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Conditions for Symmetric Running in Single- and Double-Support

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ABSTRACT

Even and odd functions of time – symmetric functions – have been used to simplify the control of running robots and have been observed in the body and leg motion of running animals. This paper explores the relationship between symmetric motion of the body and symmetric use of the legs in three planar models. The algebra of symmetric functions is applied to the equations of motion to show that symmetric leg actuation is required for symmetric body motion in the single-support case, but not in the double-support case unless additional constraints are imposed.

1. Introduction

There are many patterns of motion a legged system can use to propel itself forward in steady-state travel. One interesting class of such patterns is the symmetric motions. When motion of the body through space is described by appropriate even and odd functions of time, the body experiences zero average acceleration in the forward, vertical, and angular directions. These kinds of symmetric motions have been used to control the behavior of legged robots (Raibert 1986a), and data from animals suggest that they sometimes run with symmetric motion (Hildebrand 1966, 1976; Raibert 1986b, 1986c).

In this paper we explore the necessity of symmetric motion for simple legged models. The models assume massless legs, mechanically symmetric bodies, and motion restricted to the sagittal plane. The main finding is that in order to run with symmetric body motion, symmetric actuation is required for the single-support model but not for the double-support model. If additional constraints are imposed, such as requiring the legs to act like springs and the hip forces to have the same direction, then symmetric actuation is also required in the double-support case.

2. Review of Symmetry

Motivation for symmetry in running comes from considering the requirements for steady-state. In steady-state, the net acceleration of the system over an entire stride is zero. There is no forward acceleration during the flight phase if we assume negligible air resistance, so the acceleration during stance must integrate to zero

$$\int_{stride} f_x \, dt = 0. \tag{1}$$

where f_x is the forward force acting on the body. If the time origin is defined so that $t = 0$ midway through the stance phase and if f_x is an odd function of time, $f_x(t) = -f_x(-t)$, then (1) is satisfied—the integral of an odd function over symmetric limits is zero.

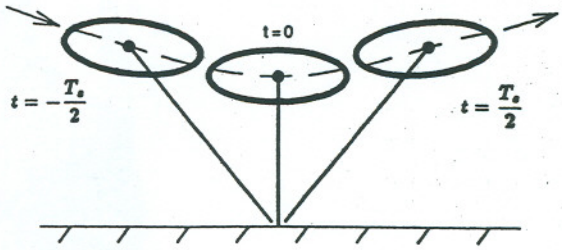


Figure 1: Symmetric running. The left-most drawing shows the system at the point of touchdown; the center drawing shows the system halfway through stance, and the rightmost drawing shows the system just before liftoff. If the origin, $x = t = 0$, is defined to be at halfway point, the trajectory of the body's center of mass can be described by symmetric functions of time.

For a legged system like the one shown in Fig. 1, a forward force that varies as an odd function of time can be achieved by arranging for the body and legs to move with patterns of even and odd symmetry. Symmetric body motion is given by

$$\begin{aligned} x(t) &= -x(-t) \\ z(t) &= z(-t) \\ \phi(t) &= -\phi(-t) \end{aligned} \tag{2}$$

where x, z, and ϕ specify the forward, vertical, and angular position of the body in the sagittal plane. Symmetric leg motion is given by

$$\begin{aligned} \theta_i(t) &= -\theta_j(-t) \\ r_i(t) &= r_j(-t) \end{aligned} \tag{3}$$

and symmetric actuation is given by

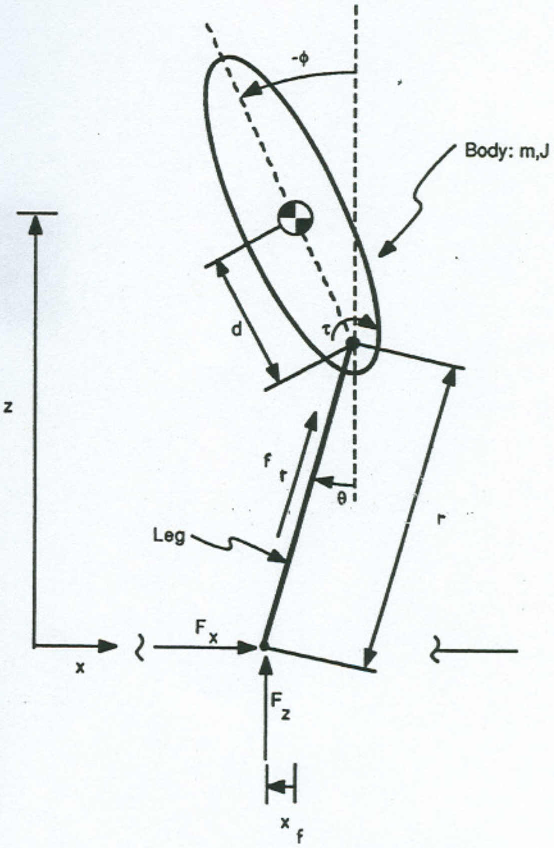
$$\begin{aligned} \tau_i(t) &= -\tau_j(-t) \\ f_i(t) &= f_j(-t). \end{aligned} \tag{4}$$

where θ is the angle of the leg with respect to the vertical, r is the length of the leg, τ is hip torque, and f is axial leg thrust. Subscripts identify the legs. For single-support $i = j = 1$. For double-support $i = 1, j = 2$.

One can understand symmetry by considering the instant during stance when the foot is directly under the center of mass and the vertical component of the body's velocity is zero (see $t = 0$ in Fig.1). If the mechanical system has left-right symmetry at this instant, then the expected behavior going forward in time is the same as the past behavior going backward in time, except for a left-right reversal. The equivalence is formulated in the symmetry equations (2). More details of symmetry theory, including the case of multiple support can be found in (Raibert 1986c).

In general, symmetric body motion is only sufficient for steady-state behavior—many patterns of body and leg motion can produce zero net acceleration of the system throughout a stride. In this paper we assume that the body moves with the symmetries given in (2) and examine the implications for motion of the legs.

3. Single-Support Model



$$m\ddot{x} = f_r \sin\theta - \frac{\tau}{r}\cos\theta \tag{5}$$

$$m\ddot{z} = f_r \cos\theta + \frac{\tau}{r}\sin\theta - mg \tag{6}$$

$$J\ddot{\phi} = -f_r d \sin(\theta - \phi) + \frac{\tau}{r} d \cos(\theta - \phi) + \tau \tag{7}$$

Figure 2. Single-support model: one leg on the ground during stance. Position and orientation of the body's center of mass is (x, z, ϕ). The offset between the hip and the center of mass is d. Distance from the center of mass to the foot is x_f. The leg has length r and makes an angle θ with the vertical. Axial leg force f_r and hip torque τ can be resolved into Cartesian components f_x and f_z.

The single-support case is illustrated in Fig. 2. A massless leg of length r is connected by a hinge joint to a rigid planar body of mass m and moment-of-inertia J. The distance between the hip and the body's center of mass is d. The body is free to move in the plane. The pitch angle of the body ϕ is defined so that the center of mass is aligned directly above the hip when $\phi = 0$. The forward displacement of the body x is defined for each stance phase so that $x(t = 0) = 0$. The angle of the leg with respect to the vertical is θ.

We focus on the stance interval, which begins at the instant the foot first touches the ground t_{td} and ends at the instant the foot last touches the ground t_{lo}. During this interval the leg can apply axial force, $f_r(t) \geq 0$ and the hip can exert arbitrary continuous torque $\tau(t)$ between the body and the leg. We assume that friction between the foot and the floor is sufficient to prevent the foot from slipping during the stance phase. The equations of motion are found in (5- 7).

3.1. Leg Symmetry in Single-Support

Assertion. If body motion is symmetric in single-support, then leg motion must be symmetric.

Proof: Since the restrictions on body motion are expressed in Cartesian coordinates, we first resolve body moment (7) into Cartesian components.

$$J\ddot{\phi}(t) = -f_x(t)z(t) - f_z(t)\Big(x_f(0) - x(t)\Big) \qquad (8)$$

where

f_x	is the horizontal force by the leg,
z	is the vertical position of the body,
f_z	is the vertical force exerted by the leg,
x_f	is the horizontal distance from
the center	of mass to the foot, and
x	is the horizontal position of the body.

The second component of body moment is due to the vertical force f_z acting through a horizontal moment arm which is the horizontal displacement of the foot from the center of mass, $x_f(t) = x_f(0) - x(t)$. Because the foot is fixed with respect to the ground during stance, the length of the moment arm is a function of $x(t)$, the distance from the origin to the body's center of mass. Rearranging (8) to solve for $x_f(0)$

$$x_f(0) = (-1/f_z)\Big(J\ddot{\phi} + f_x z\Big) + x. \qquad (9)$$

Differentiating (2), we note that body moment $J\ddot{\phi}$, horizontal force $f_x = m\ddot{x}$, and position x must be odd functions of time and that vertical force $f_z = m(\ddot{z} + g)$ and position z are even functions.

The algebra of symmetric functions, summarized in the Appendix, can be used to show that the right hand side of equation (9) is odd during stance. The left hand side is constant. Thus $x_f(0) = 0$. The only way to satisfy the symmetry constraints and the equations of motion is for the foot to be located directly under the center of mass when $t = 0$.

With the foot constrained to be at the origin, leg angle and leg length depend only on body position

$$\tan\theta = \frac{(x - d\sin\phi)}{(z - d\cos\phi)} \qquad (10)$$
$$r = \sqrt{(x - d\sin\phi)^2 + (z - d\cos\phi)^2}$$

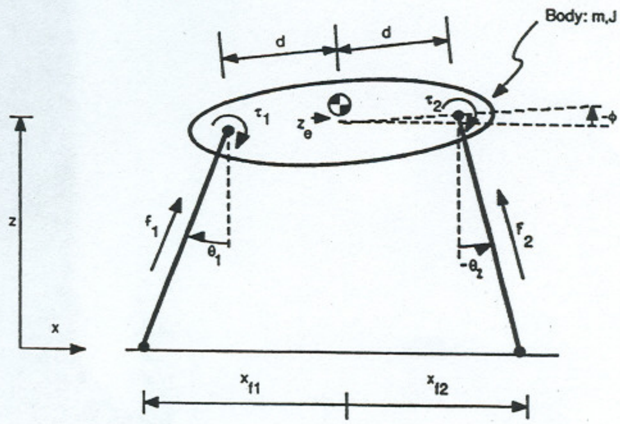
Applying the rules in the Appendix to (10), θ must be odd and r even. This establishes that leg motion must be symmetric. Writing actuator forces in terms of body forces and leg angle θ

$$\begin{aligned} \tau/r &= -f_x\cos\theta + f_z\sin\theta \\ f_r &= f_x\sin\theta + f_z\cos\theta. \end{aligned} \qquad (11)$$

Substituting odd θ and even r in (11) shows that τ is odd and f_r is even.

We next consider the case where two legs provide support simultaneously during the stance phase. For the case of double-support, we find that leg motion does not have to be symmetric in order for a legged system to travel with symmetric body motion.

4. Double-Support



$$\begin{aligned} m\ddot{x} &= f_{r,1}\sin\theta_1 - \frac{\tau_1}{r_1}\cos\theta_1 + f_{r,2}\sin\theta_2 - \frac{\tau_2}{r_2}\cos\theta_2, \\ m\ddot{z} &= f_{r,1}\cos\theta_1 + \frac{\tau_1}{r_1}\sin\theta_1 + f_{r,2}\cos\theta_2 + \frac{\tau_2}{r_2}\sin\theta_2 - mg, \\ J\ddot{\phi} &= f_{r,1}\Big(d\cos\delta_1 - z_e\sin\delta_1\Big) + \frac{\tau_1}{r_1}\Big(d\sin\delta_1 + z_e\cos\delta_1 + r_1\Big) \\ &\quad - f_{r,2}\Big(d\cos\delta_2 + z_e\sin\delta_2\Big) + \frac{\tau_2}{r_2}\Big(-d\sin\delta_2 + z_e\cos\delta_2 + r_2\Big). \end{aligned} \qquad (12)$$

Figure 3. Double-support model: two legs on the ground at a time. Position of the center of mass of the body is $[x, z, \phi]$. The horizontal distance between each hip and the center of mass is d. The vertical offset of the body's center of mass from the center of the hips is z_e, and $\delta_i = \theta_i - \phi$ is the angle between leg i and the line connecting the hips. Other leg variables are defined as in Fig. 1.

Consider the model shown in Fig. 3. The center of mass is equidistant from the two hips. The legs and hips are like those of the previous model and motion is again planar. Equations of motion for the system during double-support are in (12).

The equations of motion for double-support in Cartesian coordinates are:

$$m\ddot{x} = f_{x,1} + f_{x,2} \tag{13}$$

$$m\ddot{z} = f_{z,1} + f_{z,2} - mg \tag{14}$$

$$J\ddot{\phi} = -\left(f_{x,1} + f_{x,2}\right)z - f_{z,1}x_{f,1} - f_{z,2}x_{f,2}, \tag{15}$$

where

$f_{x,i} = f_{r,i}\cos\theta_i + \frac{\tau_i}{r_i}\sin\theta_i$	is the horizontal force exerted by leg i,
$f_{z,i} = f_{r,i}\sin\theta_i - \frac{\tau_i}{r_i}\cos\theta_i$	is the vertical force exerted leg i,
z	is the vertical position of the body's c.o.m.,
$x_{f,i} = x_{f,i}(0) - x(t)$	is the horizontal distance from the com to foot i,

4.1. Asymmetric Leg Motion in Double-Support

To preserve symmetric body motion in double-support, asymmetries in leg and hip actuation can compensate for asymmetries in leg motion. We start by finding vertical leg forces that satisfy the requirement for odd body torque.

We can rewrite the definition of symmetric body moment, $J\ddot{\phi}(t) = -J\ddot{\phi}(-t)$, in terms of Cartesian forces (15) and simplify by taking advantage of symmetries in the vertical and horizontal forces. Applying body symmetry to (13) and (14),

$$f_{x,1}(t) + f_{x,2}(t) = -f_{x,1}(-t) - f_{x,2}(-t), \tag{16}$$

$$f_{z,1}(t) + f_{z,2}(t) = f_{z,1}(-t) + f_{z,2}(-t). \tag{17}$$

Cancelling terms and rearranging, the equation for symmetric body moment can be solved for the vertical leg forces,

$$f_{z,1}(t) = -\mathrm{A}f_{z,1}(-t) + (1-\mathrm{A})f_{z,2}(-t) \tag{18}$$

$$f_{z,2}(t) = \mathrm{A}f_{z,2}(-t) + (1+\mathrm{A})f_{z,1}(-t), \tag{19}$$

where

$$\mathrm{A} = \frac{x_{f,1}(0) + x_{f,2}(0)}{x_{f,1}(0) - x_{f,2}(0)}.$$

A is a measure of the asymmetry in position of the feet. Since the feet do not move during stance, A is constant over the stance period. When the feet are symmetrically placed about the origin, $\mathrm{A} = 0$. The degenerate case where $x_{f,1}(0) = x_{f,2}(0)$ occurs when the feet are placed together at the origin, in which case the system behaves as though it were in single-support. If vertical leg forces obey (18) and (19), symmetric body motion can be produced without symmetric leg motion.

4.2. Leg Symmetry under Additional Constraints

In double-support there are enough control variables to allow symmetric body motion without symmetric placement of the feet. However, if additional constraints are put on the model these extra control freedoms are lost. For instance, assume that the legs are springy in the axial direction so they cannot exert arbitrary axial force. They are modelled as springs in compression, with their axial force related to length by $f_i = k/r_i$. We assume that both legs have the same spring constant, $k = k_1 = k_2$. Further assume that the control system restricts hip torques so that they do not fight each other during the support phase. This prevents the system from doing isometric exercises with itself during running. With the addition of these constraints, symmetric body motion once again implies symmetric leg motion in double-support.

Assertion. If body motion is symmetric in double-support, with legs such that $f_i = k/r_i$ and $\mathrm{sgn}(f_{x,1}) = \mathrm{sgn}(f_{x,2})$, then leg motion must be symmetric.

Proof: The fact that both legs have the same stiffness is used to establish that the feet must be placed symmetrically about the origin in order to generate symmetric body moment when $t = 0$.

Expressing axial leg force in terms of Cartesian body forces (11) when $t = 0$,

$$f_i(0) = f_{x,i}(0)\sin\theta_i(0) + f_{z,i}(0)\cos\theta_i(0). \tag{20}$$

Eliminate the first term by observing that the total horizontal force must be zero as required by the assumption of body symmetry, $0 = f_x(0) = f_{x,1}(0) + f_{x,2}(0)$. Because the legs do not oppose each other, $\mathrm{sgn}(f_{x,1}) = \mathrm{sgn}(f_{x,2})$, so $f_{x,1}(0) = f_{x,2}(0) = 0$. Thus axial leg force is solely a function of vertical force and leg angle,

$$f_i(0) = f_{z,i}(0)\cos\theta_i(0). \tag{21}$$

Rearrange (21) noting that $\cos\theta_i = z_i/r_i$, and apply the assumption that each leg acts like a spring $k = f_i(0)r_i(0)$. Both legs have the same stiffness, k, so the product of axial leg force and leg length is constant and equal for both legs. The legs are related by

$$f_{z,1}(0)z_1(0) = f_{z,2}(0)z_2(0). \qquad (22)$$

Body attitude is an odd function, so when $t = 0$ the body must be horizontal and both hips have the same height. Thus vertical forces from each leg must be equal, $f_{z,1}(0) = f_{z,2}(0)$. Odd body moment must equal zero at $t = 0$. Applying body symmetry to (15)

$$0 = -f_{z,1}(0)x_{f,1}(0) - f_{z,2}(0)x_{f,2}(0). \qquad (23)$$

Because vertical forces are equal as established above, the only way to maintain zero body moment when $t = 0$ is for the feet to be symmetrically placed about the midpoint of stance.

$$x_{f,1}(0) = -x_{f,2}(0). \qquad (24)$$

Since the feet cannot move with respect to the ground during stance, leg positions are a function of body position and must have reciprocating symmetry throughout stance, $\theta_1(t) = -\theta_2(-t)$, and $r_1(t) = r_2(-t)$. Because axial leg force is proportional to leg length, axial leg force must be an even function, $f_{r,1}(t) = f_{r,2}(-t)$.

It remains to show that hip torques must exhibit reciprocating symmetry. Recall that (18) with symmetric leg positions reduces to reciprocating symmetry, so vertical leg forces are related by $f_{z,1}(t) = f_{z,2}(-t)$. Converting back to machine coordinates using (11), and eliminating horizontal forces, $f_{x,i}$,

$$\tau_i = \frac{r_i}{\sin\theta_i}\left(f_{z,i} - f_{r,i}\cos\theta_i\right). \qquad (25)$$

It follows that $\tau_1(t) = -\tau_2(-t)$. Thus leg motion must be symmetric for the constrained model to move with symmetric body motion.

5. Conclusion

Symmetric patterns of body and leg motion are interesting because they help simplify the control of running machines, and because they may help us to understand the behavior of running animals. This paper explores the degree to which leg motion must be symmetric, given symmetric body motion, for a simple class of planar models with massless legs. We have shown that symmetric actuation is not always necessary to provide symmetric body motion.

For the single-support case, symmetric body motion requires symmetric motion and actuation of the leg. In double-support, motion of the leg may or may not be symmetric. Symmetric leg motion becomes a necessity once again when the legs are constrained to act like springs and hips do not generate opposing ground forces.

This result may be understood in terms of the degrees of freedom available to each model. The single-support model has two actuators controlling motion in three dimensions. The double-support model has four actuators controlling the same motion. The redundancy allows asymmetry in actuation to compensate for asymmetry in leg motion. When the actuators of the double-support model are constrained by to act like springs and the hip torques do not oppose, the extra degrees of freedom are lost — symmetric actuation is necessary to produce symmetric motion.

6. References

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7. Acknowledgements

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8. Appendix: Symmetric Functions

Symmetric functions are either odd, $f(t) = -f(-t)$, or even, $f(t) = f(-t)$. The following are properties of symmetric functions:

- Any function can be written as the sum of an odd and an even part,

$$f(t) = {}^{e}f(t) + {}^{o}f(t)$$

where

$$ {}^{e}f(t) = 1/2\Big(f(t) + f(-t)\Big),$$
$$ {}^{o}f(t) = 1/2\Big(f(t) - f(-t)\Big).$$

- Constants are even, except $f(t) = 0$, which may be considered both odd and even.

- The cosine of an even or odd function is even. The sine of an even function is even. The sine of an odd function is odd.

- The derivative of an even function is odd. The derivative of an odd function is even.

- All functions can be categorized as either Odd, Even, Mixed, or Zero. Addition and multiplication obey the following rules:

$$\begin{array}{ll}
O+O \rightarrow O|Z & O \cdot O \rightarrow E \\
O+E \rightarrow M & O \cdot E \rightarrow O \\
O+M \rightarrow E|M & O \cdot M \rightarrow M \\
O+Z \rightarrow O & O \cdot Z \rightarrow Z \\
E+E \rightarrow E|Z & E \cdot E \rightarrow E \\
E+M \rightarrow O|M & E \cdot M \rightarrow M \\
E+Z \rightarrow E & E \cdot Z \rightarrow Z \\
M+M \rightarrow Z|E|O|M & M \cdot M \rightarrow E|O|M. \\
M+Z \rightarrow M & M \cdot Z \rightarrow Z \\
Z+Z \rightarrow Z & Z \cdot Z \rightarrow Z
\end{array}$$

- *Reciprocating symmetry* describes pairs of functions that together form either an odd reciprocating pair, $f_1(t) = -f_2(-t)$, or an even reciprocating pair, $f_1(t) = f_2(-t)$.