Exam

Due Date: 4/18/2005 - 1:30 PM

Important notes: This is an open-book, open-notes exam. Do not discuss the contents of this exam or your solutions with anyone. Show that you understand the course material very well. Make sure that your answers are legible, statements are precise, and notation is well-defined. Good luck!

1. (30pnts)

(a) Consider the polytope

\[ P = \left\{ x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_{ij} \leq 1, \ i = 1, 2, \ldots, n; \ \sum_{i=1}^{n} x_{ij} \geq 1, \ j = 1, 2, \ldots, n \right\}. \]

Determine the dimension of \( P \) and its facets.

(b) Consider the polytope

\[ Q = \left\{ x \in \mathbb{R}^n_+ : \sum_{j=1}^{n} x_{ij} = 1, \ i = 1, 2, \ldots, n; \ \sum_{i=1}^{n} x_{ij} = 1, \ j = 1, 2, \ldots, n \right\}. \]

Determine the dimension of \( Q \) and its facets.

2. (70pnts) Consider the following small instance of the uncapacitated facility location problem

\[
\begin{align*}
\text{(UFL)} \quad \min & \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} y_{ij} + \sum_{i=1}^{m} f_i x_i \\
\text{s.t.} & \quad (1) \quad \sum_{i=1}^{m} y_{ij} = 1, \ j = 1, \ldots, n \\
& \quad (2) \quad y_{ij} - x_i \leq 0, \ i = 1, \ldots, m; j = 1, \ldots, n \\
& \quad (3) \quad y \in \mathbb{R}^{m \times n}, x \in \{0, 1\}^n
\end{align*}
\]

with \( n = m = 3, f_i = 15, 1 \leq i \leq 3, \) and \( c_{ij} = \begin{cases} 10, & \text{if } i = j, 1 \leq i, j \leq 3, \\ 1, & \text{if } i \neq j, 1 \leq i, j \leq 3. \end{cases} \)

(a) Formulate and solve the Lagrangian dual of UFL with respect to the demand constraints (1) using the subgradient optimization method with step length \( 1/k \) for \( k \)th iteration. Is the complementary primal solution for the optimal dual solution feasible? Explain.

(b) Formulate the Benders reformulation of UFL by projecting out the continuous variables and solve it with the Benders decomposition algorithm. Use \( x = (1, 1, 1) \) as the initial solution for the restricted master problem.
(c) Formulate the Dantzig-Wolfe reformulation of UFL using the extreme points of the convex hull of solutions to (2) and (3) and solve it with the branch-and-price algorithm.

(d) Can you find a valid inequality for UFL that cuts the optimal solution of its LP relaxation? What is the optimal value of the new LP relaxation after adding this cut?

Note 1: This problem is to be done by hand to show your understanding of the material. Do not use AMPL implementations of the decomposition algorithms to do the problem. You can use an LP/MIP solver to solve the master and subproblems. But clearly illustrate the progress of each algorithm, i.e., describe the updated instances of the master and subproblems, and corresponding solutions and objectives.

Note 2: Solve the subproblems to optimality and indicate the relevant lower bound, upper bound, and the optimality gap at each iteration.

Note 3: Limit the maximum number of iterations to 10.