adequate return on capital. This includes as special cases maximization of the net revenue function Equation (1) and total surplus maximization. The net revenue, given a vector of prices $\mathbf{m}$, was defined previously. Total welfare is expressed as net revenue plus consumer surplus, or as the consumers' total utility minus the supplier's cost. Defining the functions

$U(\mathbf{q})$ = total consumer utility for consuming the product quantities $\mathbf{q} = q_1, \ldots, q_n$
$CS(\mathbf{m})$ = total consumer surplus, when the vector of prices is $\mathbf{m} = m_1, \ldots, m_n$

total welfare is given by

$$NR(\mathbf{m}) + CS(\mathbf{m}) = U(\mathbf{q}(\mathbf{m})) - C(\mathbf{q}(\mathbf{m})) \tag{7}$$

Let us add the constraint that $NR(\mathbf{m}) \geqslant N_0$, where N_0 is an externally imposed revenue constraint. If we associate the Lagrange multiplier μ with this constraint, the resulting Lagrangian is of the form

$$U(\mathbf{q}(\mathbf{m})) - C(\mathbf{q}(\mathbf{m})) + \mu NR(\mathbf{m})$$

This will be maximized over $\mathbf{m}$ by the supplier, subject to the condition that the demands $\mathbf{q}(\mathbf{m})$ constitute an optimal consumer response to the prices $\mathbf{m}$. Using the utility function $U(\mathbf{q})$ and neglecting income effects, this is equivalent to the condition

$$\mathbf{q}(\mathbf{m}) = \arg\max_{\mathbf{q}} \left[U(\mathbf{q}) - \sum_i q_i m_i\right] \tag{8}$$

or

$$\partial U(\mathbf{q}(\mathbf{m}))/\partial q_i = m_i \qquad i = 1, 2, \ldots, n$$

Thus the pricing problem may be stated as

$$\max[U(\mathbf{q}(\mathbf{m})) - C(\mathbf{q}(\mathbf{m})) + \mu NR(\mathbf{m})] \tag{9}$$

subject to $\partial U(\mathbf{q}(\mathbf{m}))/\partial q_i = m_i \qquad i = 1, 2, \ldots, n$

The first-order necessary conditions for this objective function are given by the following:

$$\partial/\partial m_i = \mu q_i + (\mu + 1) \sum_i [m_i - c_i(\mathbf{q})](\partial q_i/\partial m_i) = 0$$
$$i = 1, \ldots, n$$

using $m_i = \partial U/\partial q_i$. These can be rewritten in the form

$$[\mathbf{m} \circ \mathbf{q}]\,(I - \beta E^{-1}) = [\mathbf{c} \circ \mathbf{q}] \tag{10}$$

where $\beta = \mu/(1 + \mu)$ and $[\mathbf{m}\circ\mathbf{q}]$, $[\mathbf{c}\circ\mathbf{q}]$ are the vectors defined previously, and E is the cross-elasticity matrix. Notice that as μ approaches 0, ie as the revenue constraint receives less weight, the optimal prices approach the marginal costs. As μ approaches ∞, the prices approach the optimal monopoly prices defined by Equation (4). The utility function $U(\mathbf{q})$, was defined only for the formulation and does not need to be determined to solve these necessary conditions, since its partial derivatives equal the optimal prices.

Demand function models

By assuming particular forms for the consumer preference structure, specific models for the demand functions $q_i(m_1, \ldots, m_n)$ can be developed and more explicit results can be obtained. We will assume that income effects can be neglected so that there is an indirect utility function that describes a customer's willingness to pay for consumption for various products. We assume that each customer's potential consumption consists of many types of individual units each having a different preference structure, both in terms of relative importance and preference values for the various products.

Each different type of unit of consumption will be identified by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ where $x_i \geqslant O$ for all i. We define the (maximum) willingness to pay function

$W(\mathbf{x}, i)$ = maximum willingness to pay for consuming a unit of type $\mathbf{x}$, under the conditions corresponding to product i.

In time of use pricing, for example, the products would correspond to particular discrete time intervals $t_{i-1} \leqslant t \leqslant t_i$, $i = 1, 2, \ldots, n$ for consumption. The t_i would be the hours of the day, with $t_0 = t_n$ corresponding to 12 midnight.

Each consumption unit will be assigned to exactly one of the n possible products (or none of them). We will use the following form for $W(\mathbf{x}, i)$:

$$W(\mathbf{x}, i) = x_i \qquad i = 1, \ldots, n \tag{11}$$

It will be illustrated shortly that there is no loss of generality in assuming this form. We next define a general distribution function $Q(\mathbf{x})$, which is non-increasing in all components of $\mathbf{x}$

$Q(\mathbf{x})$ = quantity of potential consumption units for which the willingness to pay is less than or equal to $\mathbf{x}$, component by component

That is, $Q(\mathbf{x})$ includes all units $y_1, \ldots, y_n$ such that $y_i \leqslant x_i$, $i = 1, \ldots, n$. The demand functions $q_i(\mathbf{m})$ can then be defined in terms of $Q(\mathbf{x})$.

Let us now show that there is no loss of generality in assuming this preference structure. Suppose that the consumption unit types are indexed by T, and we let $u_i(T)$ correspond to the willingness to pay for assigning a unit of type T to product i. Furthermore, let $H(T)$ be the density function for the number of units of type T. Then $Q(\mathbf{x})$ is defined as

$$Q(\mathbf{x}) = \int_{\{T \mid u_i(T) \leqslant x_i,\ i = 1, 2, \ldots, n\}} H(T)dT$$

Without loss of generality, the index T can be suppressed and units can be identified directly by the willingness to pay vector $(x_1, x_2, \ldots, x_n) = (u_1(T), u_2(T), \ldots, u_n(T))$. That is, $Q(\mathbf{x})$ is simply the quantity of units whose willingness to pay vectors are less than $\mathbf{x}$, component by component.

Let us now suppose that a price schedule $\mathbf{m} = m_1, m_2, \ldots, m_n$ is offered and we wish to determine the resulting demands $q_i(\mathbf{m})$, $i = 1, \ldots, n$. Assuming that customers specify their consumption patterns so as to maximize benefit minus cost, we have that a unit $\mathbf{x}$ is assigned to product i if and only if

$$x_i \geqslant m_i$$

and

$$x_i - m_i \geqslant x_j - m_j, \quad \text{for all } j \tag{12}$$

To obtain the demand function $q_i(\mathbf{m})$ we simply integrate the density $dQ(\mathbf{x})$ over the region for which these conditions hold. It can be seen that this leads to an iterated integral over the following ranges of x_j variables:

$$0 \leqslant x_j \leqslant x_i - m_i + m_j \qquad j \neq i$$
$$m_i \leqslant x_i < \infty$$

where the intergration over x_i is performed last. Writing this expression, for $i = 1$, for example, we have

$$q_1(\mathbf{m}) = \int_{m_1}^{\infty} \int_0^{x_1 - m_1 + m_2} \cdots \int_0^{x_1 - m_1 + m_n} dQ(\mathbf{x}) \tag{13}$$

Assuming the integrations can be performed, this explicitly determines the demand functions.

Given the demand functions $q_1, \ldots, q_n$, the total consumer surplus can be obtained by Willig's method of line integrals (Willig [19]). The surplus is obtained as the value of a line integral in the n dimensional space of prices, whose path goes to the point $m_1, \ldots, m_n$ from the point $(\infty, \ldots, \infty)$. That is, we are using a path independent line integral to compute the area 'under the demand curve' in this n dimensional space. The path that leads to the simplest integration formula is to integrate along the following sequence of line segments: $(\infty, \ldots, \infty)$ to $(m_1, \infty, \ldots, \infty)$, then $(m_1, \infty, \ldots, \infty)$ to $(m_1, m_2, \ldots, \infty)$, $\ldots$, and finally $(m_1, \ldots, m_{n-1}, \infty)$ to $(m_1, \ldots, m_n)$. Expressing these as a sum integrals, we have

$$CS_1(\mathbf{m}) = \int_{m_1}^{\infty} q_1(\mathbf{x})|_{\substack{x_i = \infty \\ i \neq 1}} dm_1 + \int_{m_2}^{\infty} q_2(\mathbf{x})|_{\substack{x_i = \infty \\ i \neq 1,2}} dm_2 + \int_{m_n}^{\infty} q_n(\mathbf{x}) dm_n \tag{14}$$

For certain special forms of $q_1, \ldots, q_n$, the integrals can be evaluated directly, as will be illustrated shortly with some examples.

Some specific cases

Our analysis techniques can be made more specific by investigating the demand functions that result from some simplifying assumptions. A 'first-order' demand model can be obtained by assuming that there exist independent density functions for the various components x_i in $Q(\mathbf{x})$. That is, we assume the following form for $Q(\mathbf{x})$:

$$Q(\mathbf{x}) = Q_0 Q_1(x_1) Q_2(x_2) \ldots Q_n(x_n)$$

where

$$\int_{x_i} dQ_i(x_i) = 1 \quad i = 1, 2, \ldots, n \tag{15}$$

and

$$\int_x dQ(\mathbf{x}) = Q_0$$

Operationally, this means that we assume that the fraction of demand that has willingness to pay x_i or less for the ith contract is the same, regardless what willingness to pay levels x_j, $j \neq i$ hold for the other contracts.

Independence, like linearity, is an assumption that seldom holds precisely in practice, but is often a reasonable first-order approximation. Let us consider its implications in the time of day pricing context, for example. In our model, typical potential demand units would be composed of various types of **x** vectors defined by the type of consumption they represent. Consumption which has a high willingness to pay and is also highly substitutable across time periods, would be characterized by **x** vectors with all components fairly large. Relatively less important, but still substitutable, consumption would be characterized by **x** vectors with all small entries. Demand for consumption that is important to maintain at a nearly constant level across all time periods would be represented by n different **x** vectors, one for each time period of intended consumption. The **x** vector corresponding to the ith time period would have a large x_i component, with small components $x_j, j \neq i$. The two situations we have described are extremes in that in the first situation, the sizes of the x_i components are positively correlated and in the second situation the components are negatively correlated. Assuming that independence holds assumes, in effect, that the relative proportions of these types of units balance out so that the magnitudes of the various components of **x** are uncorrelated in the sample space.

By assuming independence, the form of $q_i(\mathbf{m})$ simplifies considerably, as illustrated for $i = 1$

$$q_1(\mathbf{m}) = \int_{m_1}^{\infty} Q_0 Q_2(x_1 - m_1 + m_2) \ldots Q_n(x_1 - m_1 + m_n) dQ_1(x_1) \qquad (16)$$

To solve some particular examples, we will choose specific functions for $Q_i(x_i)$.

Exponential case

Let us consider the case in which

$$Q_i(m_i) = 1 - e^{-\lambda_i m_i}, \ i = 1, 2, \ldots, n \qquad (17)$$

This simple form turns out to have very appealing properties from a practical standpoint and is analytically convenient because it involves only the n parameters $\lambda_1, \ldots, \lambda_n$. These functions lead to densities for the m_i which have progressively smaller amounts of demand as m_i increases. That is, $d^2Q_i(m_i)/dm_i^2 = -\lambda_i^2 e^{-\lambda_i m_i}$ is negative. The value $1/\lambda_i$ equals the average willingness to pay per unit for product i.

The resulting $q_i(\mathbf{m})$ functions are

$$q_i(\mathbf{m}) = Q_0 e^{-\lambda_i m_i} \int_{m_i}^{\infty} \left(\prod_{j \neq i} [1 - e^{-\lambda_j(x_i + m_j)}] \right) \lambda_i e^{-\lambda_i x_i} dx_i, \ i = 1, 2, \ldots, n \qquad (18)$$

Although somewhat complex algebraicly, these expressions can be integrated out. For example, we have for $n = 3$,

$$\begin{aligned} q_1(m_1, m_2, m_3) = \lambda_1 Q_0 e^{-\lambda_1 m_1} [& (1/\lambda_1) \\ & - e^{-\lambda_2 m_2}/(\lambda_1 + \lambda_2) \\ & - e^{-\lambda_3 m_3}/(\lambda_1 + \lambda_3) \\ & + e^{-\lambda_2 m_2 - \lambda_3 m_3}/(\lambda_1 + \lambda_2 + \lambda_3)] \end{aligned} \qquad (19)$$

The general form of this expression, which holds for all values of n, can be seen from this formula. For $q_i(\mathbf{m})$, we have a sum of terms, multiplied by $Q_0 e^{-\lambda_i m_i}$. The terms in the sum are of the form

$$(-1)^k \lambda_i e^{-\Sigma_{j \varepsilon L} \lambda_j x_j} / \left(\lambda_i + \sum_{j \varepsilon L} \lambda_j \right)$$

where L ranges over all subsets of indices j, $1 \leqslant j \leqslant n$ and $j \neq i$, and k is the number of indices in L.

One easy way to characterize all the sets L for a given i is to use $n - 1$ digit binary numbers from 0 to $2^{n-1} - 1$, where each position containing a 1 selects an entry $j \neq i$ to be included in that particular L.

The partial derivatives have simple expressions in terms of the q_i. It can be verified, using integration by parts, that

$$\begin{aligned} & \partial q_i / \partial m_i = \lambda_i q_i \qquad i, j = 1, \ldots, n \\ & \partial q_i / \partial m_j = \lambda_j (q_i |_{m_j = \infty} - q_i) \end{aligned} \qquad (20)$$

where

$$q_i |_{m_j = \infty} = q_i(m_1, \ldots, m_{j-1}, \infty, m_{j+1}, \ldots, m_n)$$

Notice that $q_i|_{m_j = \infty} - q_i$ is the maximum potential new demand for product i that can result from increasing price m_j. The expression for $\partial q_i / \partial m_i$ really has this same form too, since $q_i|_{m_i = \infty} = 0$. To interpret these derivatives then, the rate of change in demand in any period due to a single price change is proportional to the maximum demand change that would occur if the price were *increased* as far as possible.

The elasticities have simple forms as well,

$$E_{ij} = \lambda_i m_i \qquad j = i$$
$$\lambda_j m_j[1 - (q_i|_{m_j=\infty}/q_i)] \quad j \neq i \tag{21}$$

The terms $\lambda_i m_i$ are dimensionless, as required, and, in terms of the exponential distribution function, correspond to the 'time constant' of the exponentially discounted demand for the current set of prices.

The elasticities in Equation (21) have intuitively appealing interpretations. The elasticity of demand for the ith product with respect to its own price m_i is a linear function of m_i. This states that, as price increases, the demand becomes more elastic at a linear rate. That is, for low price levels, demand is relatively inelastic. However, as prices become higher, demand is more noticeably affected by price. In the electric power context, this increase in elasticity might be attributed to increased substitution of other energy resources or increased conservation, for example.

The elasticity of the ith demand with respect to the jth price, ie the cross-elasticity, is proportional to the jth price as well. This is reasonable, since as the jth price increases, the jth product becomes less attractive as a substitute. Thus we would expect that this rate of substitution would decline. The other multiplicative term is equal to the maximum fractional increase possible in the ith product's demand that could result from a price increase in m_j. That is, when $m_j = \infty$, this factor is the fractional demand change in q_i that would result. Thus the local elasticity value is assumed to be proportional to the maximum percentage demand change that could occur. As a first-order model in which elasticity effects are separable in i and j, these appear to be reasonable assumptions.

The consumer surplus can be integrated directly in this case. By using the facts that

$$\partial q_i/\partial m_i = -\lambda_i q_i \qquad \text{and} \qquad q_i|_{m_i=\infty} = 0$$

the integrals can be evaluated easily. The resulting consumer surplus can be expressed as follows:

$$\mathrm{CS}(\mathbf{m}) = q_1(m_1)/\lambda_1 + q_2(m_1, m_2)/\lambda_2 + \cdots + q_n(m_1, m_2, \ldots, m_n)/\lambda_n \tag{22}$$

In this equation, the number of arguments in m_i that are included indicate the appropriate number of products in the expression for q_i. That is, $q_1(m_1)$ is the demand for product 1 when there are no other products, $q_2(m_1, m_2)$ is the demand for product 2 when there is one other product priced at m_1, etc. Since the line integral is independent of the path, this expression should be invariant to permutations in the indices of the products. It can be verified algebraically that this is, in fact, the case.

Let us consider these demand functions in the context of our previous first-order necessary conditions (10). The key factors in obtaining the solution are the entries E_{ij}, namely $\lambda_i m_i$ and $1 - (q_i|_{m_j=\infty})/q_i$. Let us consider the second expression first. By referring to the expression for $Q(\mathbf{m})$, it can be seen that for $n = 4$, for example,

$$\begin{aligned} q_1|_{m_2=\infty} - q_1 = Q_0 e^{-\lambda_1 m_1}[&\lambda_1 e^{-\lambda_2 m_2}/(\lambda_1 + \lambda_2) \\ &- \lambda_1 e^{-\lambda_2 m_2 - \lambda_3 m_3}/(\lambda_1 + \lambda_2 + \lambda_3) \\ &- \lambda_1 e^{-\lambda_2 m_2 - \lambda_4 m_4}/(\lambda_1 + \lambda_2 + \lambda_4) \\ &- \lambda_1 e^{-\lambda_2 m_2 - \lambda_3 m_3 - \lambda_4 m_4}/(\lambda_1 + \lambda_2 \\ &+ \lambda_3 + \lambda_4)] \end{aligned} \tag{23}$$

That is, the difference consists of all terms from q_1 that contain m_2. For this model these terms are always less than q_1 itself. Thus, the largest terms in the matrix E are the $\lambda_i m_i$.

For the Newton method solution approach discussed in the Appendix, it is simpler to determine the equations for $\nabla\mathbf{q}$ and $\nabla^2\mathbf{q}$ directly. We have

$$(\nabla\mathbf{q})_{ki} = \lambda_j(q_k|_{m_j=\infty} - q_k)$$
$$(\nabla^2\mathbf{q})_{kij} = \lambda_i\lambda_j[q_k|_{\substack{m_i=\infty \\ m_j=\infty}} - q_k|_{m_j=\infty} - q_k|_{m_i=\infty} + q_k] \tag{24}$$

Thus the equations for ∇f and $\nabla^2 f$ can be evaluated explicitly at any vector $\mathbf{m}$.

Specific example

The exponential demand functions have been programmed as part of a computer algorithm for determining optimal prices for the n products. Let us consider the following example for time of use pricing for electric power with four products (time periods). Let the 24 hour day be divided into 4 equal periods of 6 hours each.

A simple generation cost model will be used for illustrative purposes only. We assume that the supplier has a basic generation capacity of 5 billion watts, ie 5 GWh per hour. Thus, during a 6 hour time period, the base power capacity would be 30 GWh. We assume that the average unit cost of generating the 30 GWh in 6 hours is 3.33¢/kWh. Thus the 30 GWh would cost \$1.0 million to generate. Suppose that capacity can be expanded beyond the base output, but that diseconomies of scale result, because more costly production facilities are brought into service. Suppose that the diseconomies are such that the marginal cost is always twice the average cost per unit. Then the production cost for the ith time period is given by

$$C_i(q_i) = C_n(q_i/B_n)^{\gamma} \tag{25}$$

where q_i is the amount demanded in that time period.

The parameters have the following interpretations:

B_n = base generating capacity for the time period
C_n = cost of generating B_n
γ = ratio of marginal cost to average cost

Thus $B_n = 30\,\text{GWh}$, $C_n = \$1.0$ million and $\gamma = 2$ in our example. We will assume that costs are equal and additively separable across products. Thus

$$C(\mathbf{q}) = \sum_{i=1}^{4} (1.0)(q_i/30)^2 \tag{26}$$

To model the product demands we will assume the separable exponential distribution over unit types, so that

$$Q(\mathbf{x}) = Q_0 Q_1(x_1) Q_2(x_2) Q_3(x_3) Q_4(x_4) \tag{27}$$

where

$$Q_i(x_i) = 1 - e^{-\lambda_i x_i}, \quad i = 1, 2, 3, 4$$

Thus the demand function parameters are λ_1, λ_2, λ_3, λ_4, and Q_0. For example, let $\lambda = 10, 15, 20, 30$ and $Q_0 = 100$ GWh. The values $1/\lambda_i = 0.1, 0.067, 0.05, 0.033$ are equal to the average willingness to pay in \$/kWh for each of the respective products. Intuitively, the parameter Q_0 is the total demand across all time periods that would result from offering service with zero prices.

Computational examples

To calculate examples of the above form, an algorithm was programmed in APL to solve iteratively for the optimal price. The algorithm uses a gradient search optimization technique, in which the user specifies the step size interactively at each stage. This is a rather crude approach but it successfully located the optimal solution to within two or three significant figures after 10 to 20 steps for the examples that were tried. The interactive approach is useful in practice, because it might be desirable to implement a major price change in several stages. This would allow time for consumers to adjust to the price change and would provide additional information about demand characteristics at each new stage. The algorithm used for solving this example would permit this sort of gradual implementation.

For the initial conditions, suppose that all time periods have equal prices of 5¢/kWh. Thus $m_i = 5.0$ for $i = 1, 2, 3, 4$. The units of m_i are equal both to millions of dollars per GWh and to dollars per kWh. Using the demand functions defined previously we have the base case set out in Table 1.

We can make several observations about this case. The total demand is 89.8 GWh or 89.8% of the maximum demand. The net revenues are higher in the intermediate time periods because, although demands are lower, the unit margins are much larger. The total net revenue is \$1.482 million per day or \$541 million per year.

By applying the iterative optimization method, we obtained Table 2 for the prices that maximize net revenue. Considering these figures, we note the following. First, the total net revenue has increased substantially to \$5.25 million per day. Despite the increase in prices in all time periods, the demand in the fourth period has actually increased. This is because the much larger price increases in the other time periods caused demand to shift into the fourth time period, even though its price increased as well. This example illustrates the cross-substitution effects that can be captured by this model, in that the demand is redistributed over the complete range of contracts in a globally optimal manner.

The optimal price plan that results in this case produced high revenues through large price increases and resulted in a significant decrease in consumption volume. This rather drastic change occurred because no constraints were placed on the consumer surplus

Table 1. Base case.

	Time periods 1	2	3	4	Totals
Prices (¢/kWh), **m**	5.0	5.0	5.0	5.0	
Demands (GWh), **q**	42.4	24.8	15.6	7.0	89.8
Marginal cost, **c**	9.4	5.5	3.5	1.6	
Average cost	4.7	2.8	1.7	0.8	
Net revenue (\$ million)	0.127	0.546	0.515	0.294	1.482

Table 2. Maximum net revenue.

	Products 1	2	3	4	Total
Prices (¢/kWh), **m**	15.9	11.7	9.5	6.9	
Marginal cost, **c**	3.9	3.0	2.5	1.9	
Average cost	2.0	1.5	1.3	1.0	
	Averages per day				
Demands (GWh), **q**	17.6	13.6	11.3	8.6	51.0
Net revenue ($ million/day)	2.45	1.39	0.92	0.51	5.28

and thus a pure monopoly price plan resulted. The weighted sum objective function which considers both net revenue and consumer surplus can be used to control this effect. For example, we can optimize the objective, total welfare + μ[net revenue]. The same optimization approach can be applied, with the modifications discussed previously, to obtain the optimal price plan in this case. For the case $\mu = 1.0$, we obtain output shown in Table 3.

Let us now consider the case in which $\mu = 0$, ie no revenue constraint is imposed. In this case we are maximizing the sum of consumer surplus and net revenue (total welfare). For the previously given demand parameters, this case is illustrated in Table 4. Notice that in this case the optimal prices are equal to the marginal costs, as discussed previously.

For comparison, let us also consider the 'one product' pricing case that would achieve the same total revenue as the case in Table 4. That is, although the products have different customer demand, we impose the requirement that they be priced identically as one product. By iterating the pricing calculation, it can be determined that this requires a constant price level of 6.13¢/kWh, as indicated in Table 5.

Conclusion

This paper's modelling assumptions make it well suited for certain types of applications and less appropriate for others. It was developed for analysing cross-substitution effects among competing electric power products, eg different time of use periods. The model has not been applied to telecommunications markets. However, it appears that there are analogies between telecommunications and electric power, both in the presence of cross-substitution effects and in some product family attributes, such as time of use differentiation. Since the nature of the product attributes is not restricted, the model can consider products defined by attribute combinations, eg time of use and power firmness in the electric power context. When time of use product differentiation is to be considered, there is typically no universal ordering of customers' preferences with regard to time period, so this model's lack

Table 3. Weighted sum ($\mu = 1.0$).

	1	2	3	4	Total
Prices (¢/kWh), **m**	10.3	8.5	7.3	5.8	
Marginal cost, **c**	6.3	4.2	3.1	2.1	
Average cost	3.2	2.1	1.6	1.0	
	Averages per day				
Demands (GWh), **q**	28.4	19.0	14.1	9.3	70.8
Net revenue ($ million/day)	2.03	1.21	0.81	0.44	4.49

Table 4. No revenue constraint ($\mu = 0$).

	Products 1	2	3	4	Total
Prices (¢/kWh), **m**	7.3	5.4	4.3	3.1	
Marginal cost, **c**	7.3	5.4	4.3	3.1	
Average cost	3.6	2.7	2.1	1.6	
	Averages per day				
Demands (GWh), **q**	32.62	24.13	19.30	13.99	90.04
Net revenue ($ million/day)	1.18	0.65	0.41	0.22	2.46

Table 5. Fixed price 6.13 ¢/kWh all products

	Products 1	2	3	4	Total
Prices (¢/kWh), **m**	6.13	6.13	6.13	6.13	
Marginal cost, **c**	8.98	5.12	3.15	1.33	
Average cost	4.49	2.56	1.57	0.67	
	Averages per day				
Demands (GWh), **q**	40.40	23.06	14.16	6.01	83.63
Net revenue ($million/day)	0.66	0.82	0.65	0.33	2.46

of restriction in this regard is particularly advantageous. Because the model considers demand functions for specific products rather than for product attributes, it provides no information about product repositioning (other than price changes). It is thus better suited for markets in which the competing products' attributes are relatively stable and well defined.

Because the optimal prices are determined by an iterative procedure, there are computational limitations on the number of products that can be considered in determining optimal prices for the product family. The computational difficulty increases geometrically in the number of products. In experimental runs on a DEC 2060 computer, seven or eight competing products appeared to be about the limit for computational feasibility of this model, when using the one parameter exponential family of demand functions.

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Appendix

Iterative methods of solution

The matrix equation (10), namely

$$[\mathbf{m}\cdot\mathbf{q}](I - \beta E^{-1}) = [\mathbf{c}\ \mathbf{q}]$$

cannot be directly solved when the explicit dependencies $q_i = q_i(\mathbf{m})$ and $c_i = C_i(\mathbf{q})$ are included. However, if we treat E and $\mathbf{q}$ as fixed, Equation (10) can be solved for $\mathbf{m}$ as long as the matrix $(I - \beta E^{-1})^{-1}$ is well defined.

This suggests an iterative solution approach. Suppose we start with some initial set of prices $\mathbf{m}^0$. These can be used to determine the parameters $q_i^0 = q_i(\mathbf{m}^0)$, $c_i^0 = c_i(q_i^0)$, and the matrix E^0. Alternatively, E^0 could be computed by using the values $\mathbf{q}^0$ and $\mathbf{m}^0$ and a model for the demand functions. Some particular models were discussed previously. These initial values can then be used to compute a new set of prices

$\mathbf{m}^1$ from the linear equations. At the kth step, we would have the following set of vector and matrix equations:

$$\begin{aligned}
[\mathbf{m}^k \circ \mathbf{q}^{k-1}] &= [\mathbf{c}^{k-1} \circ \mathbf{q}^{k-1}][1 - \beta(E^{k-1})^{-1}]^{-1} \\
q_i^k &= g_i(\mathbf{m}^k) \\
c_i^k &= c_i(q_i^k) \\
E^k &= E(\mathbf{q}^k, \mathbf{m}^k)
\end{aligned} \tag{28}$$

Operationally, a step by step procedure could be implemented to obtain incremental pricing improvements over a period of time. For example, at a given point in time, $\mathbf{m}^0$, $\mathbf{q}^0$, $\mathbf{c}^0$ could be observed and E^0 could be estimated. The linear equations could be used to obtain a new set of prices m^1, which could be implemented on an experimental basis.

After a period of observation, new quantities q^1, c^1 could be determined. These might be estimated using some weighted average method over time to reduce random fluctuation effects. Updated elasticity estimates E^1 could also be prepared, based upon empirical observations, or from a specific demand model.

Newton algorithm

The above iterative procedure, although theoretically correct, does not consider the convergence or efficiency of the procedure. Various efficient iterative optimization techniques have been developed for problems of this type. We will discuss a Newton method approach in this section, which appears well suited to our later examples.

The optimization problem will be stated as one of maximizing a general unconstrained function $f(\mathbf{m})$ over the price vector $\mathbf{m}$. We will discuss the case in which $f(\mathbf{m})$ equals the supplier's net revenue, ie a profit maximizing criterion, first. However, a similar approach can also be used for the other criteria mentioned previously, such as minimizing total production cost, subject to a constraint on total revenue or a constraint on rate of return.

For the profit maximization case,

$$f(\mathbf{m}) = \sum_k m_k q_k(\mathbf{m}) - C(\mathbf{q}(\mathbf{m})) \tag{29}$$

where $q_k(\mathbf{m})$ is the demand for the kth product as function of the price vector $\mathbf{m}$. We wish to compute the vector ∇f and the matrix $\nabla^2 f$, where

$$(\nabla f)_j = \partial f/\partial m_j \quad \text{and} \quad (\nabla^2 f)_{ij} = \partial^2 f/\partial m_i \partial m_j$$

Taking partial derivatives, we have

$$(\nabla f)_j = q_j + \sum_k (\partial q_k/\partial m_j)(m_k - c_k) \tag{30}$$

and

$$\begin{aligned}
(\nabla^2 f)_{ij} = {} & \partial q_j/\partial m_i + \partial q_i/\partial m_j \\
& + \sum_k (\partial^2 q_k/\partial m_i \partial m_j)(m_k - c_k) \\
& - \sum_k (\partial q_k/\partial m_j) \sum_l (\partial c_k/\partial q_l)(\partial q_l/\partial m_i)
\end{aligned} \tag{31}$$

where
$c_k = \partial C/\partial q_k$

If we define the matrices

$$(\nabla q)_{kj} = \partial q_k/\partial \mathrm{m}_j, \quad (\nabla^2 C)_{kl} = \partial^2 C/\partial q_k \partial q_l \tag{32}$$

and the three dimensional array

$$(\nabla^2 q)_{kij} = \partial^2 q_k/\partial m_i \partial m_j \tag{33}$$

we can write these equations as

$$\nabla f = \mathbf{q} + (\mathbf{m} - \nabla C)\nabla \mathbf{q}$$

$$\nabla^2 f = \nabla q + \nabla q' + (\mathbf{m} - \nabla C)(\nabla^2 \mathbf{q}) - (\nabla \mathbf{q})(\nabla^2 C)\nabla \mathbf{q})' \tag{34}$$

It can be seen that $\nabla^2 f$ is symmetric, as required.

If both ∇f and $\nabla^2 f$ can be obtained, we can use a 'Newton' method of maximizing $f(\mathbf{m})$. The basic approach is to assume that $f(\mathbf{m})$ is locally a quadratic function and thus has a three term Taylor expansion about any vector $\mathbf{m}$, ie

$$f(\mathbf{m} + \mathbf{a}) = f(\mathbf{m}) + \mathbf{a}\nabla \mathbf{f}(\mathbf{m}) + \mathbf{a}\nabla^2 f(\mathbf{m})\mathbf{a}'/2$$

Thus the vector direction $\mathbf{a}$ which produces the greatest improvement in $\mathbf{m}$ would satisfy

$$\nabla f(\mathbf{m}) + (\nabla^2 f(\mathbf{m}))a' = 0$$

or

$$a = -(\nabla f(\mathbf{m}))'[\nabla^2 f(\mathbf{m})]^{-1} \tag{35}$$

Assuming f is concave, ie the matrix $\nabla^2 f(\mathbf{m})$ is negative semidefinite, the vector $\mathbf{m} + \mathbf{a}$ will produce an improve functional value. Often a 'scale factor' α is applied so that steps of the form $\mathbf{m} + \alpha\mathbf{q}$ are taken. The value of α can be adjusted experimentally to achieve the best improvement at each step.