BCOL RESEARCH REPORT 05.03

Industrial Engineering & Operations Research
University of California, Berkeley, CA

Forthcoming in *Mathematics of Operations Research*

THE FLOW SET WITH PARTIAL ORDER

ALPER ATAMTÜRK AND MUHONG ZHANG

ABSTRACT. The flow set with partial order is a mixed-integer set described by a budget on total flow and a partial order on the arcs that may carry positive flow. This set is a common substructure of resource allocation and scheduling problems with precedence constraints and robust network flow problems under demand/capacity uncertainty.

We give a polyhedral analysis of the convex hull of the flow set with partial order. Unlike for the flow set without partial order, cover-type inequalities based on partial order structure are a function of a lifting sequence. We study the lifting sequences and describe structural results on the lifting coefficients for general and simpler special cases. We show that all lifting coefficients can be computed in polynomial time by solving maximum weight closure problems in general. For the special case of induced-minimal covers, we give a sequence-dependent characterization of the lifting coefficients. We prove, however, if the partial order is defined by an arborescence, then lifting is sequence-independent and all lifting coefficients can be computed in linear time. Moreover, if the partial order is defined by a path (total order), then the coefficients can be expressed explicitly. We also give a complete polyhedral description of the flow set with partial order for the polynomially-solvable total order case. We show that finding an optimal lifting order for a given induced-minimal cover and a given fractional solution is a submodular optimization problem, which is solved greedily. Finally, we present preliminary computational results with a cutting-plane algorithm based on the lifting and separation results.

Keywords: Partially ordered sets; polyhedra; precedence constraints; fixed-charge flow.

December 2005
Revised: December 2007

Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720-1777 USA. atamturk@berkeley.edu, mhzhang@ieor.berkeley.edu

This research is supported, in part, by Grant DMI0218265 from the National Science Foundation.

1. Introduction

In this paper we give a polyhedral analysis for a flow set with partial order described by a budget on total flow and a partial order on the arcs that may carry positive flow. The flow set with partial order is a substructure of resource allocation and scheduling problems with precedence constraints and robust network flow problems under demand/capacity uncertainty.

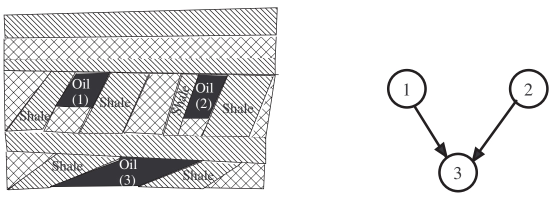
Given a set N of arcs with capacities $u \in \mathbb{Q}^N$, a lower or an upper bound $b \in \mathbb{Q}$ on the total flow on the arcs, and a set A of pairwise relations defining a partial order on the arcs, the convex hull of the *flow set with partial order* is

$$\mathcal{F}_\triangle := \operatorname{conv}\left\{(x,y) \in \mathbb{B}^N \times \mathbb{R}^N : y(N) \triangle b,\ \mathbf{0} \le y \le u \circ x,\ x_i \ge x_j,\ (ij) \in A\right\},$$

where $\mathbb{B} = \{0,1\}$ and $\triangle \in \{\le, \ge\}$. In this formulation, y_i is the amount of flow on arc $i \in N$ and x_i is the binary variable indicating whether arc i may carry positive flow or not. Here $y(N)$ denotes the sum $\sum_{i \in N} y_i$, and the symbol '$\circ$' the Hadamard product; thus, $(u \circ x)_i = u_i x_i$ for $i \in N$. The pairwise relations A represent precedences on the arcs that may carry positive flow: if $(ij) \in A$, then $x_i = 0$ implies $x_j = 0$.

Motivating examples. In this section we describe formulations of a couple of motivating applications, in which the flow set with partial order $\mathcal{F}_\triangle$ arises naturally as a substructure.

Application 1. (Oil extraction) In an oil field, oil deposits are distributed in different layers of rock formations as illustrated in Figure 1(a). Figure 1(b) represents the precedence relationship for extracting oil from deposits, labeled as 1, 2, and 3, indicating that one must drill into deposits 1 and 2 before deposit 3.



(a) (b)

Figure 1. Oil extraction.

Given the fixed cost of drilling from surface into an oil deposit and between deposits, revenue per barrel of oil extracted, and estimated amount of oil reserve in each deposit, an optimal extraction plan that maximizes profits in an oil field can be modeled using the flow set with partial order.

To illustrate, let the f_1 and f_2 be the fixed costs of drilling into the oil deposits 1 and 2 in Figure 1(a) from the surface, and f_{13} and f_{23} be the fixed costs of drilling between deposits 1 and 3, and 2 and 3, and r be revenue for per barrel of oil. Then an optimal extraction plan with bounds b and b' on the total amount of extracted oil can be found by solving

$$\begin{array}{rl}\max & ry_1 + ry_2 + ry_3 - f_1x_1 - f_2x_2 - (f_{13} + f_{23})x_3 \\ \text{s.t.} & b \geq y_1 + y_2 + y_3 \geq b' \\ & x_1 \geq x_3 \\ & x_2 \geq x_3 \\ & y_i \leq u_i x_i, \; i = 1, 2, 3 \\ & x \in \{0,1\}^3, y \in \mathbb{R}^3.\end{array}$$

Note that the oil exploration problem described here has a similar structure to that of open-pit mining problem (Hochbaum and Chen, 2000). In the open-pit mining problem, however, there are no bounds on the total amount extracted and sections of a mine cannot be extracted partially. Note that if the constraints on the continuous variables y are removed from $\mathcal{F}_\triangle$, then the constraint matrix becomes totally unimodular.

Application 2. (Generator scheduling) Another application of our model is scheduling generators on a power grid. In order to avoid unbalanced load on the grid, generators must be started in a certain order and shut down in the reverse order. For example, consider the generators illustrated in Figure 2: generator 1 has to be started before generators 2 and 3 and shut down after 2 and 3.

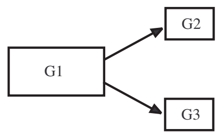


FIGURE 2. Generator scheduling.

Let f_i and g_i be the fixed cost of starting and shutting down generator i and p_i be the profit for each unit of electricity from generator i, $i = 1, 2, 3$. Moreover, let b_t be the minimum total output required from these generators in time period t. Then, we can formulate the minimum cost scheduling problem over a horizon of n time periods as

$$\begin{array}{rl}\min & \sum_{t=1}^{n} \sum_{i=1}^{3} (f_i z_{it} + g_i w_{it} - p_i y_{it}) \\ \text{s.t.} & y_{1t} + y_{2t} + y_{3t} \geq b_t, \; t = 1, \dots, n \\ & y_{it} \leq u_i x_{it}, \; i = 1, 2, 3, \; t = 1, \dots, n \\ & x_{1t} \geq x_{2t}, \; x_{1t} \geq x_{3t}, \; t = 1, \dots, n \\ & z_{it} \geq x_{it} - x_{it-1} \geq -w_{it}, \; i = 1, 2, 3, \; t = 1, \dots, n \\ & x, w, z \in \mathbb{B}^{3n}, y \in \mathbb{R}^{3n}.\end{array}$$

Many other problems with precedence relationships between activities and a budget or total requirement on the activities can be formulated using the flow set with partial order. Kis (2005) describes a production scheduling problem with a special case of $\mathcal{F}_\leq$ as a substructure. The flow set with partial order is also used for modeling separation problems for robust network flows under demand/capacity uncertainty (Atamtürk and Zhang, 2007).

Relevant literature. The flow set with partial order generalizes two very interesting sets that have been studied earlier. The first one is the 0-1 knapsack set with precedence constraints, which is the face of $\mathcal{F}_\triangle$ obtained by setting $y = u \circ x$. The polyhedral structure of the 0-1 knapsack set with precedence constraints is studied by Boyd (1993), Park and Park (1997), and van de Leensel et al. (1999). Boyd (1993) gives several classes of facet-defining inequalities of its convex hull. Park and Park (1997) and van de Leensel et al. (1999) describe inequalities obtained by lifting the cover inequalities for the knapsack problem. Aghezzaf et al. (1995) study the related problem of packing subtrees with cardinality constraints. The second related set is the (fixed-charge) flow set, which is the relaxation of $\mathcal{F}_\triangle$ for $A = \emptyset$. Strong valid inequalities for the flow set are given by Padberg et al. (1984), Gu et al. (1999, 2000), Atamtürk (2001). These inequalities are very effective in strengthening linear programming bounds of mixed-integer programs with fixed-charges and have become standard features of commercial solvers. Kis (2005) studies a special case of $\mathcal{F}_\leq$ arising in a scheduling problem, in which precedence relations A form a path (total order) and capacities $u_i,\ i \in N$ are constant.

Because the flow set is a relaxation of $\mathcal{F}_\triangle$ by dropping the precedence constraints $x_i \geq x_j,\ (ij) \in A$, valid inequalities for it are also valid for $\mathcal{F}_\triangle$. The basic inequality for the flow set in $\leq$ form is the *flow cover inequality* (Padberg et al., 1984)

$$\sum_{i \in C} [y_i + (u_i - \lambda)^+ (1 - x_i)] \leq b, \tag{1}$$

where C is a cover, i.e., a subset of N satisfying $\lambda := \sum_{i \in C} u_i - b > 0$. In the presence of a partial order, cover inequalities (1) can be strengthened by lifting. However, such inequalities cannot be written explicitly except for special cases. Moreover, cover inequalities for $\mathcal{F}_\triangle$ that are based on the partial order structure are not unique for a given cover C and are themselves a function of a lifting sequence.

The example below illustrates that valid inequalities for the flow set may not cut off any fractional extreme point of the linear programming (LP) relaxations of $\mathcal{F}_\triangle$.

Example 1. Consider the instance of $\mathcal{F}_\leq$ given by

$$y_1 + y_2 + y_3 + y_4 \leq 3,\ y_i \leq x_i,\ x_1 \geq x_3,\ x_1 \geq x_4,\ x_2 \geq x_3,\ x_2 \geq x_4,\ y \in \mathbb{R}^4,\ x \in \mathbb{B}^4$$

with the corresponding precedence relationship illustrated in Figure 3. The LP relaxation of $\mathcal{F}_\leq$ has exactly three fractional extreme points:

$$x = y = (1, 2/3, 2/3, 2/3), (2/3, 1, 2/3, 2/3), (3/4, 3/4, 3/4, 3/4).$$

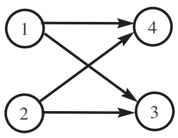


FIGURE 3. Precedence graph for Example 1.

It is easy to see that, in this example, the LP relaxation of the flow set obtained by dropping the precedence constraints is integral. Therefore, no valid inequality for the flow set can cut off the fractional points listed above.

On the other hand, inequalities

$$y_1 + y_2 + y_3 + y_4 + 2(1 - x_1) + 1(1 - x_2) \leq 3 \tag{2}$$
$$y_1 + y_2 + y_3 + y_4 + 1(1 - x_1) + 2(1 - x_2) \leq 3 \tag{3}$$

which are special cases of the cover inequalities (5) introduced in Section 4.1 are valid for $\mathcal{F}_{\leq}$ and cut-off the fractional points listed above.

Outline. This paper is organized as follows. In Section 2, we introduce the notation and assumptions used throughout the paper. In Section 3, we discuss the complexity of a linear optimization problem over the flow set with partial order and characterize the structure of the extreme points of its linear programming relaxation. Sections 4–6 are devoted to the main polyhedral results of the paper, where we identify strong valid inequalities for the flow set with partial order through lifting arguments. Unlike for the flow set without partial order ($A = \emptyset$), the cover inequalities for $\mathcal{F}_{\triangle}$ are themselves a function of the lifting sequence. We study the lifting sequences and describe structural results on the lifting coefficients for general and simpler special cases. We show that all lifting coefficients can be computed in polynomial time by solving maximum weight closure problems in general. For the special case of induced-minimal covers, we give a sequence-dependent characterization of the lifting coefficients. We prove that if the partial order is defined by an arborescence, then lifting is sequence-independent and lifting coefficients can be described recursively. Moreover, if it is defined by a path (total order), then the coefficients can be expressed explicitly. We also give a complete polyhedral description of the flow set with partial order for the polynomially-solvable total order case. In Section 7, we study the separation problem of the identified inequalities and show that finding an optimal lifting order for a given induced-minimal cover and a given fractional solution is a submodular optimization problem, which is solved by the greedy algorithm. In Section 8, we present a summary of preliminary computational experiments that illustrate the effectiveness of the lifting and separation results when using the inequalities as cutting planes. Finally, we conclude with Section 9.

2. Definitions and Assumptions

In this section we introduce the notation and assumptions used throughout the paper. Let $(N, \preccurlyeq)$ be a partially ordered set (*poset*); that is, N is equipped with the binary relation $\preccurlyeq$ which is reflexive, antisymmetric, and transitive. The relation $i \prec j$ denotes that $i \preccurlyeq j$ and $i \neq j$. An element k *covers* element i if $i \prec k$ and there is no element j such that $i \prec j \prec k$. A *Hasse diagram* of poset $(N, \preccurlyeq)$ is an acyclic directed graph $G = (N, A)$ with node set N and arc set A, where $(ij) \in A$ if and only if j covers i. Thus G is the minimal graph representing poset $(N, \preccurlyeq)$. For $C \subseteq N$ let $(C, \preccurlyeq)$ be the sub-poset with the same relation and $\mathrm{H}(C)$ to be the corresponding Hasse diagram, i.e., minimal graph representing poset $(C, \preccurlyeq)$.

We refer to G as the *precedence graph.* If there is a directed path from node i to node j in G, we say that i as a *predecessor* of j and j is a *successor* of i. Let $\mathrm{P}(i)$ denote the set of all predecessors of node i and $\mathrm{S}(i)$ denote the set of all successors of node i. For $C \subseteq N$ we define

$$\mathrm{P}(C) := \bigcup_{i \in C} \mathrm{P}(i), \quad \overline{\mathrm{P}}(C) := \mathrm{P}(C) \cup C, \quad \text{and} \quad \mathrm{S}(C) := \bigcup_{i \in C} \mathrm{S}(i).$$

For $C \subseteq N$ let $\mathrm{L}(C)$ denote the leaf nodes of $\mathrm{H}(C)$, that is,

$$\mathrm{L}(C) := \{i \in C : C \cap \mathrm{S}(i) = \emptyset\}.$$

For C, $T \subseteq N$ let $\mathcal{C}(C,T)$ denote the subset of T and its successors contained in C, that is,

$$\mathcal{C}(C,T) := C \cap (T \cup \mathrm{S}(T)).$$

We denote the component-wise multiplication of two vectors a and b of the same dimension as $a \circ b$. We use $\mathbf{0}$ to denote a vector of zeros and $\mathbf{1}$ for a vector of ones. For simplicity of notation, we denote a singleton set $\{i\}$ by its element i. Finally, a vector $a \in \mathbb{R}^N$, we define $a(C) := \sum_{i \in C} a_i$ for $C \subseteq N$.

Example 2. We illustrate the definitions for the graph in Figure 4(a). Here $\mathrm{S}(4) = \{6, 7\}$, $\mathrm{P}(4) = \{1, 2\}$, and $\mathrm{L}(\{1, 3, 4, 7\}) = \{3, 7\}$. $H(\{1, 2, 3, 5, 6\})$ is shown in Figure 4(b). Also $\mathcal{C}(\{1, 2, 3, 5, 6\}, 1) = \{1, 3, 5, 6\}$.

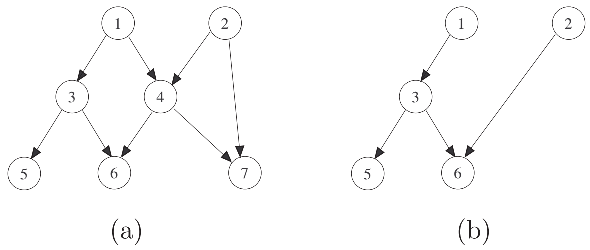


FIGURE 4. Precedence graph: Nodes have a topological labeling.

Definition 1. A pair $i, j \in N$ is *incomparable* in poset $(N, \preccurlyeq)$ if neither $i \preccurlyeq j$ nor $j \preccurlyeq i$.

Definition 2. A subset C of N is called a *cover* if $\lambda := u(C) - b > 0$. A cover C is *minimal* if $u(C \setminus i) \leq b$ for all $i \in C$; it is *induced-minimal* if $u(C \setminus \mathcal{C}(C, i)) \leq b$ for all $i \in C$.

Observe that a minimal cover is an induced-minimal cover, but not the opposite.

Definition 3. Given an acyclic directed graph $G = (N, A)$, a bijection $\pi : C \subseteq N \to \{1, 2, \ldots, |C|\}$ is called a *labeling* of C and π_i is the corresponding *label* of $i \in C$. We denote the inverse function of π with σ and refer to $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{|C|})$ as an *order* of C. A labeling π of C is called a *topological labeling* if $\pi_i < \pi_j$ for all $i, j \in C$ with $(ij) \in A$; it is called a *reverse topological labeling* if $\pi_i > \pi_j$ for all $i, j \in C$ with $(ij) \in A$. We refer to the inverse function σ of a (reverse) topological labeling π as a *(reverse) topological order* of C.

Assumption 1. For brevity of presentation, we assume that $0 < u_i \leq b$ for all $i \in N$. If $u_i > b$, we can replace u_i with b without changing the feasible set. If $u_i = 0$, then y_i can be dropped and x_i can be removed.

We use $\mathcal{F}^I_\triangle$ to denote $\mathcal{F}_\triangle \cap \{(x, y) \in \mathbb{B}^N \times \mathbb{R}^N\}$. In the remainder of the paper, we will restrict our attention to $\mathcal{F}_\leq$ because, as shown in the Appendix, valid inequalities for $\mathcal{F}_\leq$ can be converted to valid inequalities for $\mathcal{F}_\geq$ with a simple transformation. Therefore, for simplicity of notation, in the remainder we use $\mathcal{F}$ to denote $\mathcal{F}_\leq$.

3. Preliminaries

In this section we give a few preliminary results on the complexity of optimization of a linear function over the flow set with partial order and on the structure of its linear programming relaxation.

3.1. **Optimization complexity.** The optimization of a linear function over the flow set with partial order is $\mathcal{NP}$-hard because the 0-1 knapsack polytope is a face of $\mathcal{F}$ when $A = \emptyset$ and $y = u \circ x$. This is true even when the precedence graph $G = (N, A)$ is a star (either an in-star or an out-star by fixing x_0 for the center node to 0 or 1).

On the other hand, if the precedence graph is a path, then the partial order defined by G reduces to a total order. In this case, the optimization problem is polynomially solvable because there are $|N| + 1$ feasible assignments for x and for fixed x the remaining problem is a continuous knapsack problem solvable in $O(|N| \log |N|)$. In Section 5 we give a complete polyhedral description of $\mathcal{F}$ for this polynomially-solvable case.

3.2. **Extreme points.** Now we give a characterization of the extreme points of the continuous relaxation

$$\widehat{\mathcal{F}} := \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : y(N) \le b,\ \mathbf{0} \le y \le u \circ x,\ \mathbf{0} \le x \le \mathbf{1},\ x_j \le x_i,\ (ij) \in A\right\}.$$

Proposition 1. Let (x, y) be an extreme point of $\widehat{\mathcal{F}}$.

1. If $y(N) < b$, then $y_i \in \{0, u_i\}$ and $x_i \in \mathbb{B}$ for $i \in N$.
2. If $y(N) = b$, then
   i. if $0 < x_i < 1$ and $0 < x_j < 1$ for $i, j \in N$, then $x_i = x_j$;
   ii. if $0 < x_i < 1$ for some $i \in N$, then $y_j \in \{0, u_j x_j\}$ for all $j \in N$;
   iii. $0 < y_i < u_i x_i$ for at most one $i \in N$; and if so x is integral.

Proof. 1. Suppose $0 < y_i < u_i x_i$ for some $i \in N$. Then (x, y) is a strict convex combination of $(x, y + \epsilon e_i)$, $(x, y - \epsilon e_i) \in \widehat{\mathcal{F}}$ for ϵ such that $0 < \epsilon \le \min\{y_i, u_i x_i - y_i, b - y(N)\}$. So we must have $y_i \in \{0, u_i x_i\}$ for all $i \in N$. Now suppose $0 < x_i < 1$ for some $i \in N$. Let $C = \{j \in N : x_j = x_i\}$ and $K = \{j \in C : y_j = u_j x_j\}$. Then for small $\epsilon > 0$, we have $(x + \sum_{j \in C} \epsilon e_j, y + \sum_{j \in K} \epsilon u_j e_j)$, $(x - \sum_{j \in C} \epsilon e_j, y - \sum_{j \in K} \epsilon u_j e_j) \in \widehat{\mathcal{F}}$ and (x, y) is a strict convex combination of them. Contradiction.
2. *i.* and *ii.* At extreme point (x, y) there are $2|N|$ linearly independent tight constraints of $\widehat{\mathcal{F}}$. Let $C = \{i \in N : 0 < x_i < 1\}$ be nonempty. There are at most $2|N \setminus C|$ linearly independent tight constraints among $0 \le x_i \le 1$, $0 \le y_i \le u_i x_i$, $i \in N \setminus C$ and $x_j \le x_i$ $i, j \in N \setminus C$. Then, the remaining $2|C|$ tight constraints are among $0 \le y_i \le u_i x_i$, $i \in C$, $x_j \le x_i$, $i, j \in C$, and $y(N) \le b$. There are at most $|C|$ tight constraints among $0 \le y_i \le u_i x_i$, $i \in C$. Since the linearly independent constraints $x_j \le x_i$, $i, j \in C \subseteq N$ form a directed forest in $G(C)$, the subgraph induced by C, there are at most $|C| - 1$ linearly independent constraints among them, given by a spanning directed tree in $G(C)$. Thus, $G(C)$ is connected and $x_i = x_j$ for all $i, j \in C$.
iii. Suppose $0 < y_i < u_i x_i$, $0 < y_j < u_j x_j$ for distinct $i, j \in N$. Then for a small enough $\epsilon > 0$, the points $(x, y - \epsilon e_i + \epsilon e_j)$, $(x, y + \epsilon e_i - \epsilon e_j) \in \widehat{\mathcal{F}}$ and (x, y) is a strict convex combination of them. The latter part is from part *ii.* $\square$

Corollary 1. If (x, y) is an extreme point of $\widehat{\mathcal{F}}$ with $x \notin \mathbb{B}^N$, then there exists a partition $\{C_0, C_f, C_1\}$ of N, such that $x_i = y_i = 0$ for $i \in C_0$, $y_i = u_i x_i = u_i f$ for $i \in C_f$, where $0 < f < 1$, and $x_i = 1$, $y_i \in \{0, u_i\}$ for $i \in C_1$ and $y(C_f \cup C_1) = b$; moreover, $G(C_f)$ is connected.

Proposition 2. $\mathcal{F}$ is full-dimensional.

Proof. Assume without loss of generality that the variables are indexed in a topological order. For $i \in N$ let $x^i := \sum_{k \in N : k \leq i} e_k$ and $y^i := u_i e_i$. Then $(x^i, \mathbf{0})$ and (x^i, y^i) for $i \in N$, and $(\mathbf{0}, \mathbf{0})$ are $2|N| + 1$ affinely independent points in $\mathcal{F}$. □

4. Facet-defining inequalities

This is the main section of the analysis, in which we identify strong inequalities for $\mathcal{F}$. The first result is on the basic inequalities.

Proposition 3. *Trivial facets.*

1. $0 \leq y_i$, $i \in N$ defines a facet of $\mathcal{F}$.
2. $x_j \leq x_i$, $(ij) \in A$ defines a facet of $\mathcal{F}$.
3. $x_i \leq 1$, $i \in N$ defines a facet of $\mathcal{F}$ if and only if node i has indegree zero.
4. $y_i \leq u_i x_i$, $i \in N$ defines a facet of $\mathcal{F}$ if and only if either $u_i < b$ or node i has outdegree zero.

Proof. Without loss of generality, assume that the nodes are labeled in a topological order. We use (x^i, y^i), $i = 1, ..., n$, as defined in the proof of Proposition 2.
1. The points $(\mathbf{0}, \mathbf{0}), (x^k, \mathbf{0})$ for $k \in N$ and (x^k, y^k) for $k \in N \setminus i$ are affinely independent points of $\mathcal{F}$ with $y_i = 0$.
2. Because G is a Hasse diagram, $(kj) \notin A$ for any $k \in \mathrm{S}(i)$. Then, without loss of generality, we may assume $i = j - 1$. The points $(\mathbf{0}, \mathbf{0}), (x^k, \mathbf{0})$ for $k \in N \setminus i$, (x^k, y^k) for $k \in N \setminus i$ and (x^j, e_i) are affinely independent points of $\mathcal{F}$ such that $x_j = x_i$.
3. If $(ki) \notin A$ for any $k \in N$, without loss of generality, we may assume $i = 1$. Then the points $(x^k, \mathbf{0})$ and (x^k, y^k) for $k \in N$ are affinely independent points of $\mathcal{F}$ with $x_i = 1$. If $(k, i) \in A$, then $x_i \leq x_k$ and $x_k \leq 1$ imply $x_i \leq 1$.
4. Note that $y^i_i = u_i = u_i x^j_i$ for $j \geq i$. If $u_i < b$, for small $\epsilon > 0$ the points $(\mathbf{0}, \mathbf{0}), (x^i, u_i e_i)$, $(x^j, \mathbf{0}), (x^j, \epsilon e_j)$ for $j < i$, and $(x^j, u_i e_i), (x^j, u_i e_i + \epsilon e_j)$ for $j > i$ are affinely independent points of $\mathcal{F}$ satisfying $y_i = u_i x_i$. If $(ik) \notin A$ for any $k \in N$, then we may assume $i = n$ and the same points are sufficient. On the other hand, if $u_i = b$ and $(ik) \in A$ for some $k \in N$, then valid inequality (5) $y_i + y_k + b(1 - x_i) \leq b$ with $C = \{i, k\}$ and $y_k \geq 0$ imply $y_i \leq u_i x_i$. □

4.1. Lifting covers.

In this section we derive non-trivial facets of $\mathcal{F}$ by sequentially lifting a simple inequality from a restriction of $\mathcal{F}$:

$$\mathcal{F}_C = \{ \ (x, y) \in \mathcal{F} : x_i = 1 \text{ for all } i \in C \ \} \text{ for } C \subseteq N.$$

For a cover C consider the simple valid inequality

$$y(C) \leq b \tag{4}$$

for $\mathcal{F}_C$. We lift (4) with binary variables x_i, $i \in C$. For a reverse topological order $\sigma := (\sigma_1, \sigma_2, \ldots, \sigma_{|C|})$ of C, a valid *cover inequality*

$$y(C) + \sum_{i=1}^{|C|} \alpha_{\sigma_i} (1 - x_{\sigma_i}) \leq b \tag{5}$$

for $\mathcal{F}$ is obtained by computing the *lifting coefficients* α_{σ_i} as

$$\alpha_{\sigma_i} := b - \max \left\{ y(C) + \sum_{k=1}^{i-1} \alpha_{\sigma_k}(1 - x_{\sigma_k}) : (x, y) \in \mathcal{F}_{C \setminus C_i},\ x_{\sigma_i} = 0 \right\}, \tag{6}$$

where $C_i := \{\sigma_1, \sigma_2, \ldots, \sigma_i\}$, one at a time in the order of σ. Observe that the *lifting problem* (6) is infeasible unless σ is a reverse topological order. Therefore, it suffices to consider only reverse topological lifting orders.

In general, lifting of (4) is sequence-dependent; that is, different reverse topological orders used for lifting may lead to different cover inequalities (5). Example 1 illustrates two inequalities obtained from $C = \{1, 2, 3, 4\}$ using different orders. Inequality (2) is obtained by lifting $y(C) \leq 3$ in the order (3,4,1,2) and inequality (3) is obtained by lifting $y(C) \leq 3$ in the order (3,4,2,1). We show, however, in Section 5 that for an arborescence precedence graph, all reverse topological lifting orders give the same inequality, i.e., lifting is sequence-independent (Atamtürk, 2004) in this case.

The cover inequality (5) may or may not be a facet of $\mathcal{F}$ if the initial inequality (4) is not a facet of $\mathcal{F}_C$. In the presence of precedence constraints, (4) typically does not define a facet of $\mathcal{F}_C$. In the following theorem we describe a sufficient condition for the cover inequality (5) to be a facet of $\mathcal{F}$. Necessity is discussed in Remark 1.

Theorem 1. *Inequality (5) is facet-defining for $\mathcal{F}$ if*

1. $C^* := \{i \in C : \alpha_i > 0\} \neq \emptyset$ *or* $C = N$*; and*
2. $R := \bigcap_{i \in C^*} \mathrm{S}(i) \subseteq C$*; and*
3. *for all* $k \in \mathrm{P}(C) \setminus C$*, there is* $j_k \in C^* \setminus \mathrm{P}(k)$ *such that* $\mathcal{C}(C, k) \subseteq \mathcal{C}(C, j_k)$ *and* $\pi_{j_k} \leq \pi_l$ *for all* $l \in \mathrm{P}(k) \cap C$.

Proof. As C is a cover, there are $|C|$ affinely independent points $(\mathbf{1}, y^k)$, $k \in C$ such that $y^k(C) = b$ and $y(N \setminus C) = 0$. Let $(\mathbf{1}, \bar{y})$ be one of these points to be used later. By definition of α_k, there exists another set of $|C|$ affinely independent points (x^k, y^k), $k \in C$ (e.g. optimal solutions for the lifting problems (6)) satisfying inequality (5) as equality and $x^k_k = 0$.

For $k \in \mathrm{S}(C) \setminus C$, consider the points $(\mathbf{1} - e_k - \sum_{i \in \mathrm{S}(k)} e_i, \bar{y})$. Since $R \subseteq C$, for any $k \in \mathrm{S}(C) \setminus C$, there exists $j \in C$, such that $j \in \mathrm{L}(C^*)$ and $k \notin \mathrm{S}(j)$. Then there is a point (x^k, y^k), such that $0 < y^k_k \leq \alpha_j$, $y^k(C) + \alpha_j = b$, $y^k_j = 0$, and $x^k_i = 0$ for all $i \in j \cup \mathrm{S}(j)$. Thus we have $2|C| + 2|\mathrm{S}(C) \setminus C|$ affinely independent points.

For $k \in \mathrm{P}(C) \setminus C$, consider the optimal solution (x^{j_k}, y^{j_k}) to the lifting problem with respect to $j_k \in C^*$ as defined in the theorem and let $T(j_k) = \{i \in N : x^{j_k}_i = 0\}$. Since $\pi_{j_k} \leq \pi_l$ for all $l \in \mathrm{P}(k) \cap C$, we have $x^{j_k}_k = 1$ and therefore $k \notin T(j_k)$. Because $y^{j_k}(C) < b$, the points $(x^{j_k}, y^{j_k} + \epsilon e_k)$ for all $k \in \mathrm{P}(C) \setminus C$ is feasible for small $\epsilon > 0$. As $\mathrm{S}(k) \cap C \subseteq \mathcal{C}(C, j_k)$ for all $k \in \mathrm{P}(C) \setminus C$ the points $(\mathbf{1} - e_k - \sum_{i \in \mathrm{S}(k) \cup T(j_k)} e_i, y^{j_k})$ are on the face defined by (5) and are affinely independent.

Finally for $k \notin C \cup \mathrm{S}(C) \cup \mathrm{P}(C)$, consider the points $(\mathbf{1} - e_k - \sum_{i \in \mathrm{S}(k)} e_i, \bar{y})$ and $(\mathbf{1} - e_j - \sum_{i \in \mathrm{S}(j)} e_i, y)$ such that $y_k = \min\{u_k, \alpha_j\},\ y_j = 0,\ y(C \setminus j) + \alpha_j = b$ for some $j \in \mathrm{L}(C^*)$. Because each point has either a unique binary variable decreased from one to zero or a unique continuous variable increased from zero to a positive value for the first time in the order they are defined, the $2|N| + 1$ direction vectors defined by these perturbations are linearly independent; hence, the listed points on the face are affinely independent. □

Remark 1. The first two conditions of Theorem 1 are necessary: If $C^* = \emptyset$ and $C \neq N$, then $y(N) \leq b$ and $y \geq 0$ implies (5). Suppose $R \not\subseteq C$ and let $i \in R \setminus C$. Consider the cover $C' = C \cup i$ and the lifting coefficients α' for C'. Since C is a cover, the lifting coefficient $\alpha'_i = 0$ and as $x_i = 0$ whenever $x_k = 0$ for $k \in C^*$, we have $\alpha'_k = \alpha_k$. Thus cover inequality for C' and $y_i \geq 0$ imply the cover inequality for C. Finally, if the last condition of Theorem 1 does not hold, a stronger inequality may be obtained by lifting with x_i, $i \in \mathrm{P}(C) \setminus C$ as well. We discuss such inequalities in Section 6.

The next two lemmas provide bounds on the lifting coefficients. These bounds are central in computing the lifting coefficients efficiently.

Lemma 1. *For any cover C and reverse topological lifting order σ, the coefficients of the cover inequality* (5) *satisfy*

1. $\alpha \geq 0$*;*
2. $\alpha(\mathcal{C}(C,T)) < u(\mathcal{C}(C,T))$ *for all nonempty* $T \subseteq C$*.*

Proof. 1. Consider an optimal solution (x, y) for the lifting problem for σ_i. By definition, $(x, y) \in \mathcal{F}$, $x_{\sigma_i} = 0$, and (x, y) satisfies (5) at equality. Then, the point $(\bar{x}, y)$ with $\bar{x}_k = 1$ for all $k \in \mathrm{P}(\sigma_i) \cup \sigma_i$ and $\bar{x}_k = x_k$ otherwise, is also in $\mathcal{F}$. As $\bar{x}_k = x_k = 1$ for all $k \in \mathrm{P}(\sigma_i) \cap C$, feasibility of $(\bar{x}, y)$ implies $\alpha_{\sigma_i} \geq 0$.
2. For $T \subseteq C$ consider a point $(x, y) \in \mathcal{F}$ with $x_i = y_i = 0$ for $i \in \mathcal{C}(C,T)$, $x_i = 1$ for $i \in C \setminus \mathcal{C}(C,T)$, and $y(C \setminus \mathcal{C}(C,T)) = \min\{b, u(C \setminus \mathcal{C}(C,T))\}$. Since inequality (5) is valid for (x, y)

$$\alpha(\mathcal{C}(C,T)) \leq b - y(C \setminus \mathcal{C}(C,T)) = \max\{0, b - u(C \setminus \mathcal{C}(C,T))\} < u(\mathcal{C}(C,T)),$$

where the last inequality follows as C is a cover and $u > 0$. □

Lemma 2. *For any cover C and reverse topological lifting order σ, the coefficients of the cover inequality* (5) *satisfy*

$$\alpha_i \begin{cases} \leq 0 & \text{if } u(C \backslash \mathcal{C}(C,i)) \geq b, \\ \geq u_i & \text{if } u(C \backslash \mathrm{S}(i)) \leq b. \end{cases}$$

Proof. Let $C_i = \{\sigma_1, \sigma_2, \ldots, \sigma_i\}$ and consider the lifting problem for x_{σ_i}, which may be restated as follows as $x_k = 0$ for all $k \in \mathcal{C}(C, \sigma_i)$:

$$\alpha_{\sigma_i} = b - \max_{(x,y) \in \mathcal{F}_{C \setminus C_i}} \left\{ y(C \backslash \mathcal{C}(C,\sigma_i)) + \alpha(\mathrm{S}(\sigma_i) \cap C) + \sum_{k \in C_i \backslash \mathcal{C}(C,\sigma_i)} \alpha_k (1 - x_k) \right\}.$$

If $u(C \backslash \mathcal{C}(C,\sigma_i)) \geq b$, there is a feasible solution (x, y) to the lifting problem with $y(C \setminus \mathcal{C}(C,\sigma_i)) = b$, $x_k = 1$ for all $k \in C \setminus \mathcal{C}(C,\sigma_i)$, and $x_k = 0$ for all $k \in \mathcal{C}(C,\sigma_i)$, implying $\alpha_{\sigma_i} + \alpha(\mathrm{S}(\sigma_i) \cap C) = \alpha(\mathcal{C}(C,\sigma_i)) \leq 0$. Then, from Lemma 1, we have $\alpha_k = 0$, for all $k \in \mathcal{C}(C,\sigma_i)$; in particular, $\alpha_{\sigma_i} = 0$.

For the second part, consider an optimal solution (x, y) to the lifting problem above. By definition, $(x, y) \in \mathcal{F}$, $x_k = 0$ for all $k \in \mathcal{C}(C,\sigma_i)$, and (x, y) satisfies (5) at equality. If $u(C \setminus \mathrm{S}(\sigma_i)) \leq b$, the point $(\bar{x}, \bar{y})$ with $\bar{x}_k = 1$ for all $k \in \mathrm{P}(\sigma_i) \cup \sigma_i$, $\bar{x}_k = x_k$ otherwise, and $\bar{y}_{\sigma_i} = u_{\sigma_i}$, $\bar{y}_k = y_k$ otherwise, is also in $\mathcal{F}$. As $\bar{x}_k = x_k = 1$ for all $k \in \mathrm{P}(\sigma_i) \cap C$, feasibility of $(\bar{x}, \bar{y})$ implies $\alpha_{\sigma_i} \geq u_{\sigma_i}$. □

Now we are ready for the main result of the section on polynomial computation of the lifting coefficients.

Theorem 2. *For a cover C and reverse topological lifting order σ, the coefficients of the cover inequality* (5) *can be computed by solving at most $|C|$ maximum flow problems.*

Proof. From Lemmas 1 and 2, if $u(C \setminus \mathcal{C}(C, i)) \geq b$ for $i \in C$, we have $\alpha_i = 0$. Otherwise, in the lifting problem (6) the constraint (4) is inactive for any feasible solution. Thus, all of the continuous variables y can be dropped from the problem by fixing them to their bounds. Hence, the lifting problem in this case reduces to the maximum weight closure problem (Ahuja et al., 1993, Hochbaum and Chen, 2000) defined only with the precedence constraints on the binary variables, which is a special case of the maximum flow problem. □

4.2. **Lifting induced-minimal covers.** In this section we give stronger results on the lifting coefficients for induced-minimal covers (Definition 2). In particular, we give an order-dependent explicit description of the lifting coefficients.

Lemma 3. *For an induced-minimal cover C and a maximal connected subset T of $C_i := \{\sigma_1, \sigma_2, \dots, \sigma_i\}$ in $\mathrm{H}(C_i)$ with $\sigma_i \in T$, we have*

$$\alpha(T \setminus \mathcal{C}(C, S)) \geq u(T \setminus \mathcal{C}(C, S)) \text{ for all nonempty } S \subseteq T.$$

Proof. We use induction on i. If $i = 1$, then $C_1 = \{\sigma_1\} = T = S$ and $T \setminus \mathcal{C}(C, S) = \emptyset$; thus, the lemma holds trivially.

Assume that the lemma holds for all C_h with $h < i$ and consider the lifting problem for x_{σ_i}:

$$\alpha_{\sigma_i} = b - \max_{(x,y) \in \mathcal{F}_{C \setminus C_i}} \left\{ y(C \setminus \mathcal{C}(C, \sigma_i)) + \alpha(\mathrm{S}(\sigma_i) \cap C) + \sum_{k \in C_i \setminus \mathcal{C}(C, \sigma_i)} \alpha_k (1 - x_k) \right\}. \quad (7)$$

Let (x, y) be an optimal solution to (7) and $Q = \{k \in C_i \setminus T : x_k = 0\}$. As T is a maximal connected set in C_i, we have $\mathcal{C}(C, Q) \cap T = \emptyset$. Then, due to precedence constraints, $\mathcal{C}(C, Q) = Q$ holds, which from Lemma 1 implies that $\alpha(Q) \leq u(Q)$. Since C is an induced-minimal cover, $u(C \setminus \mathcal{C}(C, \sigma_i)) \leq b$, and therefore we may assume that $Q = \emptyset$ and there is an optimal solution (x, y) to the lifting problem with $y(C \setminus T) = u(C \setminus T)$. Then, this solution satisfies

$$\alpha(\mathcal{C}(C, \sigma_i)) + u(C \setminus T) + \sum_{k \in T \setminus \mathcal{C}(C, \sigma_i)} \alpha_k (1 - x_k) + y(T \setminus \mathcal{C}(C, \sigma_i)) = b. \quad (8)$$

Next we will show that there is indeed an optimal solution with $x(T \setminus \mathcal{C}(C, \sigma_i)) = 0$ if $T \setminus \mathcal{C}(C, \sigma_i) \neq \emptyset$. Let $j = \max\{k : \sigma_k \in T \setminus \mathcal{C}(C, \sigma_i)\}$ and consider $\mathrm{H}(C_j)$. From the choice of j and reverse topological order σ, we have $\mathcal{C}(C, T \setminus \mathcal{C}(C, \sigma_i)) \subseteq C_j$. Let T' be the node set of any connected component of $\mathrm{H}(C_j)$.

Claim. Any feasible solution to the lifting problem (7) has $x(\mathcal{C}(C, S')) = 0$ for some nonempty $S' \subseteq T'$.

Proof. Consider $S' = T' \cap \mathcal{C}(C, \sigma_i)$. If $S' = \emptyset$, since $T' = \mathcal{C}(C, T')$, then $\mathcal{C}(C, T') \cap \mathcal{C}(C, \sigma_i) = \emptyset$, which means σ_i and nodes in T' have no common successors in C_i, thus $T' \subseteq T \setminus \mathcal{C}(C, \sigma_i)$. But σ_i has no predecessor in C_i and no node in $T \setminus \mathcal{C}(C, \sigma_i)$ has σ_i as its predecessor either. Contradiction with connectedness of T in $\mathrm{H}(C_i)$. Thus, $S' \neq \emptyset$. Moreover, because $S' \subseteq \mathcal{C}(C, \sigma_i)$, we have $x(\mathcal{C}(C, S')) = 0$. □

For the optimal solution (x, y) satisfying (8) let $S' = \{k \in T' : x_k = 0\}$; thus $y_k = u_k$ and $x_k = 1$ for $k \in T' \setminus \mathcal{C}(C, S')$. From the claim above $S' \neq \emptyset$. Then, by the induction hypothesis, as T' is the node set of some connected component $\mathrm{H}(C_h)$ of $\mathrm{H}(C_j)$ with $h < i$,

$$\alpha(T' \setminus \mathcal{C}(C, S')) \geq u(T' \setminus \mathcal{C}(C, S')).$$

Therefore, since $x_k = 0$ for all $k \in \mathcal{C}(C, S')$, we may set $x_k = 0$ for all $k \in T' \setminus \mathcal{C}(C, S')$ as well without decreasing the objective and violating feasibility. Hence, we may assume that $x(\mathcal{C}(C,T) \setminus \mathcal{C}(C, \sigma_i)) = y(T \setminus \mathcal{C}(C, \sigma_i)) = 0$. Consequently, (x, y) satisfies

$$\alpha(T) + u(C \setminus T) = b. \tag{9}$$

On the other hand, for any nonempty $S \subseteq C$, induced-minimality of C and validity of (5) implies

$$\alpha(\mathcal{C}(C, S)) + u(C \setminus \mathcal{C}(C, S)) \leq b \tag{10}$$

as $x_k = y_k = 0$ for $k \in \mathcal{C}(C, S)$ and $x_k = 1,\ y_k = u_k$ for $k \in C \setminus \mathcal{C}(C, S)$ is a feasible point. Subtracting (9) from (10) gives

$$\alpha(T \setminus \mathcal{C}(C, S)) \geq u(T \setminus \mathcal{C}(C, S))$$

as desired. □

Remark 2. It is shown in Lemma 1 that Lemma 3 does not hold for $S = \emptyset$. If C is not induced-minimal, Lemma 3 does not hold either.

Proposition 4. For an induced-minimal cover C and a reverse topological order σ, the lifting coefficient of x_{σ_i} in (5) is

$$\alpha_{\sigma_i} = b - u(C \setminus T_i) - \alpha(T_i \setminus \sigma_i), \tag{11}$$

where T_i is the maximal connected subset of C_i in $\mathrm{H}(C_i)$ containing σ_i.

Proof. The proof of Lemma 3 shows that there is an optimal solution to the lifting problem for σ_i with $y(C \setminus T) = u(C \setminus T)$ and $x(T \setminus \sigma_i) = 0$. Evaluating the objective function for such a solution gives the result. □

Proposition 4 suggests that the lifting coefficients of (5) can be computed efficiently for an induced-minimal cover. In the next theorem we give an order-dependent explicit characterization of the lifting coefficients for an induced-minimal cover.

To this end, for a given a reverse topological order $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{|C|})$ of C and poset $(C, \preccurlyeq)$, we construct another poset $(C, \propto)$ in which $\sigma_i \propto \sigma_k$ if and only if $\sigma_i \preccurlyeq \sigma_k$ or $\sigma_k \in T_i \setminus \mathcal{C}(C, \sigma_i)$, where T_i is defined as in Proposition 4. Relation $\propto$ defines a partial order on C as $\sigma_k \not\preccurlyeq \sigma_i$ for any $\sigma_k \in T_i \setminus \mathcal{C}(C, \sigma_i)$, Thus, $\propto$ imposes an order on each T_i so that $\sigma_i \propto \sigma_h$ for all $h \in T_i \setminus \sigma_i$, as desired. Observe that $(C, \propto)$ can be constructed from $(C, \preccurlyeq)$ iteratively in the order of σ.

Lemma 4. *The Hasse diagram $H_\propto$ of poset $(C, \propto)$ consists of an arborescence for each connected component of* $\mathrm{H}(C)$.

Proof. Because $i \preccurlyeq k$ implies $i \propto k$ and additional binary relations in $\propto$ are within each connected component of $\mathrm{H}(C)$, nodes i and k are connected in $H_\propto$ if and only if they are connected in $\mathrm{H}(C)$. By definition of a Hasse diagram, each component of $H_\propto$ is an acyclic directed graph. If a connected component of $H_\propto$ has distinct root nodes σ_h and σ_i (without loss of generality $h < i$), then σ_h and σ_i are incomparable in poset $(C, \propto)$, contradicting with $\sigma_i \propto \sigma_h$ as $\sigma_h \in T_i$. □

Theorem 3. *For an induced-minimal cover C the coefficients of the cover inequality* (5) *are*

$$\alpha_i = u_i + \lambda(\gamma_i - 1), \quad i \in C, \tag{12}$$

where γ_i is the number of children of node i in the Hasse diagram $H_\propto$.

Proof. We prove the theorem by induction on i. For $i = 1$, σ_i is a leaf node of $H_\propto$. Then $\alpha_{\sigma_1} = b - u(C \setminus \sigma_1) = u_{\sigma_1} - \lambda$ as C is an induced-minimal cover. Suppose the result is true for all $h < i$. From Proposition 4

$$\begin{aligned}
\alpha_{\sigma_i} &= b - u(C \setminus T_i) - \alpha(T_i \setminus \sigma_i) \\
&= b - u(C \setminus T_i) - \sum_{h \in T_i \setminus \sigma_i} (u_h + \lambda(\gamma_h - 1)) \\
&= b - u(C \setminus T_i) - u(T_i \setminus \sigma_i) + \lambda\gamma_i \\
&= u_{\sigma_i} + \lambda(\gamma_i - 1).
\end{aligned}$$

The last two equations follow from Lemma 4, the fact that T_i corresponds to an arborescence in $H_\propto$ for which

$$\sum_{h \in T_i} (\gamma_h - 1) = -1$$

holds, and induced-minimality of cover C. $\square$

Remark 3. Because $\sum_{i \in C}(\gamma_i - 1) = -1$ for arborescence $H_\propto$, it follows that

$$\alpha(C) = u(C) - \lambda = b > 0$$

for an induced-minimal cover C.

Remark 4. In this remark we show that inequalities (5) reduce to a subset of the inequalities given in van de Leensel et al. (1999) and Park and Park (1997) for the 0-1 knapsack polytope with a partial order, by restricting (5) to the face of $\mathcal{F}$ obtained by setting $y = u \circ x$.

Park and Park (1997) define $S \subseteq N$ a *minimal induced cover* if (1) every pair in S is incomparable, (2) $u(\overline{\mathrm{P}}(S)) > b$, and (3) $u(\overline{\mathrm{P}}(S \setminus i)) \leq b$ for all $i \in S$; and they lift knapsack cover inequalities

$$\sum_{i \in S} x_i \leq |S| - 1 \tag{13}$$

for minimal induced covers S.

Note the difference between a minimal induced cover and an induced-minimal cover per Definition 2. If $C \subseteq N$ is an induced-minimal cover, then $\mathrm{L}(C)$ is a minimal induced cover. On the other hand, for a minimal induced cover S, $\overline{\mathrm{P}}(S)$ is not necessarily an induced-minimal cover.

For a given minimal induced cover S let $C = \overline{\mathrm{P}}(S) = P(S) \cup S$. If C is an induced-minimal cover , from Theorem 3, the corresponding cover inequality (5) is

$$y(C) + \sum_{i \in C} (u_i + \lambda(\gamma_i - 1))(1 - x_i) \leq b.$$

Letting $y = u \circ x$, we have

$$\sum_{i \in C} u_i x_i + \sum_{i \in C} (u_i + \lambda(\gamma_i - 1))(1 - x_i) \leq b$$

on the knapsack face. After rewriting the inequality as

$$\sum_{i \in C} \lambda(\gamma_i - 1)(1 - x_i) \leq -\lambda$$

and observing that $\gamma_i = 0$ for $i \in \mathrm{L}(C) = S$ and that $S \cap P(S) = \emptyset$, we obtain the lifted minimal induced cover inequalities

$$\sum_{i \in S} x_i + \sum_{i \in P(S)} (\gamma_i - 1)(1 - x_i) \leq |S| - 1 \tag{14}$$

for the 0-1 knapsack polytope with a partial order, as given in Park and Park (1997) and van de Leensel et al. (1999).

On the other hand, if $C = \overline{\mathrm{P}}(S)$ is not an induced-minimal cover, then we cannot obtain inequalities (14) from cover inequalities (5). For the example with $G = (N, A), N = \{1, \ldots, 5\}, A = \{(3,1), (4,1), (4,2), (5,2)\}$, $u_1 = u_2 = 1$, $u_3 = u_4 = u_5 = 10$ and $b = 25$, minimal induced cover $S = \{1, 2\}$ gives the facet-defining inequality $x_1 + x_2 + (1 - x_4) \leq 1$ for $\mathcal{F}$. In this case, $C = P(S) \cup S = \{1, \ldots, 5\}$ is a cover, but it is not induced-minimal. From Section 4.1, all lifted cover inequalities (5) for this cover are $y(N) + 14(1 - x_1) + 11(1 - x_2) \leq 25$ and $y(N) + 11(1 - x_1) + 14(1 - x_2) \leq 25$, and they do not reduce to inequalities of the form (14) when restricted to the face $y = u \circ x$.

5. Special cases

In this section we consider special graphs for which we have stronger results for lifting covers (not necessarily induced-minimal) than for the general case. The two graphs considered here are arborescences and simple paths.

5.1. Arborescence case.

An arborescence is a directed tree in which every node except the root has an indegree one. Consequently, two nodes on different branches of an arborescence do not have a common successor. Using this property, for an arborescence $G = (N, A)$, we first show that lifting covers is sequence-independent and then give a recursive expression for computing all lifting coefficients in linear time.

Theorem 4. *Consider a cover C and two reverse topological orders σ and $\hat{\sigma}$ of C, where $\hat{\sigma}$ is obtained from σ by interchanging σ_i and σ_j, i.e., $\hat{\sigma}_i = \sigma_j$ and $\hat{\sigma}_j = \sigma_i$. Then the cover inequalities with respect to orders σ and $\hat{\sigma}$ are the same.*

Proof. Without loss of generality, we may assume that $i < j$. First, consider the case $j = i + 1$. Let α and $\hat{\alpha}$ denote the lifting coefficients with respect to orders σ and $\hat{\sigma}$. By definition of σ and $\hat{\sigma}$, we have $\alpha_{\sigma_k} = \hat{\alpha}_{\sigma_k}$ for all $k < i$. Below we argue that $\alpha_{\sigma_i} = \hat{\alpha}_{\sigma_i}$ and $\alpha_{\sigma_{i+1}} = \hat{\alpha}_{\sigma_{i+1}}$, which implies $\alpha_{\sigma_k} = \hat{\alpha}_{\sigma_k}$for $k > i + 1$.

Let $C_i = \{\sigma_1, \sigma_2, \ldots, \sigma_i\}$ and consider

$$\alpha_{\sigma_i} = b - \max_{(x,y) \in \mathcal{F}_{C \setminus C_i}} \left\{ y(C \setminus \mathcal{C}(C, \sigma_i)) + \alpha(\mathrm{S}(\sigma_i) \cap C) + \sum_{k \in C_i \setminus \mathcal{C}(C, \sigma_i)} \alpha_k (1 - x_k) \right\}.$$

If $b \leq u(C \setminus \mathcal{C}(C, \sigma_i))$, it follows Lemmas 1 and 2 that $\alpha_{\sigma_i} = 0$ and, by the same reasoning, $\alpha_k = 0$ for all $k \in \mathrm{S}(\sigma_i) \cap C$ as well. Otherwise, since for an arborescence we have $C_i \setminus \mathcal{C}(C,\sigma_i) = \mathcal{C}(C, C_i \setminus \mathcal{C}(C,\sigma_i))$, it follows from Lemma 1 that $\alpha(C_i \setminus \mathcal{C}(C,\sigma_i)) \le u(C_i \setminus \mathcal{C}(C,\sigma_i))$. Therefore, for both cases,

$$\alpha_{\sigma_i} = (b - u(C \setminus \mathcal{C}(C,\sigma_i)))^+ - \alpha(\mathrm{S}(\sigma_i) \cap C).$$

Now consider $\hat{\alpha}_{\sigma_i}$, or equivalently $\hat{\alpha}_{\hat{\sigma}_{i+1}}$:

$$\hat{\alpha}_{\sigma_i} = b - \max_{(x,y)\in\mathcal{F}_{C\setminus C_{i+1}}} \left\{ y(C \setminus \mathcal{C}(C,\sigma_i)) + \hat{\alpha}(\mathrm{S}(\sigma_i)\cap C) + \sum_{k\in C_{i+1}\setminus\mathcal{C}(C,\sigma_i)} \hat{\alpha}_k(1-x_k) \right\}.$$

Similarly, we have

$$\hat{\alpha}_{\sigma_i} = (b - u(C \setminus \mathcal{C}(C,\sigma_i)))^+ - \hat{\alpha}(\mathrm{S}(\sigma_i)\cap C) = \alpha_{\sigma_i},$$

where the last equality follows from $\alpha_{\sigma_k} = \hat{\alpha}_{\sigma_k}$ for all $k < i$.

That $\alpha_{i+1} = \hat{\alpha}_{i+1}$ follows from symmetry. For the case of $j > i+1$, observe that since both σ and $\hat{\sigma}$ are reverse topological orders, $\sigma_k \notin \mathrm{P}(i)$ for $k = i+1,\dots,j$ and $\sigma_k \notin \mathrm{S}(j)$ for $k = i,\dots,j-1$. Therefore, the result follows by iteratively interchanging σ_i and σ_k, $k = i+1,\dots,j$, and then iteratively interchanging σ_j and σ_k, $k = j-1,\dots,i+1$. □

Theorem 4 shows that the lifting coefficients of inequality (5) are independent of the lifting order σ. Moreover, the proof of the theorem gives a recursive expression

$$\alpha_i = (b - u(C \setminus \mathcal{C}(C,i)))^+ - \alpha(\mathrm{S}(i)\cap C) \tag{15}$$

for the lifting coefficients, which allows all α_i, $i \in C$ to be computed in $O(|N|)$. Inequality (15) can alternatively be written as

$$\alpha(\mathcal{C}(C,i)) = (b - u(C \setminus \mathcal{C}(C,i)))^+ \text{ for } i \in C,$$

which states that the sum of the lifting coefficients for the nodes of C in the subtree rooted at node i equals exactly $(b - u(C \setminus \mathcal{C}(C,i)))^+$.

5.2. **Path case.** Specializing the graphs further, we now consider simple paths, in which case the partial order defined by G reduces a total order. We show that the lifting coefficients can, in this case, be stated explicitly. Moreover, we give a complete polyhedral description of $\mathcal{F}$.

Theorem 5. *Let $G = (N, A)$ be a simple path, where $N = \{1, 2, \dots, n\}$ and $A = \{(i, i+1) : i = 1, ..., n-1\}$. Then for a cover $C = \{i_1, \dots, i_{|C|}\}$ with $i_1 < \cdots < i_{|C|}$, the lifting coefficients of* (5) *can be stated explicitly as*

$$\alpha_{i_k} = \begin{cases} u_{i_k}, & k < k^*, \\ u_{i_k} - \lambda_k, & k = k^*, \\ 0, & k > k^*, \end{cases} \tag{16}$$

where $k^* := \min\{k \in \{1, 2, \dots, |C|\} : \lambda_k := \sum\limits_{j=1}^{k} u_{i_j} - b \ge 0\}$.

Proof. Because a path is a special case of an arborescence, using (15), the recursive expression for the lifting coefficient α_{i_k} is

$$\alpha_{i_k} = (b - u(\mathrm{P}(i_k)\cap C))^+ - \alpha(\mathrm{S}(i_k)\cap C).$$

Since G is a path, we have $S(i_k) \cap C = \{i_{k+1}, ..., i_{|C|}\}$. By definition of k^*, for any $k > k^*$, we have $\sum_{j=1}^{k-1} u_{i_j} \ge \sum_{j=1}^{k^*} u_{i_j} \ge b$. So $\alpha_{i_k} = -\alpha(\mathrm{S}(i_k)\cap C)$. Since $\alpha_{i_k} \ge 0$ (Lemma 1) $\alpha_{i_k} = 0$ holds.

For $k = k^*$, $\alpha(\mathrm{S}(i_{k^*}) \cap C) = \sum_{j=k^*+1}^{|C|} \alpha_{i_j} = 0$. Also, by definition of k^*, $\sum_{j=1}^{k} u_{i_j} < b$ for $k < k^*$. Thus $\alpha_{i_{k^*}} = b - \sum_{j=1}^{k^*-1} u_{i_j} = u_{i_{k^*}} - \lambda_{k^*}$.

Finally for $k < k^*$, $\alpha_{i_k} = b - \sum_{j=1}^{k-1} u_{i_j} - \sum_{j=k+1}^{k^*-1} \alpha_{i_j} - (u_{i_{k^*}} - \lambda_{k^*})$. Therefore,

$$\sum_{j=k}^{k^*-1} \alpha_{i_j} = b - \sum_{j=1}^{k-1} u_{i_j} - (u_{i_{k^*}} - \lambda_{k^*}) = \sum_{j=k}^{k^*-1} u_{i_j}, \text{ for } k = 1, ..., k^* - 1.$$

Solving these equations, we find that $\alpha_{i_k} = u_{i_k}$ for $k = 1, ..., k^* - 1$. □

Remark 5. For an arbitrary acyclic graph, if all nodes in cover $C = \{i_1, i_2, \ldots, i_{|C|}\}$ are on a directed path, then there is obviously a unique reverse topological lifting order of C and the coefficients of inequality (5) are as defined in (16). Furthermore, in this case, $C^* \neq \emptyset$ (defined in Theorem 1) as, in particular, $\alpha_{i_1} = u_{i_1} > 0$.

Also by Remark 1, in this case, for inequality (5) to define a facet of $\mathcal{F}$ it is necessary to have $\{i_{k^*} + 1, \ldots, n\} \subseteq C$ because $\{i_{k^*} + 1, \ldots, n\} \subseteq R$. Furthermore, the third condition of Theorem 1 is satisfied in this case.

Corollary 2. Let $G = (N, A)$ be a simple path, where $N = \{1, 2, \ldots, n\}$ and $A = \{(i, i+1) : i = 1, ..., n-1\}$. Then cover inequality (5) defines a facet of $\mathcal{F}$ if and only if $\{i_{k^*} + 1, \ldots, n\} \subseteq C$.

Theorem 6. *Let $G = (N, A)$ be a simple path, where $N = \{1, 2, \ldots, n\}$ and $A = \{(i, i+1) : i = 1, ..., n-1\}$. Then the trivial inequalities (Proposition 3) and the cover inequalities* (5) *give a complete description of $\mathcal{F}$.*

Proof. Consider an optimization problem $\max\{\pi x + \mu y : (x, y) \in \mathcal{F}\}$ with objective $(\pi, \mu) \neq (\mathbf{0}, \mathbf{0})$ so that the optimal face is proper. We will show that the optimal face is included in a face defined by either one of the trivial inequalities or by a cover inequality to establish the result.

If $\mu_i < 0$, then $y_i = 0$ for all optimal solutions. Therefore, we may assume $\mu \geq 0$. If $\mu_i = 0$ and $\pi_i > 0$, then $x_i = x_{i-1}$ ($= 1$ if $i = 1$); if $\mu_i = 0$ and $\pi_i < 0$, then $x_i = x_{i+1}$ ($= 0$ if $i = n$) for all optimal solutions. Therefore, we may assume that $\mu_i = 0$ implies $\pi_i = 0$. Then, because $(\pi, \mu) \neq (\mathbf{0}, \mathbf{0})$, we have $\mu \neq \mathbf{0}$.

Let $C := \{i \in \{1, \ldots, n\} : \mu_i > 0\}$. If C is not a cover, then $y_i = u_i x_i$ for all $i \in C$ for all optimal solutions. Therefore, we may assume that C is a cover. Consider inequality (5) for the cover C. Let (x, y) be an optimal solution and $k = \max\{i = 1, \ldots, n : x_i = 1\}$. Due to the path structure $x_1 = \cdots = x_k = 1$ and $x_{k+1} = \cdots = x_n = 0$. If $k \geq k^*$ (defined in Theorem 5), then $y(C) = b$, because $\mu_i > 0$ for $i \in C$. On the other hand, if $k < k^*$, then (x, y) is the optimal solution of the lifting problem with respect to x_{k+1}. Therefore, for any optimal solution, inequality (5) is tight. □

Using a combinatorial argument Kis (2005) proves a special case of Theorem 6 for which all capacities u_i, $i \in N$ are constant. Recall from Section 3.1 that polynomial solvability of optimization over $\mathcal{F}$ does not depend on the capacities for the path case.

6. Lifting with P(C)

In this section we generalize inequality (5) by lifting (4) with C as well as $\mathrm{P}(C) \setminus C$. To this end, consider the restriction

$$\overline{\mathcal{F}} := \left\{(x, y) \in \mathbb{B}^N \times \mathbb{R}^N : y(N) \leq b,\ \mathbf{0} \leq y \leq \overline{u} \circ x,\ x_j \leq x_i,\ (ij) \in A\right\}$$

of $\mathcal{F}$, where $\overline{u}_i = 0$ for $i \in \mathrm{P}(C) \setminus C$ and $\overline{u}_i = u_i$ otherwise. Clearly $C \subseteq N$ is a cover for $\mathcal{F}$ if and only if $\overline{\mathrm{P}}(C) = C \cup \mathrm{P}(C)$ is a cover for $\overline{\mathcal{F}}$ and C is an induced-minimal cover for $\mathcal{F}$ if and only if $\overline{\mathrm{P}}(C)$ is an induced-minimal cover for $\overline{\mathcal{F}}$. Therefore, we can directly apply the results in Section 4.1 to lift inequality $y(\overline{\mathrm{P}}(C)) \leq b$ for

$$\overline{\mathcal{F}}_{\overline{\mathrm{P}}(C)} = \left\{ \ (x, y) \in \overline{\mathcal{F}} : x_i = 1 \text{ for all } i \in \overline{\mathrm{P}}(C) \ \right\} \text{ for } C \subseteq N$$

in a reverse topological order σ of $\overline{\mathrm{P}}(C)$ to obtain the cover inequality

$$y(\overline{\mathrm{P}}(C)) + \sum_{i=1}^{|\overline{\mathrm{P}}(C)|} \overline{\alpha}_{\sigma_i}(1 - x_{\sigma_i}) \leq b. \tag{17}$$

Proposition 5. For a cover C and a reverse topological lifting order σ of $\overline{\mathrm{P}}(C)$, inequality

$$y(C) + \sum_{i=1}^{|\overline{\mathrm{P}}(C)|} \overline{\alpha}_{\sigma_i}(1 - x_{\sigma_i}) \leq b \tag{18}$$

is valid for $\mathcal{F}$.

Proof. Observe that $\overline{\mathcal{F}}$ is the projection of $\mathcal{F}$ onto the subspace $\mathcal{S} = \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : y(\mathrm{P}(C) \setminus C) = 0\}$. Then, for $(x, y) \in \mathcal{F}$, the point $(x, \overline{y}) := \mathrm{proj}_{\mathcal{S}}(x, y)$ is satisfied by (17) and its weakening (18). Because the left-hand-side of (18) is the same for $(x, \overline{y})$ and (x, y), inequality (18) is valid for (x, y). □

Theorem 7. *Inequality (18) is facet-defining for $\mathcal{F}$ if*

1. $C^* := \{i \in \overline{\mathrm{P}}(C) : \overline{\alpha}_i > 0\} \neq \emptyset$ *or* $C = N$*;*
2. $R := \bigcap_{i \in C^*} \mathrm{S}(i) \subseteq C$*; and*
3. $C^* \cap S(i) \neq \emptyset$ *for all* $i \in \mathrm{P}(C) \setminus C$*.*

Proof. Follows from the proof of Theorem 1 and lifting the inequality with $x_i,\ i \in \mathrm{P}(C) \setminus C$. The set of points for $i \in \mathrm{P}(C) \setminus C$ in the proof of Theorem 1 are replaced with the following: optimal (x^i, y^i) for the lifting problem with respect to x_i and $(x^k, y^k + \epsilon e_i)$, where (x^k, y^k) is an optimal solution for the lifting problem with respect to x_k for some $k \in C^* \cap S(i)$. □

The next example shows that both of inequalities (5) and (18) can be strong for a given cover C.

Example 3. Consider an instance of $\mathcal{F}$ with $u_i = 2$ for $i = 1, 2, 3, 4$ and $b = 5$ and the precedence graph G given in Figure 3. For the cover $C = \{1, 3, 4\}$ and reverse topological order $\sigma = (4, 3, 1)$, inequality (5) is

$$y_1 + y_3 + y_4 + 3(1 - x_1) + (1 - x_3) + (1 - x_4) \leq 5. \tag{19}$$

For C we have $\overline{\mathrm{P}}(C) = \{1, 2, 3, 4\}$. Then for reverse topological order $\sigma' = (4, 3, 2, 1)$, inequality (18) is

$$y_1 + y_3 + y_4 + 2(1 - x_1) + (1 - x_2) + (1 - x_3) + (1 - x_4) \leq 5. \tag{20}$$

It is easy to check that both of these inequalities are facet-defining for $\mathcal{F}$.

On the other hand, for reverse topological order $\sigma'' = (4, 3, 2, 1)$, inequality (18) gives (19).

Example 3 raises the question whether or when inequalities (5) are implied by inequalities (18). We will show below that if C is an induced-minimal cover, then introducing variables $x_i, i \in \mathrm{P}(C)\setminus C$ to lifting as late as possible, while maintaining the order on C, gives an inequality (18) that is at least as strong as (5). To see this, suppose inequality (5) is obtained by using lifting order σ on C. We define a reverse topological order $\hat{\sigma}$ on $\overline{\mathrm{P}}(C)$ using the labelings π and $\hat{\pi}$ per Definition 3:

- $\hat{\pi}_i < \hat{\pi}_j$ if and only if $\pi_i < \pi_j$ for distinct $i, j \in C$;
- For $i \in \mathrm{P}(C) \setminus C$, $\hat{\pi}_i > \hat{\pi}_j$ for all $j \in C \setminus \mathrm{P}(i)$ s.t. $\pi_k > \pi_j, \forall k \in \mathrm{P}(i) \cap C$.

In other words, elements of $\mathrm{P}(C)\setminus C$ are as late as possible in the reverse topological order $\hat{\sigma}$ while maintaining the order σ among the elements of C. For instance, for $i \in \mathrm{P}(C) \setminus C$ if $\mathrm{P}(i) \cap C = \emptyset$, then $\hat{\pi}_i > \hat{\pi}_j$ for all $j \in C \setminus P(i) = C$.

Lemma 5. *For an induced-minimal cover C, let α and $\hat{\alpha}$ denote the lifting coefficients of* (5) *and* (18) *with respect to σ and $\hat{\sigma}$. Then α and $\hat{\alpha}$ satisfy*

$$\alpha_i = \hat{\alpha}_i + \hat{\alpha}(\mathcal{D}_i), \text{ for } i \in C,$$

where $\mathcal{D}_i := \{j \in \mathrm{P}(C) \setminus C : \hat{\pi}_k < \hat{\pi}_j < \hat{\pi}_i\}$ and $k \in C$ such that $\pi_k = \pi_i - 1$.

Proof. We show the lemma by induction on the lifting order. First, from the construction of the lifting order $\hat{\sigma}_i$, $\hat{\sigma}_1 = \sigma_1 \in C$ and $\mathcal{D}_{\sigma_1} = \emptyset$. So $\alpha_{\sigma_1} = \hat{\alpha}_{\sigma_1} + \hat{\alpha}(\mathcal{D}_{\sigma_1})$.

Assume that $\alpha_{\sigma_i} = \hat{\alpha}_{\sigma_i} + \hat{\alpha}(\mathcal{D}_{\sigma_i})$ holds for all $i < t$. From Proposition 4,

$$\alpha_{\sigma_t} = b - u(C \setminus T_t) - \alpha(T_t \setminus \sigma_t).$$

Let T'_t be the maximal connected subset of $C_{\pi'_{\sigma_t}}$ containing σ_t as in Proposition 4. Then $u(C \setminus T_t) = \overline{u}(\overline{\mathrm{P}}(C) \setminus T'_t)$. By induction hypothesis, $\alpha(T_t \setminus \sigma_t) = \hat{\alpha}(T'_t \setminus \sigma_t) - \hat{\alpha}(\mathcal{D}_{\sigma_t})$. Hence $\alpha_{\sigma_t} = b - \overline{u}(\overline{\mathrm{P}}(C) \setminus T'_t) - \hat{\alpha}(T'_t \setminus \sigma_t) + \hat{\alpha}(\mathcal{D}_{\sigma_t}) = \hat{\alpha}_{\sigma_t} + \hat{\alpha}(\mathcal{D}_{\sigma_t})$. □

Now we are ready to describe the relationship between inequalities (5) and (18). If C is an induced-minimal cover, then there exists a lifting order for $\overline{\mathrm{P}}(C)$ for which inequality (18) is at least as strong as (5).

Theorem 8. *If C is an induced-minimal cover, then inequality* (5) *with lifting order σ is implied by inequality* (18) *with lifting order $\hat{\sigma}$, the precedence and bound constraints.*

Proof. Consider inequality (18) obtained by using the lifting order $\hat{\sigma}$ as defined above:

$$y(C) + \sum_{i \in \overline{\mathrm{P}}(C)} \hat{\alpha}_i (1 - x_i) \leq b,$$

which is equivalent to

$$\sum_{i \in P(C)\setminus C : P(i) \cap C = \emptyset} \hat{\alpha}_i (1 - x_i) + y(C) + \sum_{i \in C} \hat{\alpha}_i (1 - x_i) + \sum_{i \in C} \sum_{j \in \mathcal{D}_i} \hat{\alpha}_j (1 - x_j) \leq b. \quad (21)$$

As $\hat{\alpha} \geq \mathbf{0}$ and $x \leq \mathbf{1}$, (21) can be relaxed to

$$y(C) + \sum_{i \in C} \hat{\alpha}_i (1 - x_i) + \sum_{i \in C} \sum_{j \in \mathcal{D}_i} \hat{\alpha}_j (1 - x_j) \leq b. \quad (22)$$

Since the precedence constraints $x_j \leq x_i$ for $j \in \mathcal{D}_i$ hold by construction and $\hat{\alpha} \geq \mathbf{0}$, inequality

$$\sum_{i \in C} \sum_{j \in \mathcal{D}_i} \hat{\alpha}_j (x_j - x_i) \leq 0 \quad (23)$$

is valid for $\mathcal{F}$. Adding (22) to (23), we obtain

$$y(C) + \sum_{i \in C} (\hat{\alpha}_i + \hat{\alpha}(D_i))(1 - x_i) \leq b,$$

which, from Lemma 5, is inequality (5) for induced-minimal cover C. Hence (5) is dominated by precedence constraints and inequality (18). $\square$

7. Separation

In this section we will discuss the separation problem for inequality (5). Given a fractional solution $(\bar{x}, \bar{y}) \in \widehat{\mathcal{F}}$, the question is to decide whether there is a cover inequality (5) violated by $(\bar{x}, \bar{y})$ and to find such an inequality if it exists.

Even for a given cover C as different lifting orders give different cover inequalities, it is not obvious how to pick the right order to cut off a point. In the following we discuss how to choose a lifting order for a given induced-minimal cover C and a fractional point $(\bar{x}, \bar{y})$.

We write the problem of choosing a lifting order as

$$z_{LO} = \max_{\sigma \in \Sigma} \left\{ \bar{y}(C) + \sum_{i \in C} \alpha_i^\sigma (1 - \bar{x}_i) - b \right\}, \tag{24}$$

where Σ is the set of all reverse topological orders on C and α^σ are the lifting coefficients for reverse topological order σ. We call an optimal solution σ^* for (24) as *an optimal lifting order* of C with respect to $(\bar{x}, \bar{y})$. There is a cover inequality (5) with C violated by $(\bar{x}, \bar{y})$ if and only if $z_{LO} > 0$.

We will construct a relaxation for (24) to solve it. To this end, for any subset $S \subseteq C$, let $l(\mathcal{C}(C, S))$ denote the number of connected components of the Hasse diagram $H_{\preceq}(\mathcal{C}(C, S))$. Let $g : 2^C \to \mathbb{R}$ be a set function defined as

$$g(S) = u(\mathcal{C}(C, S)) - l(\mathcal{C}(C, S))\lambda.$$

From Theorem 3 and Remark 3, $g(S)$ is the sum of the lifting coefficients of x_i, $i \in \mathcal{C}(C, S)$ for any reverse topological order σ such that $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,S)|}\} = \mathcal{C}(C, S)$; that is, reverse topological orders in which the first $|\mathcal{C}(C, S)|$ elements are $\mathcal{C}(C, S)$. Such an order σ exists because S has no successors in $C \setminus \mathcal{C}(C, S)$. Note that $g(S)$ is independent of the lifting order on $\mathcal{C}(C, S)$.

Lemma 6. *For any $S \subseteq C$ and reverse topological order σ, $g(S) \geq \alpha^\sigma(\mathcal{C}(C, S))$.*

Proof. Let π denote the inverse function of σ as in Definition 3. There exists a reverse topological order σ' with inverse function π' such that $\{\sigma'_1, ..., \sigma'_{|\mathcal{C}(C,S)|}\} = \mathcal{C}(C, S)$ and $\pi'_i < \pi'_j$ if and only if $\pi_i < \pi_j$ for any $(i, j) \in \mathcal{C}(C, S)$; that is, the elements of $\mathcal{C}(C, S)$ are earlier in the lifting sequence σ' than any element in $C \setminus \mathcal{C}(C, S)$. Consider the Hasse diagrams $H_\propto$ and $H_{\propto'}$ corresponding to σ and σ', respectively. Because in $H_\propto$ each $k \in C \setminus \mathcal{C}(C, S)$ with a parent $i \in \mathcal{C}(C, S)$ must have at least one child in $\mathcal{C}(C, S)$, moving k to a later position in the sequence than its parent, does not decrease the number of children of the parent i in $H_\propto$. Thus, $\gamma'_i \geq \gamma_i$ for all $i \in \mathcal{C}(C, S)$. Then $g(S) = \alpha^{\sigma'}(\mathcal{C}(C, S)) \geq \alpha^\sigma(\mathcal{C}(C, S))$ by Theorem 3. $\square$

Now we define the optimization problem

$$\begin{aligned} z_R = \max \quad & cw + \bar{y}(C) - b \\ \text{s.t.} \quad & w(S) \le g(S), \ \forall S \subseteq C \\ & w \in \mathbb{R}_+^C, \end{aligned} \tag{25}$$

where $c_i = 1 - \bar{x}_i$ for $i \in C$. Let us compare problem (25) with (24). For any $\sigma \in \Sigma$, we have $\alpha^\sigma \ge 0$, and from Lemma 6, for any $S \subseteq C$, $\alpha^\sigma(S) \le \alpha^\sigma(\mathcal{C}(C,S)) \le g(S)$; thus, α^σ is feasible for (25). Then, (25) is a relaxation of (24) and $z_R \ge z_{LO}$.

Proposition 6. The function g is nondecreasing and submodular.

Proof. For any $S \subseteq B \subseteq C$, we have $\mathcal{C}(C,S) \subseteq \mathcal{C}(C,B)$. There exists a reverse topological order σ such that $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,S)|}\} = \mathcal{C}(C,S)$ and $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,B)|}\} = \mathcal{C}(C,B)$. Let α^σ denote the lifting coefficients with lifting order σ. From Lemma 1, $\alpha^\sigma \ge 0$. So

$$g(B) = \alpha^\sigma(\mathcal{C}(C,B)) \ge \alpha^\sigma(\mathcal{C}(C,S)) = g(S).$$

Thus g is nondecreasing.

For any $S \subseteq B \subseteq C$ and $k \notin B$, there exists a reverse topological order σ such that $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,S)|}\} = \mathcal{C}(C,S)$, $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,S\cup k)|}\} = \mathcal{C}(C,S\cup k)$ and $\{\sigma_1, ..., \sigma_{|\mathcal{C}(C,B\cup k)|}\} = \mathcal{C}(C,B\cup k)$. Let α^σ denote the lifting coefficients with respect to σ. From Lemma 6, $g(B) \ge \alpha^\sigma(\mathcal{C}(C,B))$. So

$$g(B\cup k) - g(B) \le \alpha^\sigma(\mathcal{C}(C,B\cup k)) - \alpha^\sigma(\mathcal{C}(C,B)) = \alpha^\sigma(\mathcal{C}(C,k) \setminus \mathrm{S}(B)),$$

$$g(S\cup k) - g(S) = \alpha^\sigma(\mathcal{C}(C,S\cup k)) - \alpha^\sigma(\mathcal{C}(C,S)) = \alpha^\sigma(\mathcal{C}(C,k) \setminus \mathrm{S}(S)).$$

Because $\mathcal{C}(C,k) \setminus \mathrm{S}(B) \subseteq \mathcal{C}(C,k) \setminus \mathrm{S}(S)$ and $\alpha^\sigma \ge 0$, we have

$$g(B\cup k) - g(B) \le g(S\cup k) - g(S).$$

Hence g is submodular. □

For an order σ on C, let $C_i = \{\sigma_1, \dots, \sigma_i\}$ and

$$w_i^\sigma = \begin{cases} g(C_i), & i = 1, \\ g(C_i) - g(C_{i-1}), & i = 2, ..., |C|. \end{cases}$$

Lemma 7. *For any order σ satisfying $\bar{x}_{\sigma_1} \le \bar{x}_{\sigma_2} \le \cdots \le \bar{x}_{\sigma_{|C|}}$, w^σ is an optimal solution for* (25)*.*

Proof. From Proposition 6, the feasible region of (25) is the polymatroid associated with (C,g). Therefore, the greedy solution w^σ is optimal (Nemhauser and Wolsey, 1988). □

Remark 6. There may be multiple orders σ satisfying the condition in Lemma 7. Let the set of all such orders be $\widetilde{\Sigma}$. There must exist a reverse topological order $\sigma^* \in \widetilde{\Sigma}$. For if $\sigma \in \widetilde{\Sigma}$ is not a reverse topological order, then there is a pair $i > j$ such that $(\sigma_j, \sigma_i) \in A$, implying $\bar{x}_{\sigma_i} \le \bar{x}_{\sigma_j}$ as $(\bar{x}, \bar{y}) \in \widehat{\mathcal{F}}$. But since $\sigma \in \widetilde{\Sigma}$, we also have $\bar{x}_{\sigma_i} \ge \bar{x}_{\sigma_j}$, implying $\bar{x}_{\sigma_i} = \bar{x}_{\sigma_j}$. Therefore, we can obtain a new order $\sigma' \in \widetilde{\Sigma}$ by swapping σ_i and σ_j. Then, repeating as necessary, from any order σ in $\widetilde{\Sigma}$, we can obtain a reverse topological order σ^* in $\widetilde{\Sigma}$.

Theorem 9. *An optimal lifting order for* (24) *is a reverse topological order σ^* satisfying $\bar{x}_{\sigma_1^*} \le \bar{x}_{\sigma_2^*} \le \cdots \le \bar{x}_{\sigma_{|C|}^*}$.*

Proof. From Remark 6, there exists a reverse topological order $\sigma^* \in \Sigma$ satisfying $\bar{x}_{\sigma_1^*} \leq \bar{x}_{\sigma_2^*} \leq \cdots \leq \bar{x}_{\sigma^*_{|C|}}$. By Lemma 7, w^{σ^*} is optimal for (25).

Now consider the lifting coefficients α^{σ^*} for σ^*. From Theorem 3, we have $\alpha^{\sigma^*}(C_i) = g(C_i)$ for all $i = 1, \ldots, |C|$. Hence $w^* = \alpha^{\sigma^*}$. Because $z_{LO} \leq z_R$, we have $z_{LO} = z_R$ and σ^* is an optimal lifting order. □

8. Computational experience

In this section we present a summary of our preliminary computational experiments performed to test the effectiveness of the inequalities introduced in the paper as cutting planes to solve optimization problems on $\mathcal{F}$. The experiments are performed using the MIP solver of CPLEX Version 10.1 on a 3 GHz Pentium4 Linux workstation.

For these experiments we randomly generate precedence graphs with varying number of nodes n and arc densities d. We use a simple heuristic to select a cover: For a given fractional linear programming solution (x, y), we let $C = \{i \in N : x_i > 0 \text{ and } y_i = u_i x_i\}$. If C is not a cover, no cut is generated. If C is a cover, then we generate a lifted cover inequality (5) by solving maximum closure problems to compute the lifting coefficients (Theorem 2) in the greedy order described in Theorem 9. In order to generate a second inequality, we remove a leaf ($\ell \in \mathrm{L}(C)$) one at a time until an induced-minimal cover C' is obtained. Then, we construct a lifted inequality (18) from C' using the greedy order on $\overline{\mathrm{P}}(C')$. Thus, in each cut generation phase, we add up to two inequalities to the formulation. Cuts are added only at the root node. CPLEX primal heuristics are turned off to eliminate their impact on the computations.

Table 1. Computational impact of the cuts.

		CPLEX		CPLEX + Cuts		
n	d	Gap (%)	Time (sec)	# Cuts	Gap (%)	Time (sec)
500	0.1	21.3	9	15	7.7	9
500	0.2	11.8	24	15	4.3	18
500	0.4	3.4	37	5	1.3	21
1000	0.1	23.2	78	10	3.0	29
1000	0.2	17.3	181	12	3.1	60
1000	0.4	10.5	378	17	1.7	211
2000	0.1	18.2	830	9	1.9	235
2000	0.2	17.2	2058	12	5.0	320
2000	0.4	10.3	2967	10	1.0	896
Average		**14.8**	**729**	**12**	**3.2**	**200**

In Table 1 we compare the integrality gap at the root node of the search tree and the computational time with and without adding the cuts. Each row of the table represents the average of five instances. We observe from the table that the integrality gap with and without cuts is smaller for dense graphs. Both the integrality gap and the computational time improve significantly after adding the cuts. The cuts reduce the average integrality gap 78% and the computation time by 73% for our test problems. Furthermore, the speed-up in computation time improves with the size of the graphs.

9. Conclusion

In this paper we studied the polyhedral structure of the flow set with partial order. This set is a common substructure of many investment and scheduling problems with precedence relations among activities. It is also a generalization of precedence-constrained knapsack set and the single-node fixed-charge flow set.

We gave a polyhedral analysis based on lifting inequalities from covers. In particular, we described structural results on the lifting coefficients, which led to efficient computation of the inequalities and, in some cases, explicit characterizations. For the polynomial-solvable total-order case, we gave a complete polyhedral description of the flow set with partial order. We showed that an optimal lifting order for a given induced-minimal cover and a fractional solution can be computed by the greedy algorithm. Our preliminary computational experience shows that the identified inequalities are effective when used as cutting planes.

10. Appendix: The relation between $\mathcal{F}_{\leq}$ and $\mathcal{F}_{\geq}$

In this section we show that inequalities described for $\mathcal{F} := \mathcal{F}_{\leq}$ in the paper can be easily converted to valid inequalities for $\mathcal{F}_{\geq}$ as in the fixed-charge flow set without the precedence constraints. The proof below follows that of Padberg et al. (1984) word by word; it is included for completeness.

Proposition 7. If $\pi x + \mu y \geq v$ is valid for $\mathcal{F}_{\leq}$, then

$$\pi x + (\mu - t\mathbf{1})y \geq v - tb$$

is valid for $\mathcal{F}_{\geq}$, where $t = \min\{\mu_j : j \in N\}$.

Proof. For any $(x, y) \in \mathcal{F}_{\geq}$, there exists a point $(\bar{x}, \bar{y}) \in \mathcal{F}_{=}$, such that $\bar{x} = x$ and $\bar{y} \leq y$. Thus, if $\pi x + \mu y \geq v$ is valid for $\mathcal{F}_{\leq}$, then $\pi \bar{x} + \mu \bar{y} \geq v$ holds. Since $\mu - t\mathbf{1} \geq 0$ and $\mathbf{1}^T \bar{y} = b$, we have

$$\pi x + (\mu - t\mathbf{1})y = \pi \bar{x} + \mu \bar{y} - tb + (\mu - t\mathbf{1})(y - \bar{y}) \geq v - tb.$$

So the inequality is a valid inequality for $\mathcal{F}_{\geq}$. □

An immediate application of Proposition 7 for the cover inequalities (5) for $\mathcal{F}_{\leq}$ gives cover inequalities for $\mathcal{F}_{\geq}$.

Corollary 3. For a cover $C \subseteq N$, the cover inequality

$$\sum_{i \in C} \alpha_i (1 - x_i) - y(N \setminus C) \leq 0, \tag{26}$$

where α is defined as in (5), is valid for $\mathcal{F}_{\geq}$.

References

Aghezzaf, E. H., Magnanti, T. L., and Wolsey, L. A. (1995). Optimizing constrained subtrees of trees. *Mathematical Programming*, 71:113–126.

Ahuja, R. K., Magnanti, T. L., and Orlin, J. B. (1993). *Network flows: theory, algorithms, and applications.* Prentice Hall.

Atamtürk, A. (2001). Flow pack facets of the single node fixed-charge flow polytope. *Operations Research Letters*, 29:107–114.

Atamtürk, A. (2004). Sequence independent lifting for mixed–integer programming. *Operations Research*, 52:487–490.

Atamtürk, A. and Zhang, M. (2007). Two-stage robust network flow and design under demand uncertainty. *Operations Research*, 55:662–673.

Boyd, E. A. (1993). Polyhedral results for the precedence-constrained knapsack problem. *Discrete Applied Mathematics*, 41:185–201.

Gu, Z., Nemhauser, G. L., and Savelsbergh, M. W. P. (1999). Lifted flow cover inequalities for mixed 0-1 integer programs. *Mathematical Programming*, 85:439–467.

Gu, Z., Nemhauser, G. L., and Savelsbergh, M. W. P. (2000). Sequence independent lifting in mixed integer programming. *Journal of Combinatorial Optimization*, 4:109–129.

Hochbaum, D. S. and Chen, A. (2000). Performance analysis and best implementations of old and new algorithms for the open-pit mining problem. *Operations Research*, 48:894–914.

van de Leensel, R. L. M. J., van Hoesel, C. P. M., and van de Klundert, J. J. (1999). Lifting valid inequalities for the precedence-constrained knapsack problem. *Mathematical Programming*, 86:161–185.

Kis, T. (2005). A branch-and-cut algorithm for scheduling of projects with variable-intensity activities. *Mathematical Programming*, 103:515–539.

Nemhauser, G. L. and Wolsey, L. A. (1988). *Integer and Combinatorial Optimization*. John Wiley and Sons, New York.

Padberg, M. W., Van Roy, T. J., and Wolsey, L. A. (1984). Valid linear inequalities for fixed charge problems. *Operations Research*, 32:842–861.

Park, K. and Park, S. (1997). Lifting cover inequalities for the precedence-constrained knapsack problems. *Discrete Applied Mathematics*, 72:219–241.