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OPTIMAL DYNAMIC PRICING FOR EXPANDING NETWORKS

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This paper analyzes the dynamic pricing decision of a monopolist marketing a new product or service whose consumption value increases with the expansion of the "network" of adopters. We characterize an optimal pricing strategy which maximizes the present value of the monopolist's profits, subject to the dynamics of the demand for network access. These dynamics depend, among other factors, on the current price and consumer anticipations about future network growth. We examine the effects of changes in the growth anticipations and the discount rate on the optimal equilibrium access price and network size. It is shown that higher growth anticipations and a lower discount rate result in a lower equilibrium price and a larger network.

1. Introduction

This paper analyzes a monopolist's intertemporal pricing decision for a new product or service whose consumption value increases with the expansion of the "network" of adopters. We refer to this demand interdependence as a *network externality*. Telecommunications networks offering services such as electronic mail, for instance, typically display this feature. In fact, the terminology used to describe the model in this paper is motivated by the telecommunications paradigm. Other examples where network externalities exist are: markets for product franchises; the leasing or renting out of lots in office blocks, residential communities and industrial parks; memberships in clubs and associations.[1,2]

In our model, the consumption of the product is restricted to those individuals who, through the act of product adoption, have become members of a network. We refer to the network as a *subscriber set* and its members as *subscribers*. Each subscriber consumes one unit of the product (e.g., a terminal), for which he pays a periodic rental fee or a *subscription price*. This price is fixed by a monopolist and may change over time. Consumers in our model differ in their *willingness-to-pay* for subscription. This heterogeneity

[1] In some instances, it may be argued that a preference for "exclusivity" may, in fact, result in decreasing benefits as the number of adopters increases. We neglect such "congestion" effects.

[2] The market for personal computers also displays somewhat similar characteristics. Here, an expanding base of owners of a specific hardware configuration provides increased incentives for the development of system-dedicated software, which, in turn, results in higher benefits for the adopters of the hardware (or its compatible "clones"). We thank one of the referees for suggesting this parallel.

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provides a basis for the ordering of the population and for the assignment of an index to each customer. In the presence of network externalities, each individual's willingness-to-pay is also a function of the size of the subscriber set. A consumer's product adoption decision depends on how his willingness-to-pay compares with the subscription price. In other words, consumer index, subscriber set size (in a dynamic setting, perhaps a forecast of the future size), and subscription price are the determinants of adoption demand. If this demand exceeds the size of the current subscriber set, then the network would grow over time; an equilibrium being attained only when the two are equal. In this paper, we characterize the optimal pricing strategy of a monopolist who is aware of the network's growth dynamics.

Our emphasis on the dynamic pricing decision constitutes an extension of the existing literature on the pricing of telecommunications services (e.g., Artle and Averous 1973, Oren and Smith 1981, Rabenau and Stahl 1974, and Rohlfs 1974). In these studies, it is generally recognized that the presence of network externalities poses a peculiar start-up problem in the case of new products or services: a "critical mass" of subscribers is needed for sustaining (nonzero) equilibrium subscription prices. One possible solution (e.g., Rohlfs 1974) is to have an initial price low enough to encourage new subscriptions. The monopolist can then raise the price gradually to the optimal steady-state level, trading off lower immediate revenues with the earlier achievement of long-run revenue levels. In fact, Rabenau and Stahl (1974) do formulate this as an optimal control problem; however, they do not actually solve it. In this paper, we characterize the optimal dynamic subscription price policy.

Not surprisingly, this pricing strategy depends on the dynamics of network growth. In this paper, we adopt a generic representation for these dynamics which is consistent with the underlying economic forces. Our modelling effort and solution methodology are based on similar models in the literature on the diffusion of innovations and dynamic pricing in growing markets. Examples are: Bass (1969), (1980), Bass and Bultez (1982), Dolan and Jeuland (1981), Feichtinger (1982), Jeuland (1981), Jeuland and Dolan (1982), Kalish (1983), (1984), and Robinson and Lakhani (1975).

We characterize the subscription demand and the dynamics of network growth in the following section. The monopolist's problem is then formulated as an optimal control problem and solved in §3, where an optimal dynamic price is obtained in a feedback form—as a function of the size of the existing subscriber set. It is shown that network growth dynamics affect both the price trajectory and the eventual steady-state equilibrium. We also discuss the importance of network growth anticipations. In §4, we apply our results to the "uniform calling" model which is frequently analyzed in the literature on telecommunications network pricing. Finally, §5 contains some concluding comments.

2. Subscription Demand and the Dynamics of Network Growth

2.1. *The Willingness-to-Pay Function*

Following Oren and Smith (1981), we consider an infinitely large, stationary population of heterogeneous consumers. These consumers have a positive income which is constant over time. Income effects are neglected and, in a partial equilibrium framework, the analysis focuses on the new product (or service) which exhibits network externalities. Consumer preferences for this product are represented by an instantaneous utility or willingness-to-pay function $W(\eta, X)$, which is individual η's reservation price for subscription, given a subscriber set X.

The consumer index η may reflect some consumer attribute (such as income). By construction, it establishes a rank ordering of the consumers according to their willingness-to-pay for subscription. Without loss of generality, it can be assumed that η is uniformly and continuously distributed over the interval [0, 1] and that $W(\eta, X)$ is monotonically

decreasing in η.[3] Formally, we assume that the consumer ranking is invariant with respect to changes in X and that $W(\eta, X)$ is strictly decreasing and continuously differentiable in η for any $X \subseteq [0, 1]$, $X \neq \varnothing$.[4] Following Faulhaber and Panzar (1977) and Mirman and Sibley (1980), we assume that the "smallest" consumer (with index $\eta = 1$) derives zero benefit from subscription, regardless of X.

Given a subscriber set X and a subscription price p, the monotonicity of $W(\eta, X)$ in η implies that the consumer surplus $[W(\eta, X) - p]$ is also monotonically decreasing in η. Furthermore, the monopolist's marketing efforts may be directed so as to "exploit" consumer heterogeneity, that is, he may concentrate on nonsubscribers with a higher willingness-to-pay before on those with a lower willingness-to-pay. In this context, we assume that nonsubscribers always join the network in sequence of increasing η.

It follows that the subscriber set X must be an interval $[0, x]$, where $x \leq 1$ is the index of the marginal subscriber. With this observation, we revise the notation for the willingness-to-pay function from $W(\eta, X)$ to $W(\eta, x)$.

The network externality effect is captured by the dependence of each consumer's willingness-to-pay on x (or X). In particular, we assume that $W(\eta, x)$ is continuously differentiable and monotonically increasing in x. The product has no value if there are no subscribers, that is, $W(\eta, 0) = 0$. Congestion effects are neglected.

The function $W(\eta, x)$, $\eta, x \in [0, 1]$, represents a surface whose cross-sectional curve $W(x, x)$ describes the marginal subscriber's willingness-to-pay as the subscriber set X increases from $\varnothing$ to $[0, 1]$. In order to facilitate our analysis, we restrict the discussion to the case where $W(x, x)$ and other linear cross-sections of $W(\eta, x)$ are unimodal.[5]

2.2. *Subscription Demand*

Given a subscription price p, and assuming that there are no costs associated with either the initiation or the renunciation of subscription,[6] a consumer η will subscribe if and only if

$$W(\eta, x) \geq p. \tag{2.1}$$

Thus, the three determinants of a consumer's subscription decision are his index, the subscriber set size, and the subscription price. Of these, consumer index, by assumption, is time-invariant; but, in a dynamic setting, subscriber set size and subscription price may change over time. Since the consumer can initiate or relinquish his subscription costlessly, only the current subscription price is relevant to his subscription decision. In particular, consumer expectations about future prices play no role in our model. A similar assertion cannot, however, be made about subscriber set size. As we shall now show, consumers may form expectations about future adopter population size. If so, these expectations should be reflected in the subscription demand.

One possible description of consumer decision making is that consumers are completely myopic (with respect to subscriber set size) and, therefore, base their decisions on the size of the existing network. In that case, the set of customers willing to subscribe is

$$Y(x, p) = \{\eta: 0 \leq \eta \leq 1,\ W(\eta, x) \geq p\}. \tag{2.2}$$

[3] This can be achieved by defining η to be the fractile of the consumer attribute chosen and transforming $W(\eta, X)$ accordingly. If, for example, we consider income to be the relevant attribute, then $\eta = 0.1$ corresponds to an income level which is reached (or exceeded) by 10% of the consumer population.

[4] The assumption that the rank ordering of $W(\eta, X)$ is preserved for any partial network $X \subseteq [0, 1]$ is, admittedly, rather strong. As pointed out by Rohlfs (1974), individual consumers or groups of consumers might exist who will benefit disproportionately from the subscription of some other consumer(s). The inclusion of such a consumer in the network may then alter the willingness-to-pay ranking of the current subscribers.

[5] Equivalently, we require that $W(\eta, x)$ is strictly quasiconcave in η and x. This guarantees that the level sets $\{\eta, x: W(\eta, x) - p \geq 0\}$ are convex and properly nested.

[6] This assumption is plausible in cases where the "hook-up" charge is small relative to, say, the monthly dues. This is true, for example, for many telephone subscribers in the United States.

The monotonicity of $W(\eta, x)$ and the assumption that $W(1, x) = 0$ imply that $Y(x, p)$ is an interval $[0, y(x, p)]$, where $y(x, p)$ is the unique root of $W(y, x) = p$.

$y(x, p)$ represents the proportion of the total population willing to join the network $[0, x]$. We note that the network will (1) shrink if $y(x, p) < x$ (i.e., $W(x, x) < p$); (2) be in equilibrium if $y(x, p) = x$ ($W(x, x) = p$); and (3) expand if $y(x, p) > x$ ($W(x, x) > p$). Given our focus on growing markets, we shall restrict the choice of subscription price so as to ensure that $W(x, x) \geq p$.

Figure 1 illustrates the locus of (x, p) pairs which satisfy the equilibrium condition $y(x, p) = x$ for a unimodal $W(x, x)$. We observe that the "demand curve" is bell-shaped and that the market for the product has multiple equilibria. This is characteristic of the network externality aspect. For a unimodal $W(x, x)$, there exists a stable equilibrium on the downward-sloping segment of the curve and an unstable equilibrium along the upward-sloping segment.[7] An immediate implication of this is that if the network is to reach some desired stable steady-state equilibrium, then it must grow beyond a "critical mass" (see, for example, Rohlfs 1974 and Oren and Smith 1981). An optimal dynamic pricing strategy must solve this start-up problem. Earlier we saw that spontaneous network growth will occur only when $y(x, p) > x$, that is, when $W(x, x) > p$. This condition defines the feasible region for start-up pricing policies. We shall refer to these policies as *interior pricing*, to distinguish them from *boundary pricing* (where $W(x, x) = p$).

The definition of $Y(x, p)$ in (2.2) implicitly assumed that consumers base their subscription decisions on the size of the current subscriber set, as if it will remain unchanged. Clearly, any network expansion will controvert this assumption. Hence, our characterization of subscription demand must be adapted to allow for the effect of active expectations or anticipations about network growth.

If each consumer has perfect knowledge about the nature of consumer ordering and his own index, then he would correctly conclude that if he finds it beneficial to subscribe, then so also would all consumers who have a lower index. Therefore, a consumer η will base his subscription decision on a network extending over (at least) $[0, \eta]$, and the set of nonsubscribers willing to join the network $[0, x]$ will be given by:

$$Z(x, p) = \{\eta: x \leq \eta \leq 1,\ W(\eta, \eta) \geq p\}. \tag{2.3}$$

For $W(x, x) \geq p$, the unimodality of $W(\eta, x)$ implies that $Z(x, p)$ is an interval $[x, z(p)]$, where $z(p)$ is the "smallest" consumer willing to subscribe at price p. Thus, the total subscription demand is represented by the interval $[0, x] \cup Z(x, p) = [0, z(p)]$. Since $W(x, x) \geq p$ and $W(1, x) = 0$, $z(p)$ must be the largest root of the equation $W(z, z) = p$ and, therefore, the eventual network $[0, z(p)]$ will be in stable equilibrium at subscription price p. The two subscription demands $y(x, p)$ and $z(p)$ are equal if and only if the network is already at the stable equilibrium, that is,

$$y(z(p), p) = z(p).$$

In reality, consumers may not be completely certain about their own index and the exact nature of consumer ordering. Such uncertainty would be perfectly justified for a new product or service. In that case, network growth will only be partially anticipated. To take such partial growth expectations into account, we introduce a *growth anticipation* parameter α, $0 \leq \alpha \leq 1$. Specifically, we assume that a nonsubscriber η bases his subscription decision on an anticipated subscriber set $[0, \alpha\eta + (1 - \alpha)x]$. Then, the set of nonsubscribers wishing to subscribe will be

$$D^{\alpha}(x, p) = \{\eta: x \leq \eta \leq 1,\ W(\eta, [\alpha\eta + (1 - \alpha)x]) \geq p\}. \tag{2.5}$$

[7] In addition, the null set $X = \varnothing$ (i.e., $x = 0$) is obviously an equilibrium for $p > 0$; but except in trivial cases, this will not be a desirable stable equilibrium.

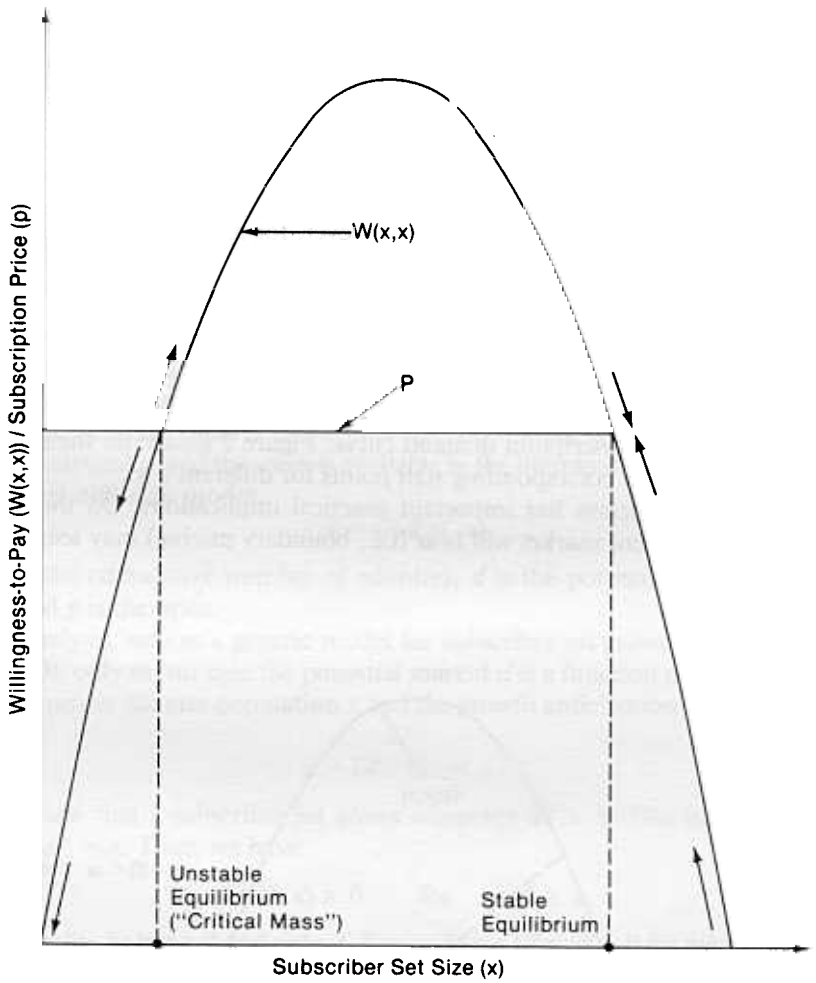

FIGURE 1. Demand Curve for the Uniform Calling Model. (Source: Rohlfs 1974)

The parameter α may be regarded as an "awareness" parameter and is assumed to be uniform across the entire population. α can be influenced through product advertising and promotional campaigns. For $W(x, x) \geq p$, the unimodality of $W(\eta, x)$ implies that $D^\alpha(x, p) = [x, d^\alpha(x, p)]$, where $d^\alpha(x, p)$ is the index of the smallest nonsubscriber willing to join the network. The total subscription demand is, therefore, $d^\alpha(x, p)$, where

$$d^\alpha(x, p) = \max \{\eta: 0 \leq \eta \leq 1, W(\eta, [\alpha\eta + (1 - \alpha)x])\} = p. \tag{2.6}$$

The demand functions $y(x, p)$ and $z(p)$ are special cases of $d^\alpha(x, p)$ for $\alpha = 0$ and $\alpha = 1$, respectively, with

$$y(x, p) \leq d^\alpha(x, p) \leq z(p). \tag{2.7}$$

Spontaneous network growth will occur only when the subscription demand exceeds the existing network size, that is, $d^\alpha(x, p) > x$; the expansion will stop when $d^\alpha(x, p) = x$. For the case of $\alpha = 0$, we showed that spontaneous growth to a stable equilibrium can be sustained only with interior pricing, that is, $W(x, x) > p$. In the presence of some growth anticipations (i.e., $\alpha > 0$), growth may occur even with boundary pricing, that is, $W(x, x) = p$, but only up to a point. In Proposition 1 in the Appendix, we show that even if $W(x, x) = p$, $d^\alpha(x, p)$ can be greater than x, but only if (x, p) lies along the upward-

sloping side of $W(x, x)$, and provided that the growth anticipation parameter exceeds some critical value $0 \leq \alpha_{cr}(x) \leq 1$, where

$$\alpha_{cr}(x) \triangleq -\left.\frac{\partial W(\eta, x)/\partial \eta}{\partial W(\eta, x)/\partial x}\right|_{\eta=x}. \tag{2.8}$$

$\alpha_{cr} = 1$ at the maximum of $W(x, x)$ and $\alpha_{cr}(0) = 0$.

In general, $\alpha_{cr}(x)$ may not be invertible. However, for any fixed $0 \leq \alpha \leq 1$, we can define the inverse function

$$x_{cr}(\alpha) \triangleq \min_x \{x\colon \alpha_{cr}(x) \geq \alpha\}. \tag{2.9}$$

We refer to $x_{cr}(\alpha)$ as the *stall point* since, for $W(x, x) = p$, network growth cannot occur beyond $x = x_{cr}(\alpha)$, that is, the network "stalls." The stall point always lies on the upward-sloping portion of the subscription demand curve. Figure 2 illustrates the subscription demand $d^{\alpha}(x, p)$ and the corresponding stall points for different α's.

The stalling phenomenon has important practical implications. On the one hand, charging what the current market will bear (i.e., boundary pricing) may seem attractive

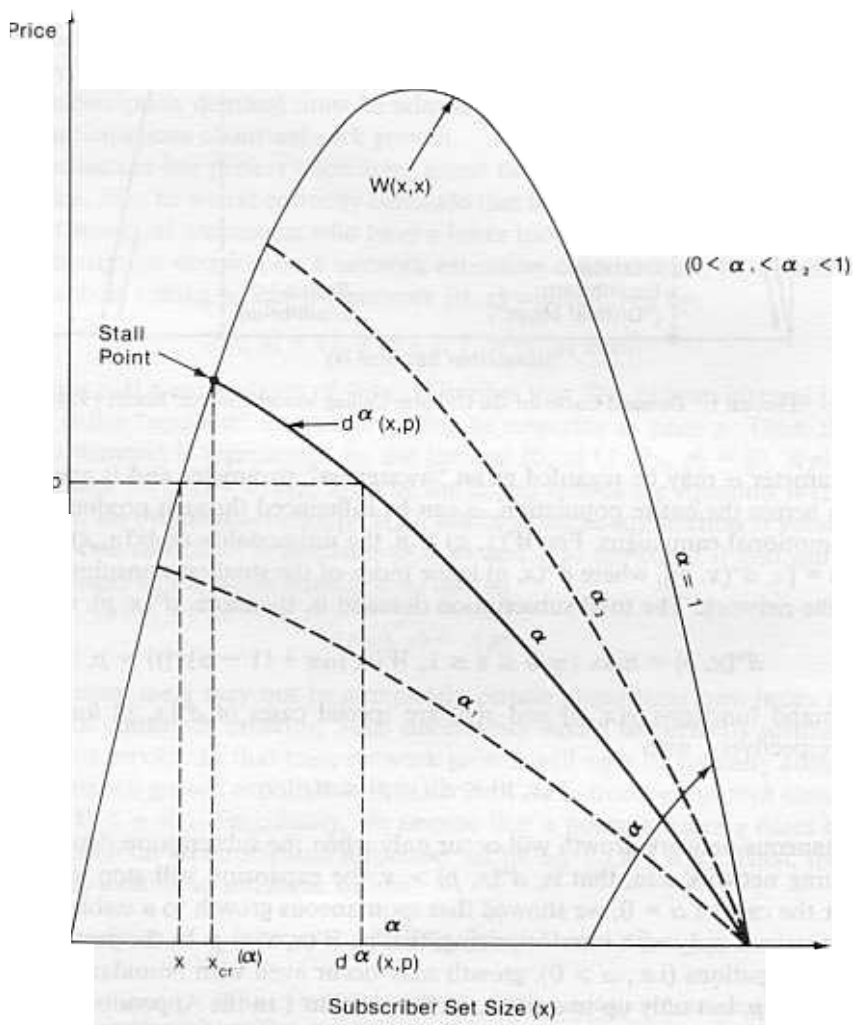

FIGURE 2. Subscription Demand and the "Stalling" of Network Growth.

as a start-up strategy since (at least in the short term) it is the most profitable. On the other hand, the above analysis suggests that the adoption of such a pricing policy in the early stages will result in the network stalling prematurely at an unstable equilibrium. This would be undesirable. Of course, the interval during which spontaneous growth occurs with boundary pricing can be prolonged by raising the consumers' growth anticipations, possibly through advertising and promotional campaigns. Nevertheless, in order to reach a stable equilibrium, the subscription price must drop below $W(x, x)$. In other words, the pricing policy must at some stage switch over to an interior pricing phase.

2.3. *The Dynamics of Network Growth*

Reference to the empirical results and the models available in the marketing literature on the diffusion of innovations and the growth of new markets suggests that the network expansion in our example will occur gradually as a dynamic adjustment process which closes the gap between x and d^α. Attempts have been made to describe such adjustment processes analytically, and the models available in the literature are typically special cases of the general diffusion model:

$$\dot{x} = G(d(p), x), \tag{2.10}$$

where x is the cumulative number of adopters, d is the potential market for the new product, and p is the price.

In our analysis, we use a generic model for subscriber set growth which is similar to that in (2.10); only in our case the potential market d is a function not only of price, but also of the current adopter population x and the growth anticipation parameter α. Thus, we have

$$\dot{x} = G(d^\alpha(x, p), x). \tag{2.11}$$

Earlier, we saw that a subscriber set grows whenever $d^\alpha > x$. This network expansion stops when $d^\alpha = x$. Thus, we have

$$G(d^\alpha, x) \geq 0 \qquad \text{for} \qquad d^\alpha \geq x, \tag{2.12}$$

with the equality holding if and only if $d^\alpha = x$. Since $G(x, x) = 0$ for all $x \geq 0$, it follows that

$$G_d(d^\alpha, x) = -G_x(d^\alpha, x) \qquad \text{for all} \qquad d^\alpha = x \geq 0, \tag{2.13}$$

where $G_d(\cdot) \triangleq \partial G/\partial d^\alpha$ and $G_x(\cdot) \triangleq \partial G/\partial x$. Finally, given some current adopter population x, it would be reasonable to expect that the greater the subscription demand d^α, the greater would be the rate of network expansion, that is,

$$G_d(d^\alpha, x) > 0. \tag{2.14}$$

3. Optimal Dynamic Pricing Policy

Let the initial network be $[0, x_0]$, where x_0 is such that $W_x(x_0, x_0) > 0$. Then, at time $t = 0$, there is potential for subscriber set growth (to a stable steady-state equilibrium). The monopolist's optimization problem is:

$$\text{Max} \int_0^\infty e^{-\delta t}[px - c(x)]dt, \tag{3.1}$$

subject to the constraints:

$$\dot{x} = G(d^\alpha, x), \qquad x(0) = x_0, \tag{3.2}$$

$$0 \leq p \leq W(x, x). \tag{3.3}$$

In the above problem, $c(x)$ is the monopolist's cost function and δ is his discount rate.

In general, a feasible price trajectory $p(t)$ for the above problem may consist of one or more of the following segments:

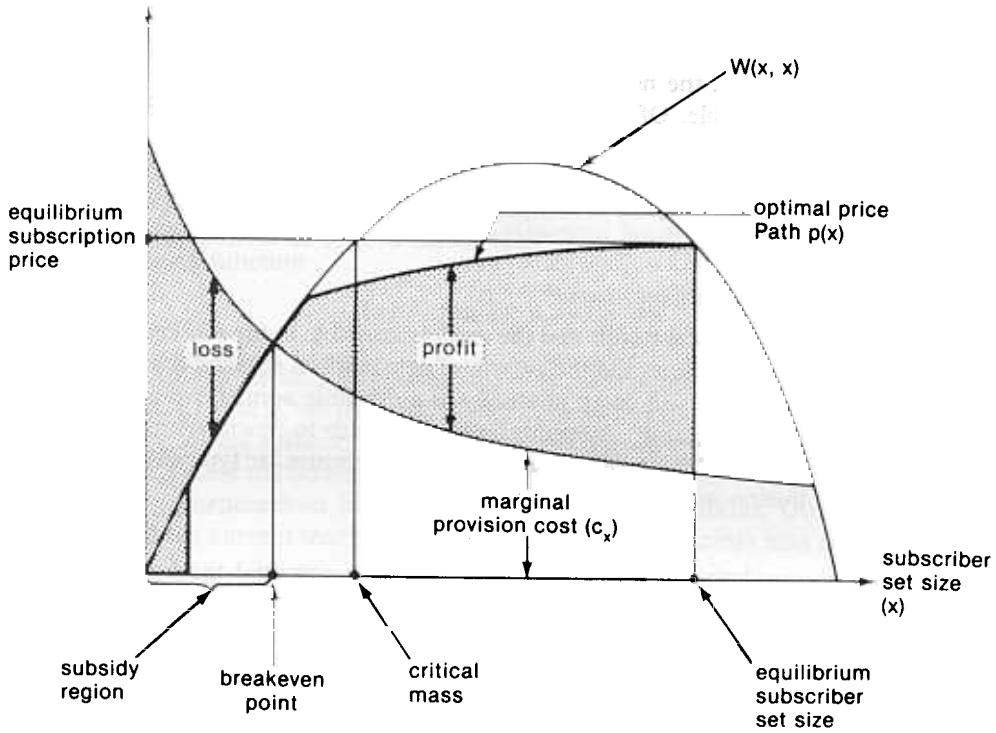

FIGURE 3. Subscription Price Evolution.

$$\text{(i)} \quad 0 < p < W(x, x), \tag{3.4}$$

$$\text{(ii)} \quad p = 0, \quad \text{and} \tag{3.5}$$

$$\text{(iii)} \quad p = W(x, x). \tag{3.6}$$

A typical pricing policy consisting of all three price regimes is illustrated in Figure 3, along with a declining marginal cost function $c_x(x)$.[8] Such a policy generates an initial subsidy region, with the subsidy terminating when the subscription price and marginal cost functions intersect.

Considering the above three feasible price regimes, we observe that, in the last of these, the monopolist continuously changes the subscription price so as to "coast" along the $W(x, x)$ curve. This constitutes a boundary pricing strategy and will be appropriate only if the current equilibrium is unstable. Since the steady-state equilibrium must be stable, the final phase of network growth cannot involve boundary pricing.[9] In fact, the optimal steady-state equilibrium can only be approached through trajectory segments defined by (3.4) or (3.5).

Of these, the former (one with $p = 0$) will involve a "jump" in the subscription price—from zero to the optimal steady-state value. Such a "bang-bang" policy will be optimal only if the function $G(\cdot)$ is linear in p. This case requires special treatment and a detailed solution is presented in Dhebar and Oren (1982). There, we show that the monopolist's optimal pricing policy is to charge a zero price until a subscriber set size x^b is reached, and to charge

[8] While the price path depicted in this figure is monotonically increasing, it is possible that it may be decreasing in some other example. This depends on δ and the functions $c(\cdot)$, $W(\cdot)$, and $G(\cdot)$.

[9] An exception to this general principle would be if $\alpha = 1$ and the optimal steady-state equilibrium exactly equals $\arg\max_{x \in [0,1]} W(x, x)$. This, however, is not a desirable situation because this equilibrium is unstable for any perturbations resulting from even an infinitesimal shrinkage of the subscriber set.

$$p^b = W(x^b, x^b) \tag{3.7}$$

thereafter. The pair (x^b, p^b) satisfies (3.7) and the condition

$$p^b\left[1 + \frac{x^b}{p^b}\left(\frac{dp^b}{dx^b}\right)\right] - c_x(x^b) - \frac{\delta x^b p^b}{G(1, x^b)} = 0. \tag{3.8}$$

If $G(\cdot)$ is strictly concave in p, then the steady-state equilibrium can only be reached through an interior solution. The subsequent discussion will focus on this case. The occurrence of switchovers between the three different regimes is illustrated through a numerical example in the next section.

To solve for the optimal interior price trajectory $(0 < p < W(x, x))$, we apply standard control theory (e.g., Kamien and Schwartz 1981) and define the Hamiltonian

$$H(x, p, \lambda) = e^{-\delta t}[px - c(x) + \lambda G(d^\alpha, x)], \tag{3.9}$$

where $e^{-\delta t}\lambda$ is the costate variable. Then for $G(\cdot)$ strictly concave in p, $H(x, p, \lambda)$ is also strictly concave in p, and the optimal trajectories for x, p and λ must satisfy differential equations (3.2) and

$$-\dot{\lambda} = e^{\delta t}H_x - \delta\lambda = p - c_x + \lambda[G_d d_x^\alpha + G_x - \delta], \tag{3.10}$$

and the first-order necessary condition

$$H_p = e^{-\delta t}[x + \lambda G_d d_p^\alpha] = 0, \tag{3.11}$$

where $d_x^\alpha \triangleq \partial d^\alpha/\partial x$ and $d_p^\alpha \triangleq \partial d^\alpha/\partial p$. Equations (3.2), (3.10) and (3.11) fully specify the interior solution. Differentiating (3.11) with respect to t and substituting for $\dot{x}$, $\dot{\lambda}$ and λ, we obtain an equation for $\dot{p}$ in terms of x and p. Since $p < W(x, x)$, $d^\alpha > x$, that is, $\dot{x} > 0$. Dividing $\dot{p}$ by $\dot{x}(=G(\cdot))$, we obtain the differential equation

$$\frac{dp}{dx} = \frac{d_p^\alpha}{x[G_{dd}(d_p^\alpha)^2 + G_d d_{pp}^\alpha]}\left[G_d\left(1 - x\frac{d_{px}^\alpha}{d_p^\alpha}\right) - x(G_{dd}d_x^\alpha + G_{dx}) - \frac{(p - c_x)G_d^2 d_p^\alpha - xG_d(G_d d_x^\alpha + G_x - \delta)}{G}\right]. \tag{3.12}$$

We note that the right-hand side of equation (3.12) involves only the subscription price p and market penetration level x. Time does not appear explicitly. Since constraint (3.3) on p also does not involve time explicitly, the optimal pricing policy can, therefore, be fully represented as a function of the market penetration x. In other words, the solution of the monopolist's pricing problem can be specified as a "feedback" policy.

This feedback policy is optimal, that is, in theory, it is superior to all other pricing strategies which induce network growth to a stable steady-state equilibrium: the feedback policy would result in the highest possible discounted profits for the monopolist. Obtaining a quantitative assessment of the relative improvement in the discounted profits is, however, very difficult because it is really during the transient stage that the feedback policy enjoys a comparative advantage, and we do not have an analytical, closed-form characterization of the pricing policy in this stage. Even for the simple example that we pursue in the next section, such a comparison would require complicated computations.

Equation (3.12) defines the subscription price path to the optimal stable steady-state equilibrium (x^*, p^*), where

$$W(x^*, x^*) = p^*. \tag{3.13}$$

Constraint (3.3) is, therefore, binding and the first-order necessary condition (3.11) for optimal pricing changes to

$$H_p = e^{\delta t}[x^* + \lambda^* G_d(d^{\alpha^*}, x^*)d_p^{\alpha^*}] \geq 0, \tag{3.14}$$

where $d^{\alpha^*} = d^\alpha(x^*, p^*) = x^*$ and λ^* is the equilibrium value of the costate variable. In Proposition 2 in the Appendix, we show that if $G(\cdot)$ is strictly concave in p, then the equality will always hold in (3.14). Furthermore, in the equilibrium, $\dot{x} = \dot{p} = \dot{\lambda} = 0$. Thus, setting (3.10) equal to zero and eliminating λ using (3.14), we obtain the condition

$$p^* - c_x^* - \frac{x^*}{G_d(x^*, x^*)d_p^{\alpha^*}}[G_d(x^*, x^*)d_x^{\alpha^*} + G_x(x^*, x^*) - \delta] = 0. \tag{3.15}$$

Using (2.13), we can restate (3.15) as

$$u(x^*) + \delta v^\alpha(x^*) = 0, \qquad \text{where} \tag{3.16}$$

$$u(x^*) = p^*\left[1 + \frac{x^*(1 - d_x^{\alpha^*})}{p^* d_p^{\alpha^*}}\right] - c_x^* \qquad \text{and} \tag{3.17}$$

$$v^\alpha(x^*) = \frac{x^*}{G_d(x^*, x^*)d_p^{\alpha^*}}. \tag{3.18}$$

Equations (3.13) and (3.16) characterize the stable steady-state equilibrium corresponding to the optimal price path in (3.12). We observe that $u(x^*) = 0$ is of the form $[1 - 1/\epsilon(p^*)] = c_x^*/p^*$, where $\epsilon(p^*) = -[p^* d_p^{\alpha^*}/x^*(1 - d_x^{\alpha^*})]$ is the price elasticity of the (stable) equilibrium subscription demand. Thus, $u(x^*) = 0$ defines the static optimum. In contrast, the function $v^\alpha(x^*)$ captures the effects of the dynamics of subscriber set growth.

Given our assumption that the function $G(\cdot)$ is strictly concave in p, there is a unique subscriber set size, subscription price pair $x^* = x(\delta, \alpha)$, $p^* = p(\delta, \alpha)$ satisfying equations (3.13) and (3.16) for every discount rate δ. $x(\delta, \alpha)$ and $p(\delta, \alpha)$ are thus monotone in δ. In Proposition 3 in the Appendix, we establish that, for $x(\delta, \alpha) \neq 0$ and $\delta \neq 0$, $x(\delta, \alpha)$ decreases with δ and increases with α:

$$\frac{\partial x(\delta, \alpha)}{\partial \delta} < 0 \qquad \text{and} \tag{3.19}$$

$$\frac{\partial x(\delta, \alpha)}{\partial \alpha} > 0. \tag{3.20}$$

Since the pair $(x(\delta, \alpha), p(\delta, \alpha))$ constitutes a stable equilibrium, it must be on the downward-sloping segment of $W(x, x)$. It follows that $dp(\delta, \alpha)/dx(\delta, \alpha) < 0$ and that $p(\delta, \alpha)$ satisfies

$$\frac{\partial p(\delta, \alpha)}{\partial \delta} > 0 \qquad \text{and} \tag{3.21}$$

$$\frac{\partial p(\delta, \alpha)}{\partial \alpha} < 0. \tag{3.22}$$

From the above analysis, we conclude that for a given discount rate $\delta > 0$, the full growth anticipation case ($\alpha = 1$) gives the largest equilibrium subscriber set and the lowest subscription price. Cases corresponding to $0 \leq \alpha < 1$ give intermediate cases. Furthermore, the discount rate is isomorphic with consumer anticipation about network growth: lesser discounting and higher growth anticipations both result in greater market penetration and lower steady-state prices. Therefore, an improvement in growth anticipations (through, say, advertising, product promotion, and other means of disseminating information about the product) can help offset the effects of high discounting.

4. An Example: The Uniform Calling Model

We shall illustrate our results by solving for the monopolist's optimal dynamic pricing policy in the case of the "uniform calling" model. This model was introduced by Artle

and Averous (1973) in the context of the telephone system. It was further explored by Rohlfs (1974) in his study of interdependent demand in communications services.

In this model, consumer η's willingness-to-pay $W(\eta, x)$ is given by

$$W(\eta, x) = w_0(1 - \eta)x, \tag{4.1}$$

where w_0 is a constant of proportionality. Thus, the consumer's reservation prices decrease linearly with consumer index and increase linearly with the number of adopters. The willingness-to-pay of the marginal subscriber, $W(x, x)$, is given by the quadratic function

$$W(x, x) = w_0 x(1 - x). \tag{4.2}$$

We assume that the network's growth rate is linear in the unsatisfied subscription demand:

$$\dot{x} = b[d^\alpha(x, p) - x]. \tag{4.3}$$

The subscription demand $d^\alpha(x, p)$ can be obtained by using (2.6). For $W(\eta, x)$ as defined in (4.1), we have

$$d^\alpha(x, p) = 1 - [p/(w_0 x)] \qquad \text{for} \qquad \alpha = 0,$$

$$= \frac{[1 - (1 - \alpha)x/\alpha] + \{[1 - (1 - \alpha)x/\alpha]^2 - 4[(p/w_0) - (1 - \alpha)x]/\alpha\}^{1/2}}{2}$$

$$\text{for} \qquad 0 < \alpha \leq 1. \tag{4.4}$$

The product under consideration is new and at $t = 0$ we have $x_0 = 0$. Finally, we assume that the monopolist's cost function is linear in the number of adopters:

$$c(x) = c_0 x, \quad \text{where} \quad c_0 \leq w_0/4. \tag{4.5}$$

From (4.3) and (4.4), we note that if network growth anticipations are totally absent (that is, $\alpha = 0$), then the Hamiltonian $H(x, p, \lambda)$ (which was defined in (3.9)) will be linear in p. The optimal policy in that case will be bang-bang pricing. Here, the monopolist will charge a zero price until the network reaches a size $x^b = x(\delta, 0)$, and then $p^b = W(x^b, x^b)$ thereafter. From (3.7) and (3.8), we have

$$x^b = x(\delta, 0) = \frac{1 + [1 - \gamma(3 + \beta)]^{1/2}}{3 + \beta}, \tag{4.6}$$

where $\beta = (\delta/b)$ and $\gamma = (c_0/w_0)$ are parameter ratios measuring, respectively, the monopolist's discount rate relative to the network's growth rate and the marginal supply cost relative to the numeraire w_0.

For $0 < \alpha \leq 1$, $H(x, p, \lambda)$ is strictly concave in the subscription price p and the results of §3 apply. Here, we shall illustrate these results for the case $\alpha = 1$, that is, when network growth anticipations are "perfect". For $\alpha = 1$, and for $W(\cdot)$, $G(\cdot)$ and $c(x)$ as defined in (4.1), (4.3) and (4.5), equation (3.12) specializes to

$$\frac{d\hat{p}}{dx} = \left(\frac{1 - 4\hat{p}}{2x}\right) 1 + \left\{2\left[\frac{\hat{p} - \gamma}{(1 - 4\hat{p})^{1/2}} - x(1 + \beta)\right] \Big/ [1 - 2x + (1 - 4\hat{p})^{1/2}]\right\}, \tag{4.7}$$

where $\hat{p} = (p/w_0)$ is the "normalized" subscription price.

Equation (4.7) characterizes the optimal interior subscription price path to the stable steady-state equilibrium. We obtain the corresponding equilibrium subscriber set size, subscription price pair $(x(\delta, 1), p(\delta, 1))$ from (3.13) and (3.15):

$$x(\delta, 1) = \frac{(2 + \beta) + [(2 + \beta)^2 - 4\gamma(3 + 2\beta)]^{1/2}}{2(3 + 2\beta)} \qquad \text{and} \tag{4.8}$$

$$p(\delta, 1) = w_0 x(\delta, 1)[1 - x(\delta, 1)]. \tag{4.9}$$

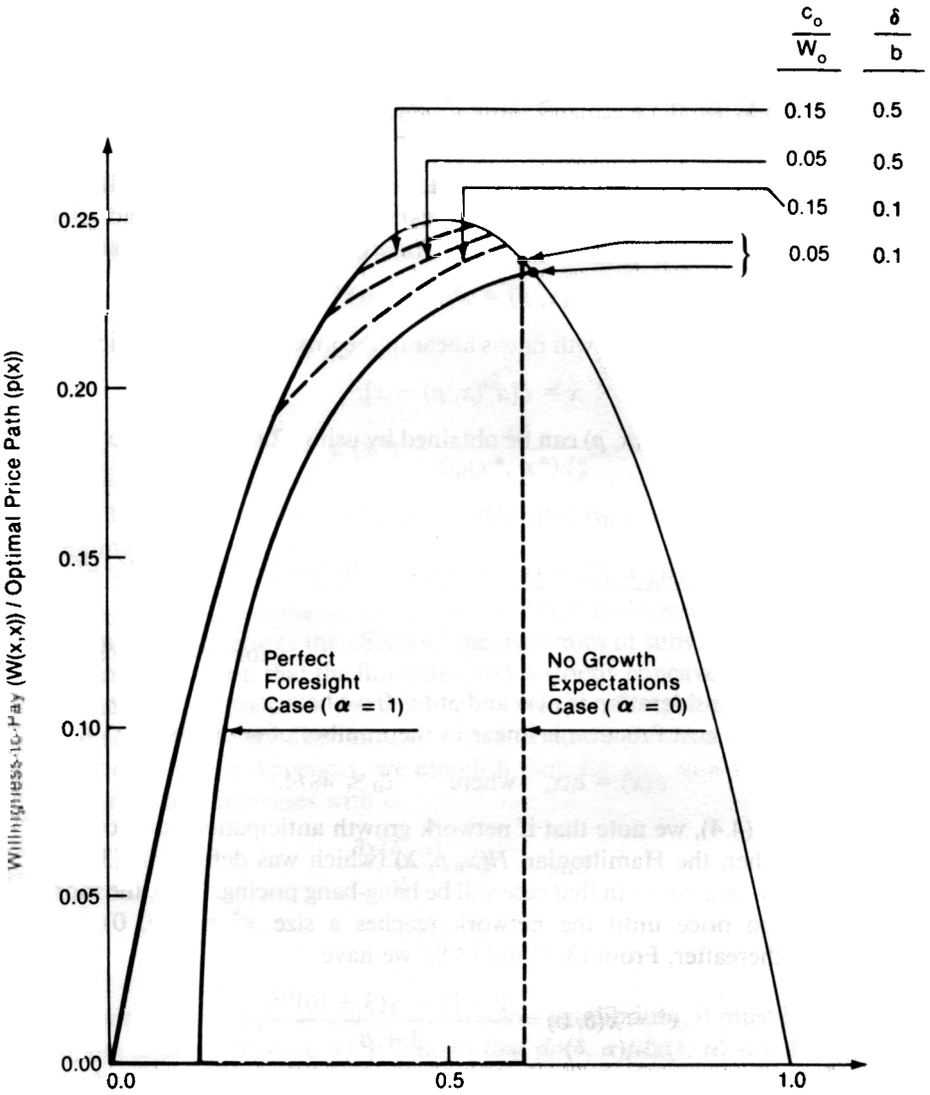

FIGURE 4. Optimal Price Paths for the Uniform Calling Model, $\alpha = 0$ or 1.

The complete optimal price path can be obtained by starting at $(x(\delta, 1), p(\delta, 1))$ and then integrating (4.7) backward in x until the resulting subscription price function $p(x)$ hits the constraint $p = W(x, x)$ or $p = 0$, whichever is earlier. From there on, $p(x)$ will coast along the active constraints all the way down to $x = 0$. Figure 4 illustrates typical price paths obtained by numerically integrating (4.7) for several values of the ratios γ and β.[10] As would be expected, a higher γ and a higher β both result in smaller networks and higher prices in the equilibrium. In Figure 4, we also illustrate the optimal bang-bang solution (corresponding to the case $\alpha = 0$) for one set of parameter values.

While the monopolist's discount rate and the network growth rate affect the subscription price path $p(x)$ similarly, they have different effects on the trajector $p(t)$. This is because the market growth rate, unlike the discount rate, affects network build-up. Figure 5

[10] Starting the numerical integration of (4.7) poses a technical problem, since at $(x(\delta, 1), p(\delta, 1))$ the derivative dp/dx given by (4.7) is not defined (the denominator is zero). This requires special treatment of the initial step which is handled by a backward iteration (see Dhebar and Oren 1982 for details).

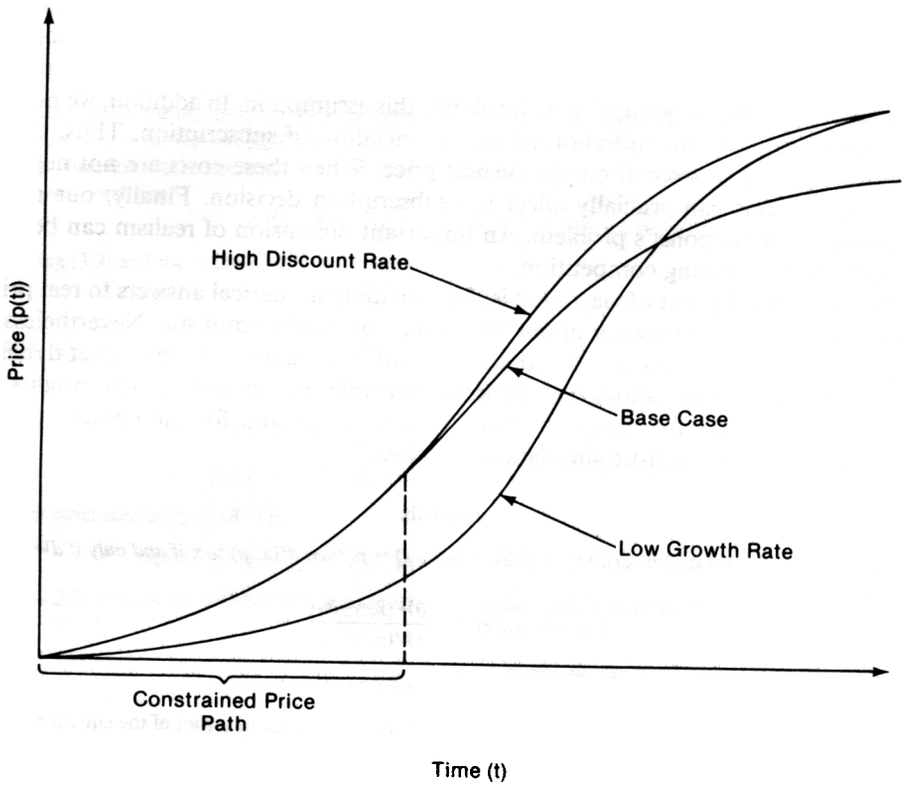

FIGURE 5. Effect of Network Growth Rate and Discount Rate δ on the Optimal Price Trajectory.

illustrates this distinction qualitatively. We note that when the higher price path is motivated by a larger discount rate, the result will be a price trajectory which is higher than the base case trajectory at all times. (The two trajectories will coincide in the region where their paths are constrained by the willingness-to-pay curve.) In contrast, a smaller growth rate will delay price increases, which are contingent on network size, and will initially produce a lower price trajectory relative to the base case. In this situation, the optimal pricing policy attempts to compensate for delays in revenue build-up by inducing a sharp rise in the price, thus overtaking the base case trajectory and approaching a higher asymptote.

5. Concluding Comments

In this paper, we analyzed the dynamic pricing problem of a monopolist marketing a new product or service which exhibits "network externalities." A number of researchers have examined this network access pricing decision in the context of communications services. However, their results concentrate on the optimal price and network size in a static setting. Here, we have extended the research to a dynamic framework. Our analysis parallels the approach adopted in the marketing literature, where dynamic pricing policies have been investigated, but not in the presence of the demand characteristics modeled in this paper.

An important component of our model was the characterization of the subscription demand. Here, we introduced the notion of consumer anticipations about future network growth. Our results suggest that a greater level of consumer awareness and information with regard to the network growth potential will result in larger equilibrium networks and lower prices.

There are several directions in which this analysis can be extended. In particular, we assumed that consumers are heterogeneous along only one dimension and that consumer ordering is invariant with respect to the size and nature of the subscriber set. In practice, the presence of "interest groups" may invalidate this assumption. In addition, we neglected costs associated with the initiation or the renunciation of subscription. Thus, the subscription decision was based on the current price. When these costs are not negligible, price expectations can crucially affect the subscription decision. Finally, our analysis addressed the monopolist's problem. An important dimension of realism can be added by explicitly considering competition.

From a practical point of view, that is, for providing numerical answers to real pricing problems, the strict framework of our model may be overly simplistic. Nevertheless, the insights gained from our analysis should be useful in situations where market dynamics and consumption externalities play an important role. For example, such insights may provide guidance for the design of market experiments and for the choice of model parameters for more realistic simulation exercises.[11]

Appendix

PROPOSITION 1. *For $d^\alpha(x, p)$ as defined in* (2.6), *if $W(x, x) = p$, then $d^\alpha(x, p) > x$ if and only if $dW(x, x)/dx > 0$ and*

$$1 \geq \alpha > \alpha_{cr}(x) = -\left.\frac{\partial W(\eta, x)/\partial \eta}{\partial W(\eta, x)/\partial x}\right|_{\eta=x} \tag{A.1.1}$$

Furthermore, $\alpha_{cr}(0) = 0$ and $\alpha_{cr}(x) = 1$ if and only if $\hat{x} = \arg\max W(x, x)$.

PROOF. Given our assumptions on $W(\eta, x)$, it follows that $d^\alpha(x, p)$ must be a root of the equation

$$W(\eta, [\alpha\eta + (1 - \eta)x]) = p. \tag{A.1.2}$$

This equation has a root at $\eta = x$. Since $W(\eta, x)$ is strictly quasiconcave, $W(\eta, [\alpha\eta + (1 - \alpha)x])$ is unimodal in η with only one possible stationary point. Thus, (A.1.2) can have a root $d^\alpha(x, p) > x$ if and only if

$$\left.\frac{dW(\eta, [\alpha\eta + (1 - \alpha)x])}{d\eta}\right|_{\eta=x} = \left.\left[\frac{\partial W(\eta, x)}{\partial \eta} + \alpha\frac{\partial W(\eta, x)}{\partial x}\right]\right|_{\eta=x} > 0. \tag{A.1.3}$$

The left-hand side in (A.1.3) will be equal to zero for $\alpha = \alpha_{cr}(x)$ and since $\partial W(\eta, x)/\partial x > 0$, it will be positive for all $\alpha > \alpha_{cr}$.

Furthermore, we have

$$\begin{aligned}\frac{dW(x, x)}{dx} &= \left.\left[\frac{\partial W(\eta, x)}{\partial \eta} + \frac{\partial W(\eta, x)}{\partial x}\right]\right|_{\eta=x} \\ &= \frac{\partial W(\eta, x)}{\partial x}[1 - \alpha_{cr(x)}]|_{\eta=x},\end{aligned} \tag{A.1.4}$$

and since $\partial W(\eta, x)/\partial x > 0$, it follows that $\alpha_{cr}(x) < 1$ if and only if $dW(x, x)/dx > 0$. Also, $\alpha_{cr}(\hat{x}) = 1$, if and only if $[dW(x, x)/dx|_{x=\hat{x}}] = 0$, that is, $\hat{x} = \arg\max W(x, x)$. Finally, since $W(\eta, 0) = 0$ for $0 \leq \eta \leq 1$, (A.1.1) implies that $\alpha_{cr}(0) = 0$. Q.E.D.

PROPOSITION 2. *Let (x^*, p^*) be the optimal steady-state market penetration level and subscription price pair satisfying $W(x^*, x^*) = p^*$ and assume $G(d^\alpha(x, p), x)$ to be strictly concave in p. Then, equality must hold in* (3.14).

PROOF. Since $W(x^*, x^*) = p^*$, constraint (3.3) is binding and, therefore, a strict inequality may hold in (3.14). On the other hand, since H_p is continuous in x, p, and λ, and since $H_p = 0$ for $0 < p(t) < W(x(t), x(t))$, the strict inequality in (3.14) can only hold at (x^*, p^*) if the steady-state price p^* is reached by a discontinuous jump at some finite time. We will show, however, that such a jump violates the optimality conditions.

Suppose that the optimal subscription price trajectory does jump to the steady-state p^* at some finite time $t = T$, that is, it satisfies $p(t) < W(x(t), x(t))$ for $t < T$ and at time $t = T$ the price jumps to $p(T) = p^* = W(x(T), x(T))$. For $\delta \neq 0$, we can then obtain the optimal $p(t)$ and the switch time T by restating monopolist's optimal pricing problem as:

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$$\operatorname*{Max}_{T,x(t),p(t)} \int_0^T e^{-\delta t}[p(t)x(t) - c(x(t))]dt + \frac{e^{-\delta T}}{\delta}[W(x(T), x(T))x(T) - c(x(T))], \tag{A.2.1}$$

subject to constraints (3.2) and (3.3).

The Hamiltonian for the above problem is again given by (3.9) and the necessary condition for the optimal subscription price in the interval $[0, T]$ is specified by (3.11). Unlike the original optimization problem, however, the revised problem has a free terminal time. This additional degree of freedom gives rise to the following transversality condition:

$$e^{-\delta T}\{-W(x, x)x + c(x)\} + H(x, p, \lambda)|_{t=T} = 0. \tag{A.2.2}$$

Substituting (3.9) and the relation $x(T) = x^*$ into (A.2.2), we have

$$e^{-\delta T}\{[p(T^-) - p^*]x^* + \lambda(T)G(d^\alpha(T), x^*)\} = 0. \tag{A.2.3}$$

However, since $p(T^-) < W(x(T), x(T))$, condition (3.11) requires that

$$x^* + \lambda(T)G_d(d^\alpha(T), x^*)d_p^\alpha(T) = 0. \tag{A.2.4}$$

Eliminating $\lambda(T)$ from (A.2.3) and (A.2.4) yields

$$e^{-\delta T}\{[p(T^-) - p^*]G_d(d^\alpha(T), x^*)d_p^\alpha(T) - G(d^\alpha(T), x^*)\} = 0. \tag{A.2.5}$$

But by the strict concavity of $G(\cdot)$ in p,

$$G(d^\alpha(T), x^*) + [p^* - p(T^-)]G_d(d^\alpha(T), x^*)d_p^\alpha(T) > G(x^*, x^*) = 0. \tag{A.2.6}$$

Hence, (A.2.5) cannot be satisfied for a finite T as assumed and, therefore, the equality must hold in (3.14). Q.E.D.

PROPOSITION 3. *Let $x(\delta, \alpha)$ satisfy* (3.16). *Also assume that it satisfies the second-order sufficiency condition for the static optimum, that is, $u_x(x) \triangleq \partial u/\partial x < 0$ at $x = x(\delta, u)$. Then, for $x(\delta, \alpha) \neq 0$, $\delta \neq 0$,*

$$\text{(i)} \quad \frac{\delta x(\delta, \alpha)}{\partial \delta} < 0 \quad \textit{and} \tag{A.3.1}$$

$$\text{(ii)} \quad \frac{\partial x(\delta, \alpha)}{\partial \alpha} > 0. \tag{A.3.2}$$

PROOF. (i) From (3.16), we have

$$\left.\frac{\partial x}{\partial \delta}\right|_{\delta=0} = -\frac{v^\alpha(x)}{u_x(x)}.$$

By assumption, $u_x(x) < 0$. From (3.18),

$$v^\alpha(x) = \frac{}{G_d(x, x)d_p^\alpha}, \quad \text{where} \tag{A.3.4}$$

$$d_p^\alpha = \frac{\partial d^\alpha}{\partial p} = \frac{1}{W_\eta(\cdot) + \alpha W_x(\cdot)}. \tag{A.3.5}$$

In (A.3.5), $W_\eta(\cdot)$ and $W_x(\cdot)$ are the partial derivatives of the willingness-to-pay function $W(\eta, x)$ with respect to η and x, respectively. Since $0 \le \alpha \le 1$ and $W_x(\cdot) > 0$, $[W_\eta(\cdot) + \alpha W_x(\cdot)] \le [W_\eta(\cdot) + W_x(\cdot)]$. We note that a stable equilibrium must be on the downward-sloping segment of $W(x, x)$, i.e., $[W_\eta(\cdot) + W_x(\cdot)] < 0$ at a stable equilibrium $\eta = x = x(\delta, \alpha)$. Therefore, $d_p^\alpha < 0$. From (2.14), $G_d(\cdot) > 0$. Thus $v^\alpha(x) < 0$ at $x = x(\delta, \alpha)$, and $\partial x/\partial\delta|_{\delta=0} < 0$. This, in conjunction with the observation that $x(\delta, \alpha)$ is monotone in δ (§3), implies (A.3.1).

(ii) To establish the second part of the proposition, we substitute for d_p^α in (A.3.4) to obtain

$$v^\alpha(x) = \frac{x[W_\eta(x, x) + \alpha W_x(x, x)]}{G_d(x, x)} \tag{A.3.6}$$

Differentiating with respect to α,

$$\frac{\partial v^\alpha(x)}{\partial \alpha} = \frac{xW_x(x, x)}{G_d(x, x)} > 0 \tag{A.3.7}$$

for $x \neq 0$. Result (A.3.2) follows from (A.3.7), (A.3.1) and the relationship (3.16). Q.E.D.

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References

Artle, R. and C. Averous (1973), "The Telephone Systems as a Public Good: Static and Dynamic Aspects," *Bell Journal of Economics and Management Science*, 4 (Spring), 89–100.

Bass, F. (1969), "A New Product Growth Model for Consumer Durables," *Management Science*, 15 (January), 215–227.

——— (1980), "The Relationship Between Diffusion Rates, Experience Curves, and Demand Elasticities for Consumer Durable Technical Innovations," *Journal of Business*, 53 (July), S51–S67.

——— and A. Bultez (1982), "A Note on Optimal Strategic Pricing of Technological Innovations," *Marketing Science*, 1 (Fall), 371–378.

Dhebar, A. and S. Oren (1982), "Dynamic Pricing and Market Equilibria for Goods with Positive Demand Externalities," PIP Report No. 42 (June), Department of Engineering-Economic Systems, Stanford University, Stanford, California, June.

Dolan, R. and A. Jeuland (1981), "Experience Curves and Dynamic Demand Models: Implications for Optimal Pricing Strategies," *Journal of Marketing*, 45 (Winter), 52–73.

Faulhaber, G. and J. Panzar (1977), "Optimal Two-Part Tariffs with Self-Selection," Bell Laboratories Economics Discussion Paper No. 74, January 1977.

Feichtinger, G. (1982), "Optimal Pricing in a Diffusion Model with Concave Price-Dependent Market Potential," *Operations Research Letters*, 1, 236–240.

Jeuland, A. (1981), "Parsimonious Models of Diffusion of Innovation, Part B: Incorporating the Variable of Price," Working Paper, University of Chicago, Graduate School of Business.

——— and R. Dolan (1982), "An Aspect of New Product Planning: Dynamic Pricing," *TIMS Studies in Management Sciences*, Special Issue on Market Planning Models, A. Zoltners, Ed.

Kalish, S. (1983), "Monopolist Pricing with Dynamic Demand and Production Cost," *Marketing Science*, 2 (Summer), 135–159.

——— (1984), "A New Product Adoption Model with Price, Advertising and Uncertainty," Working Paper, University of Rochester.

Kamien, M. and N. Schwartz (1981), *Dynamic Optimization: The Calculus of Variations and Optimal Control in Economics and Management*, New York: North-Holland.

Mirman, L. and D. Sibley (1980), "Optimal Nonlinear Price for Multi-product Monopolies," *Bell Journal of Economics*, 12 (Autumn), 659–670.

Oren, S. and S. Smith (1981), "Critical Mass and Tariff Structure in Electronic Communications Markets," *Bell Journal of Economics* 12 (Autumn), 467–487.

Rabenau, B. and K. Stahl (1974), "Dynamic Aspects of Public Goods: A Further Analysis of the Telephone System," *Bell Journal of Economics and Management Science*, 5 (Autumn), 651–669.

Robinson, S. and C. Lakhani (1975), "Dynamic Price Models for New Product Planning," *Management Science*, 21 (June), 1113–1122.

Rohlfs, J. (1974), "A Theory of Interdependent Demand for a Communication Service," *Bell Journal of Economics and Management Science*, 5 (Spring), 16–37.